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GEOMETRIC SOLUTIONS OF A QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEM HAVING THE FIRST INTEGRAL

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ABSTRACT. In this paper, we classify the geometric solutions of a class of integrable quadratic polynomial differential systems. We provide a new class of quadratic polynomial differential system that exhibits the so-called *cubic egg* curve as a first integral, we analyze the geometric solutions of this class in the Poincaré disc.

1. Introduction and statement of the main results

In the plane all polynomial differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where P, $Q \in \mathbb{R}[x, y]$, such that $\max\{\deg(P), \deg(Q)\} = 2$ are called *Quadratic polynomial differential systems*. Here, the dot denotes, as usual, differentiation with respect to time t. For such systems, we can always associate the quadratic vector field

$$\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y.$$

Let U be an open subset of $\mathbb{R}^2$. A function $H : U \to \mathbb{R}$ is called a first integral of (1) if H is constant on all solution curves of (1), that is,

$$\frac{dH}{\mathrm{d}t} = \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q = 0.$$

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We say that a quadratic system is integrable if it has a first integral $H : U \to \mathbb{R}$. For more details on the first integral of a polynomial differential system, see Chapter 8 of [6].

Quadratic differential systems have been intensively studied, and more than a thousand papers have been published on these differential systems; for more details, see the references cited in the books of Ye [16] and Reyn [13]. However, even so, the issue of classifying all integrable quadratic systems remains open.

Since the 1960s, there have been several studies interesting in classifying the global phase portraits of quadratic systems, which are not easy to study because these systems are dependent on twelve parameters.

Therefore, the authors studied the global phase portraits of some subcases of these systems. One of the first of these classes studied was the classification of the quadratic centers and their first integrals which started with the works of Dulac [5], Kapteyn [7,8], Bautin [2], Lunkevich and Sibirskii [9], Schlomiuk [14], Wei and Ye [15, 16], Artés, Llibre and Vulpe [1]. In [3] Benterki and Llibre classified the global phase portraits of quadratic polynomial differential systems that exhibit some relevant classic quartic algebraic curves as invariant algebraic curves. In [4] Belfar and Benterki classified the global phase portraits of six quadratic polynomial differential systems, which exhibit six well-known algebraic curves of degree six, etc.

In this paper, we study the global phase portraits in the Poincaré disc of a new class of integrable quadratic systems. More precisely we realize that these differential systems have seventeen phase portraits topologically non--equivalent.

Our first main result is the following.

Theorem 1.1. *The algebraic Cubic egg curve given by*

$$H(x,y) = x^2 + (1+x)y^2 - 1 = 0$$

is the first integral of the quadratic system,

$$\dot{x} = -1 + x^2 + ay + axy, \quad \dot{y} = \tfrac{a}{2} + \left(b - \tfrac{a}{2}\right)x + \tfrac{y}{2} + \tfrac{xy}{2} + by^2, \tag{2}$$

so, its associated cofactor $k(x,y) = 0$.

Proof of Theorem 1.1. To prove this theorem, we can easily verify that for the system (2) we have the partial differential equation

$$\frac{dH}{\mathrm{d}t} = \dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} = 0.$$

So H is the first integral of the system (2). □

In the next theorem, we present the topological classification of all phase portraits of the planar polynomial differential system of degree 2 having the first integral Cubic egg curve in the Poincaré disc. For a definition of singular points and Poincaré disc and for a definition of topological equivalent phase portraits of a polynomial differential system in the Poincaré disc see Sections 2.

Theorem 1.2. *The global phase portraits of the quadratic system* (2) *are topologically equivalent to the phase portrait*

(1) *for* $a = 2b$ *and* $b \in \left(0, \frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, +\infty\right)$;

(2) *for* $a = 0$ *and* $b = 0$;

(3) *for* $a = \sqrt{2}$ *and* $b = \frac{\sqrt{2}}{2}$;

(4) *for* $a = b$ *and* $b \neq 0$;

(5) *for* $a = \sqrt{2}$ *and* $b = 0$;

(6) *for* $a \in (0, \sqrt{2}) \cup (\sqrt{2}, +\infty)$ *and* $b = 0$;

(7) *for* $a \in (0, \sqrt{2})$ *and* $b \in (0, b_2)$;
$a \in \left(\sqrt{2}, \frac{9}{4}\right)$ *and* $b \in (b_1, b_2)$ *or* $a \in (0, b_0)$ *and* $b \in (-\infty, b_1)$
with $b_0 = b + \sqrt{b^2 + 2}$, $b_1 = \frac{-9+8a^2-\sqrt{81-16a^2}}{16a}$ *and* $b_2 = \frac{-9+8a^2+\sqrt{81-16a^2}}{16a}$;

(8) *for* $a \in \left(0, \sqrt{2}\right)$ *and* $b \in \left(b_2, \frac{\sqrt{2}}{2}\right)$;

(9) *for* $a \in \left(\sqrt{2}, \frac{9}{4}\right)$ *and* $b \in \left(0, \frac{a^2-2}{2a}\right) \cup \left(\frac{a^2-2}{2a}, b_1\right) \cup \left(b_2, \frac{a}{2}\right)$;
$a = \frac{9}{4}$ *and* $b \in \left(0, \frac{49}{72}\right) \cup \left(\frac{49}{72}, \frac{9}{8}\right)$;
$a \in \left(\frac{9}{4}, b_0\right)$ *and* $b \in \left(\frac{a^2-2}{2a}, \frac{a}{2}\right)$ *or* $a \in (b_0, +\infty)$ *and* $b \in \left(0, \frac{a^2-2}{2a}\right)$ *or*
$a \in (0, b_0)$ *and* $b \in (b_1, 0)$ *or* $a \in (b_0, +\infty)$ *and* $b \in \left(\frac{a}{2}, a\right) \cup (a, +\infty)$;

(10) *for* $a = \sqrt{2}$ *and* $b \in \left(0, \frac{7}{8\sqrt{2}}\right) \cup \left(\frac{7}{8\sqrt{2}}, \frac{\sqrt{2}}{2}\right)$;

(11) *for* $a = 0$ *and* $b \in (-\infty, 0)$ *or* $a = \frac{9}{4}$ *and* $b \in \left(\frac{9}{8}, \frac{9}{4}\right) \cup \left(\frac{9}{4}, +\infty\right)$;

(12) *for* $a = \sqrt{2}$ *and* $b \in \left(\frac{\sqrt{2}}{2}, \sqrt{2}\right) \cup \left(\sqrt{2}, +\infty\right)$;

(13) *for* $a = \sqrt{2}$ *and* $b \in (-\infty, 0)$;

(14) *for* $a \in \left(\sqrt{2}, +\infty\right)$ *and* $b \in (-\infty, 0)$;

(15) *for* $a \in (b_0, \sqrt{2})$ *and* $b \in (-\infty, b_1) \cup (b_1, 0)$;

(16) *for* $a = b_0$ *and* $b \in (-\infty, 0)$;

(17) *for* $a = b_0$ *and* $b \in \left(0, \frac{b_0}{2}\right) \cup \left(\frac{b_0}{2}, b_0\right) \cup (b_0, +\infty)$.

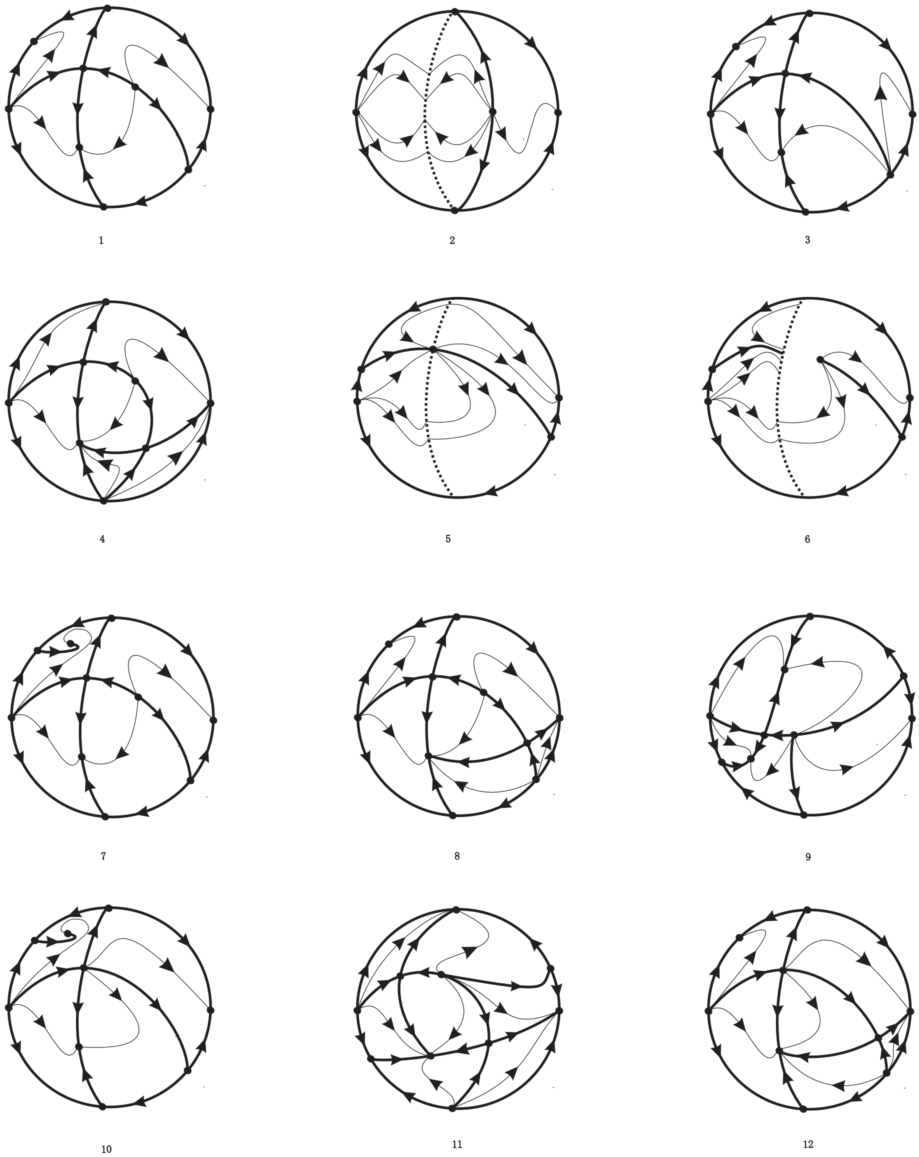

FIGURE 1. Phase portraits in the Poincaré disc of the system 2.

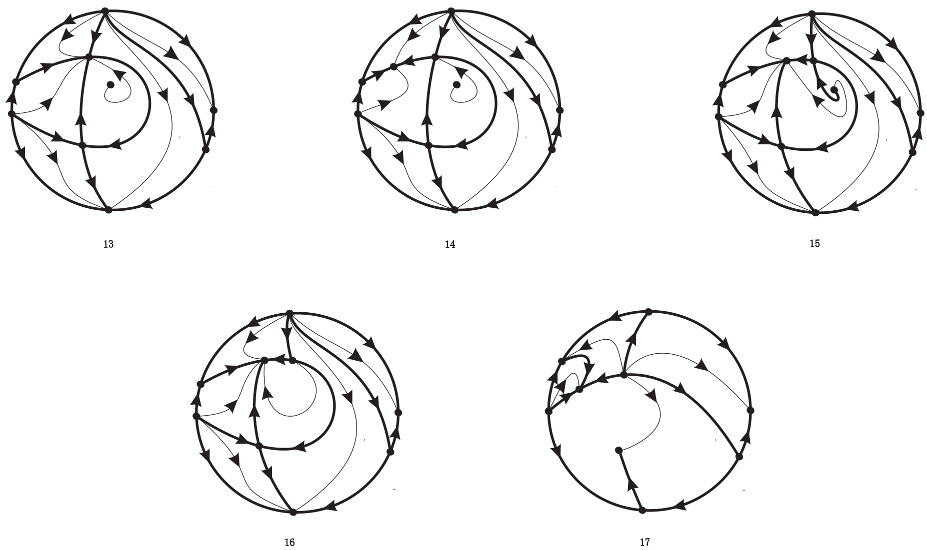

FIGURE 2. Continuation of Figure 1.

2. Preliminaries and basic results

In this section, we present some basic results and notations that are necessary to prove our results.

2.1. Phase portraits on the Poincaré disc

In this subsection, we are going to see how to characterize the global phase portraits in the Poincaré disc of the differential system (2).

A *s*eparatrix of $p(\mathcal{X})$ is an orbit that is either a singular point, or a limit cycle, or a trajectory that lies in the boundary of a hyperbolic sector at a singular point. Neumann [11] proved that the set formed by all separatrices of $p(\mathcal{X})$; denoted by $S\big(p(\mathcal{X})\big)$ is closed.

The open connected components of $\mathbb{D}^2 \setminus S(p(\mathcal{X}))$ are called *canonical regions* of $p(\mathcal{X})$: We define a *separatrix configuration* as a union of $S\big(p(\mathcal{X})\big)$ plus one solution chosen from each canonical region. Two separatrices configurations $S\big(p(\mathcal{X})\big)$ and $S\big(p(\mathcal{Y})\big)$ are said to be *t*opologically equivalent if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S\big(p(\mathcal{X})\big)$ into the trajectories of $S\big(p(\mathcal{Y})\big)$.

The following result is due to [10] Markus, [11] Neumann and [12] Peixoto.

Theorem 2.1. *The phase portraits in the Poincaré disc of the two compactified polynomial differential systems $p(\mathcal{X})$ and $p(\mathcal{Y})$ are topologically equivalent if and only if their separatrix configurations $S\big(p(\mathcal{X})\big)$ and $S\big(p(\mathcal{Y})\big)$ are topologically equivalent.*

Due to this theorem in the phase portraits in the Poincaré disc of Figures 1 and 2 we plot at least one orbit in each canonical region.

Remark 1. The system (2) is invariant under the change

$$(x, y, t, a, b) \longrightarrow (x, -y, t, -a, -b),$$

then we only need to study it for $a > 0$ or $a = 0$ and $b \geq 0$.

The method for studying the phase portraits of the systems (2)is the following. First, we characterize all the finite singular points of this system together with their local phase portraits. We then do the same for the infinite singularities.

2.2. The finite singular points

Taking into account the symmetry given in Remark (1), then the finite singular points of system (2) are given in the following proposition.

Proposition 2.2. *The following statements hold for the quadratic system* (2)*.*

(I) *Assume $a = 2b$.*
- (i) *If $b \in \big(0, \frac{\sqrt{2}}{2}\big)$ system* (2) *has three hyperbolic singularities: a stable node at p_1, a saddle at p_2 and an unstable node at q_1, such that $p_1 = (-1, -\sqrt{2})$, $p_2 = (-1, \sqrt{2})$ and $q_1 = (1 - 4b^2, 2b)$.*
- (ii) *If $b \in \big(\frac{\sqrt{2}}{2}, +\infty\big)$ system* (2) *has three hyperbolic singularities: a stable node at p_1, an unstable node at p_2 and a saddle at q_1.*
- (iii) *If $b = 0$ and $a = 0$ the system has a line of singularities $x + 1 = 0$, and a hyperbolic unstable node at q_2, with $q_2 = (1, 0)$.*
- (iv) *If $b = \frac{\sqrt{2}}{2}$ and $a = \sqrt{2}$ the system has two singularities: a stable node at p_1 and saddle-node at p_2.*

(II) *Assume $a \neq 2b$.*
- (i) *For $a \neq \sqrt{2}$ and $b \neq 0$ the system has four hyperbolic singularities p_1, p_2, p_3 and p_4, such that $p_1 = (-1, -\sqrt{2})$, $p_2 = (-1, \sqrt{2})$, $p_3 = (1 - a^2, a)$ and $p_4 = \big(1 - \frac{2a}{a-2b}, \frac{2}{a-2b}\big)$, and we have the following subcases:*
  - (a) *If $a \in (0, b_0)$ and $b \in (-\infty, 0)$ system* (2) *has a saddle at p_1, a stable node at p_2 and an unstable node at p_3, with $b_0 = b + \sqrt{b^2 + 2}$.*
  - (b) *If $a \in (b_0, \sqrt{2})$ and $b \in (-\infty, 0)$ it has two saddles at p_1 and p_3, and a stable node at p_2.*

(c) *If* $a \in (\sqrt{2}, +\infty)$ *and* $b \in (-\infty, 0)$ *it has two saddles at* p_1 *and* p_2*, and a stable node at* p_3*.*
(d) *If* $a \in [0, \sqrt{2})$ *and* $b \in (0, +\infty)$ *system* (2) *has a stable node at* p_1*, a saddle at* p_2 *and an unstable node at* p_3*.*
(e) *If* $a \in (\sqrt{2}, b_0)$ *and* $b \in (0, +\infty)$ *it has a stable node at* p_1*, an unstable node at* p_2 *and a saddle at* p_3*.*
(f) *If* $a \in (b_0, +\infty)$ *and* $b \in (0, +\infty)$ *it has two stable nodes at* p_1 *and* p_3*, and an unstable node at* p_2*.*

The nature of the fourth singular point p_4 *for all the six previous cases is given in Table 1.*

TABLE 1. The local phase portraits of the singular point p_4.

$0 < a < \sqrt{2}$	$b \in (-\infty, b_1)$	*unstable focus*
	$b \in (0, b_2)$	*stable focus*
	$b \in (b_1, 0)$	*unstable node*
	$b \in (b_2, +\infty)$	*saddle*
$\sqrt{2} < a < \frac{9}{4}$	$b \in (-\infty, 0)$	*unstable focus*
	$b \in \big(0, (a^2-2)/2a\big)$	*saddle*
	$b \in \big((a^2-2)/2a, b_1\big)$	*stable node*
	$b \in (b_1, b_2)$	*stable focus*
	$b \in (b_2, a/2)$	*stable node*
	$b \in (a/2, +\infty)$	*saddle*
$a > \frac{9}{4}$	$b \in (-\infty, 0)$	*unstable focus*
	$b \in \big((a^2-2)/2a, a/2\big)$	*stable node*
	$b \in (0, (a^2-2)/2a) \cup (a/2, +\infty)$	*saddle*
$a = 0$	*saddle*	
$a = \frac{9}{4}$	$b \in (-\infty, 0)$	*unstable focus*
	$b \in (0, 49/72)$ *or* $b \in (9/8, +\infty)$	*saddle*
	$b \in (49/72, 9/8)$	*stable node*
$b_1 = \frac{-9+8a^2-\sqrt{81-16a^2}}{16a}$, $\quad b_2 = \frac{-9+8a^2+\sqrt{81-16a^2}}{16a}$		

(ii) *If* $a \neq \sqrt{2}$ *and* $b = 0$ *it has a hyperbolic node at* p_3 *and a line of singularities* $x + 1 = 0$.

(iii) *If* $a = \sqrt{2}$ *and* $b \in (-\infty, 0)$ *it has two hyperbolic singularities a saddle at* p_1 *and an unstable focus at* q_3 *and a semi-hyperbolic saddle-node at* p_2, *where* $q_3 = \left(1 + \frac{2}{\sqrt{2}b-1}, \frac{\sqrt{2}}{1-\sqrt{2}b}\right)$.

(iv) *If* $a = \sqrt{2}$ *and* $b \in \left(0, \frac{7}{8\sqrt{2}}\right)$ *it has two hyperbolic singularities a stable node at* p_1 *and a stable focus at* q_3 *and a semi-hyperbolic saddle-node at* p_2.

(v) *If* $a = \sqrt{2}$ *and* $b \in \left(\frac{7}{8\sqrt{2}}, \frac{\sqrt{2}}{2}\right)$ *it has two hyperbolic stable nodes at* p_1 *and* q_3 *and a semi-hyperbolic saddle-node at* p_2.

(vi) *If* $a = \sqrt{2}$ *and* $b \in \left(\frac{\sqrt{2}}{2}, +\infty\right)$ *it has two hyperbolic singularities a stable node at* p_1 *and a saddle at* q_3 *and a semi-hyperbolic saddle-node at* p_2.

(vii) *If* $a = \sqrt{2}$ *and* $b = 0$ *it has a hyperbolic unstable node at* p_2 *and a line of singularities* $x + 1 = 0$.

(viii) *If* $a = b_0$ *and* $b \in (-\infty, 0)$ *it has two hyperbolic singularities a saddle at* p_1 *and a stable node at* p_2 *and a semi-hyperbolic saddle-node at* q_4, *where* $q_4 = (-1 - 2b(b + \sqrt{2 + b^2}, b + \sqrt{2 + b^2})$.

(ix) *If* $a = b_0$ *and* $b \in (0, +\infty)$ *it has two hyperbolic singularities a stable node at* p_1 *and an unstable node at* p_2 *and a semi-hyperbolic saddle-node at* q_4.

3. Proof of Proposition 2.2

Proof of statement (I) of Proposition 2.2. If $a = 2b$ the differential system (2) becomes

$$\dot{x} = -1 + x^2 + 2by + 2bxy, \quad \dot{y} = -2b + \tfrac{y}{2} + \tfrac{xy}{2} + by^2. \tag{3}$$

This system is invariant under the change

$$(x, y, t, b) \longrightarrow (x, -y, t, -b),$$

then we only need to study system (3) for $b \geq 0$. The differential system (3) has three singularities p_1, p_2 and q_1, such that

$$p_1 = \left(-1, -\sqrt{2}\right), \quad p_2 = \left(-1, \sqrt{2}\right) \quad \text{and} \quad q_1 = (1 - 4b^2, 2b).$$

The eigenvalues of the matrix of vector field in (3) at p_1 are $-2\sqrt{2}b$ and $-2(1 + \sqrt{2}b)$, at are $2\sqrt{2}b$ and $(-2 + \sqrt{2}b)$ and at q_1 are $(1 - 2b^2)$ and 2.

Then we have the following results:

- If $b \in \left(0, \frac{\sqrt{2}}{2}\right)$, p_1 is a hyperbolic stable node, p_2 is a hyperbolic saddle, and q_1 is a hyperbolic unstable node. So, the statement (i) holds for $b \in \left(0, \frac{\sqrt{2}}{2}\right)$.
- If $b \in \left(\frac{\sqrt{2}}{2}, +\infty\right)$, p_1 has the same nature as in the previous case, p_2 is a hyperbolic unstable node and q_1 is a hyperbolic saddle. So, statement (ii) holds for $b \in \left(\frac{\sqrt{2}}{2}, +\infty\right)$.
- If $b = 0$ and $a = 0$ the differential system (3) takes the from

$$\dot{x} = x^2 - 1, \quad \dot{y} = \frac{y}{2} + \frac{xy}{2}. \tag{4}$$

This system has a line of singularities $x + 1 = 0$, then if we take the change of variable $\mathrm{d}s = (x+1)\mathrm{d}t$, we get the following system:

$$\dot{x} = x - 1, \quad \dot{y} = \frac{y}{2} \tag{5}$$

The system (5) has a hyperbolic unstable node at $q_2 = (1, 0)$ with its corresponding eigenvalues 1 and $\frac{1}{2}$, which means that q_2 is a hyperbolic unstable node. So, statement (iii) is valid for $b = 0$ and $a = 0$.

- If $b = \frac{\sqrt{2}}{2}$ and $a = \sqrt{2}$ the differential system (3) becomes

$$\dot{x} = -1 + x^2 + \sqrt{2}y + \sqrt{2}xy, \quad \dot{y} = -\sqrt{2} + \tfrac{y}{2} + \tfrac{xy}{2} + \tfrac{y^2}{\sqrt{2}}. \tag{6}$$

This system has a hyperbolic stable node at p_1 with eigenvalues -4 and -2, and semi-hyperbolic singularity at p_2 with eigenvalues 0 and 2.

In order to know the local phase portrait at p_2, first we must translate p_2 at the origin of coordinates by doing the change of variables $(x, y) = (x_1 - 1, y_1 + \sqrt{2})$, and we get the following system

$$\dot{x_1} = x_1^2 + \sqrt{2}x_1y_1, \quad \dot{y_1} = \frac{\sqrt{2}}{2}x_1 + 2y_1 + \frac{x_1y_1}{2} + \frac{y_1^2}{\sqrt{2}}. \tag{7}$$

Second, we put the system (7) into the normal form

$$\dot{x} = P(x, y), \quad \dot{y} = y + Q(x, y). \tag{8}$$

To get the normal form of system (7) we consider the following change of variables

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad P = \begin{pmatrix} -4 & 0 \\ \sqrt{2} & 1 \end{pmatrix} \tag{9}$$

presents the passage matrix of the linear part of the system (7).

By doing the change of the variables $\frac{1}{2}\mathrm{d}t = \mathrm{d}s$ and by considering the change of the variables (9), system (7) becomes

$$\dot{x} = \tfrac{1}{2}\big(\sqrt{2}xy + 2x^2\big), \quad \dot{y} = y + \tfrac{y^2}{2\sqrt{2}} - xy + \tfrac{x^2}{\sqrt{2}}. \tag{10}$$

This is the normal form of the differential system (7), where

$$P(x,y) = \frac{1}{2}\big(\sqrt{2}xy - 2x^2\big) \quad \text{and} \quad Q(x,y) = -xy + \frac{x^2}{\sqrt{2}} + \frac{y^2}{2\sqrt{2}}.$$

By solving the equation $y + Q(x,y) = 0$ with respect to the variable y, we obtain

$$y = \psi(x) = -\frac{x^2}{\sqrt{2}}.$$

We substitute the value of $y = \psi(x)$ in the expression of $P(x,y)$ and get

$$\varphi(x) = P\big(x, \psi(x)\big) = -2x^2 - x^3.$$

By applying [6, Theorem 2.19] we know that p_2 is a semi-hyperbolic saddle-node. Then the statement (iv) holds for $b = \frac{\sqrt{2}}{2}$ and $a = \sqrt{2}$. $\square$

P r o o f o f s t a t e m e n t (II) o f P r o p o s i t i o n 2.2. If $a \neq \sqrt{2}$ and $b \neq 0$ the differential system (2) has in addition to p_1 which has the eigenvalues $-2\sqrt{2}b$ and $-2 - \sqrt{2}a$, and p_2 with its corresponding eigenvalues $2\sqrt{2}b$ and $-2 + \sqrt{2}a$, a third singular point p_3 with eigenvalues $-a^2 + 2ab + 2$ and $\frac{-a^2}{2} + 1$. Then we have the following results:

- If $a \in (0, b_0)$ and $b \in (-\infty, 0)$, p_1, p_2 and p_3 are a hyperbolic saddle, a stable node and an unstable node respectively, where $b_0 = b + \sqrt{b^2 + 2}$.
- If $a \in (b_0, \sqrt{2})$ and $b \in (-\infty, 0)$, p_1 and p_3 are two hyperbolic saddles and p_2 is a hyperbolic stable node.
- If $a \in (\sqrt{2}, +\infty)$ and $b \in (-\infty, 0)$, p_1 and p_2 are two hyperbolic saddles, p_3 is a hyperbolic stable node.
- If $a \in [0, \sqrt{2})$ and $b \in (0, +\infty)$, then p_1, p_2 and p_3 are a hyperbolic stable node, a saddle and an unstable node, respectively.
- If $a \in (\sqrt{2}, b_0)$ and $b \in (0, +\infty)$, p_1, p_2 and p_3 are a hyperbolic stable node, an unstable node, and a saddle, respectively.
- If $a \in (b_0, +\infty)$ and $b \in (0, +\infty)$, p_1 and p_3 are two hyperbolic stable nodes and p_2 is a hyperbolic unstable node. Then the statement (a), (b), (c), (d), (e) and (f) hold.

Now we are going to study the local phase portraits of the singular point p_4. At p_4 the Jacobian matrix of the vector field defined in (2) is given by:

$$A = D\chi\left(\frac{-a-2b}{a-2b}, \frac{2}{a-2b}\right) = \begin{pmatrix} -\frac{4b}{a-2b} & \frac{4ab}{a-2b} \\ -\frac{a}{2}+\frac{1}{a-2b}+b & \frac{2b}{a-2b} \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_{1,2} = \frac{b \pm \sqrt{b(2a(-2+(a-2b)^2)+9b)}}{2b-a}.$$

We distinguish the following cases:

- If $a \in (0,\sqrt{2})$ and $b \in (-\infty, b_1)$, $a \in \left(\sqrt{2}, \frac{9}{4}\right)$ and $b \in (-\infty, 0)$ or $a \in \left[\frac{9}{4}, +\infty\right)$ and $b \in (-\infty, 0)$, the eigenvalues λ_1 and λ_2 are complex and their real parts are positive. Then p_4 is an unstable focus, where $b_1 = \frac{-9+8a^2-\sqrt{81-16a^2}}{16a}$.
- If $a \in (0,\sqrt{2})$ and $b \in (0, b_2)$ or $a \in \left(\sqrt{2}, \frac{9}{4}\right)$ and $b \in (b_1, b_2)$, the eigenvalues λ_1 and λ_2 are complex and their real parts are negative. Then p_4 is a stable focus, where
$$b_2 = \frac{-9+8a^2+\sqrt{81-16a^2}}{16a}.$$

  For the remaining cases λ_1 and λ_2 are reals, and
$$\lambda_1 + \lambda_2 = -\frac{2b}{a-2b} \quad \text{and} \quad \lambda_1.\lambda_2 = -\frac{2b(-2+a^2-2ab)}{a-2b}.$$
  So the local phase portrait of p_4 is detailed in Table 1.
- If $a \neq \sqrt{2}$ and $b = 0$ the differential system (2) becomes
$$\dot{x} = -1 + x^2 + ay + axy, \qquad \dot{y} = \frac{1}{2}(-a - ax + y + xy). \tag{11}$$
  This system has a line of singularities $x+1=0$. Then, we take the change of variable $\mathrm{d}s = (x+1)\mathrm{d}t$, and we have
$$\dot{x} = (1-x) + ay, \qquad \dot{y} = \frac{y-a}{2}. \tag{12}$$
  This system has a singular point p_3 with eigenvalues $2-a^2$ and $\frac{1}{2}(2-a^2)$. Then p_3 is a hyperbolic unstable node if $a \in [0,\sqrt{2})$ and a hyperbolic stable node if $a \in (\sqrt{2}, +\infty)$. So statement (ii) holds.
- If $a = \sqrt{2}$ the differential system (2) becomes
$$\dot{x} = -1+x^2+\sqrt{2}y+\sqrt{2}xy, \quad \dot{y} = -\tfrac{1}{\sqrt{2}}-b+\left(b-\tfrac{1}{\sqrt{2}}\right)x+\tfrac{y}{2}+\tfrac{xy}{2}+by^2. \tag{13}$$

This system has three singularities p_1, p_2 and q_3, with

$$q_3 = \left(1 + \frac{2}{\sqrt{2}b - 1}, \frac{\sqrt{2}}{1 - \sqrt{2}b}\right).$$

At p_1 has the eigenvalues -4 and $-2\sqrt{2}b$. So it's a hyperbolic saddle if $b \in (-\infty, 0)$ and a hyperbolic stable node if

$$b \in \mathbb{R}_*^+ \Big\backslash \left\{\frac{7}{8\sqrt{2}}, \frac{\sqrt{2}}{2}\right\}.$$

the singularity p_2 is a semi-hyperbolic singular point with eigenvalues

$$2\sqrt{2}b \quad \text{and} \quad 0.$$

By applying [6, Theorem 2.19] we know that p_2 is a semi-hyperbolic saddle-node.

For the third singularity q_3 which has

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2(8\sqrt{2}b - 7)}}{\sqrt{2} - 2b},$$

where

$$\lambda_1.\lambda_2 = \frac{b^2(8 - 8\sqrt{2}b)}{(\sqrt{2 - 2b^2} - 2b)^2}, \quad \text{and} \quad \lambda_1 + \lambda_2 = \frac{-2b}{\sqrt{2} - 2b}.$$

According of the sign of b we know that p_3 is: a Hyperbolic unstable focus, if $b \in (-\infty, 0)$, a hyperbolic stable focus, if $b \in \left(0, \frac{7}{8\sqrt{2}}\right)$, a hyperbolic stable node if $b \in \left(\frac{7}{8\sqrt{2}}, \frac{\sqrt{2}}{2}\right)$ and a hyperbolic saddle if $b \in \left(\frac{\sqrt{2}}{2}, +\infty\right)$.

In summary, statements (iv), (v) and (vi) hold.

- If $a = \sqrt{2}$ and $b = 0$ the differential system (2) becomes

$$\dot{x} = (1 + x)(-1 + x + \sqrt{2}y), \quad \dot{y} = -\tfrac{1}{2}(1 + x)(\sqrt{2} - y). \tag{14}$$

This system has a line of singularities $x + 1 = 0$. We take the change of variable $\mathrm{d}s = (x + 1)\mathrm{d}t$, we get the following system

$$\dot{x} = -1 + x + \sqrt{2}y, \quad \dot{y} = -\tfrac{1}{2}(\sqrt{2} - y). \tag{15}$$

which has a hyperbolic saddle at the p_2 with its corresponded eigenvalues 1 and $\frac{1}{2}$. Then the statement (vii) holds.

- If $a = b_0$ with $b_0 = b + \sqrt{b^2+2}$ the differential system (2) becomes

$$\begin{aligned} \dot{x} &= -1 + x^2 + (b + \sqrt{2+b^2})y + (b + \sqrt{2+b^2})xy, \\ \dot{y} &= -b - \frac{1}{2}(b + \sqrt{2+b^2}) + \Big(b - \frac{1}{2}(b + \sqrt{2+b^2})\Big)x \\ &\quad + \frac{y}{2} + \frac{xy}{2} + by^2. \end{aligned} \tag{16}$$

This system has three singularities p_1, p_2 and q_4, where

$$q_4 = \Big(-1 - 2b(b + \sqrt{2+b^2}, b + \sqrt{2+b^2})\Big).$$

- The singularities p_1 and p_2 have the eigenvalues

$$-2\sqrt{2}b \quad \text{and} \quad -2 - \sqrt{2}(b + \sqrt{2+b^2}),$$

and

$$2\sqrt{2}b \quad \text{and} \quad \lambda_2 = -2 + \sqrt{2}(b + \sqrt{2+b^2}),$$

respectively.
So p_1 is a hyperbolic saddle and p_2 is a hyperbolic stable node, if $b \in (-\infty, 0)$, and p_1 is a hyperbolic stable node and p_2 is an unstable node, if $b \in (0, +\infty)$.

The third point q_4 is a semi-hyperbolic singular point with eigenvalues $-b(b + \sqrt{2+b^2})$ and 0. By applying [6, Theorem 2.19] we know that q_4 is a semi-hyperbolic saddle-node. Then the statements (viii) and (ix) hold.

□

3.1. Infinite singular points

The main goal of this part is to give the local phase portraits of system (2) at its infinite singular points. To give a full study of the infinite singular points in the Poincaré disc we present the analysis of the vector field at infinity.

Proposition 3.1. *The local phase portraits at the infinite singular points of system* (2) *in the local chart U_1 consist of*

(i) *Two singular points: the first singularity is a hyperbolic stable node at $q_1 = (0,0)$ and the second singularity at $q_2 = \big(\frac{1}{2(b-a)}, 0\big)$, where $a \neq b$, which is a hyperbolic unstable node if $b \in (a/2, a)$, a hyperbolic saddle if $b \in (-\infty, a/2) \cup (a, +\infty)$, and a semi-hyperbolic saddle-node if $a = 2b$.*

(ii) *One singular point at $q_1 = (0,0)$, which is a stable node, if $a = b$.*

Now, we give the local phase portraits of the origin of the local chart U_2.

Proposition 3.2. *The origin of the local chart U_2 is:*

(i) *An unstable node if $b \in (-\infty, 0)$, a saddle if $b \in (0, a)$ and a stable node if $b \in (a, +\infty)$.*

(ii) *Not a singularity if $b = 0$ and $a \neq 0$.*

(iii) *A semi-hyperbolic saddle-node if $a = b$ and $b \neq 0$.*

(iv) *A linearly zero singularity where its local phase portrait consists of four hyperbolic sectors if $a = 0$ and $b = 0$.*

4. Proof of Propositions 3.1 and 3.2

Proof of Proposition 3.1. The expression of system (2) in the local chart U_1 is given by

$$\begin{aligned} \dot{u} &= \left(\tfrac{1}{2}\right)\Big(-2u^2(a-b+av)+u(-1+v+2v^2)-v\big(2b(-1+v)+a(1+v)\big)\Big), \\ \dot{v} &= -v(1+au-v)(1+v). \end{aligned} \tag{17}$$

Any arbitrary infinite singular point of the differential system (17) takes the form $(u_0, 0)$. If $a \neq b$ the differential system (17) has two singular points a hyperbolic stable node at $q_1 = (0, 0)$ with eigenvalues

$$-1 \quad \text{and} \quad -\frac{1}{2}$$

and the second one at $q_2 = \left(\frac{1}{2(b-a)}, 0\right)$ with eigenvalues

$$\lambda_1 = \frac{1}{2} \quad \text{and} \quad \lambda_2 = -\frac{(a-2b)}{2(a-b)}.$$

So q_2 is an unstable node if $b \in (a/2, a)$, a saddle if $b \in (-\infty, a/2) \cup (a, +\infty)$ and a semi-hyperbolic singularity if $a = 2b$. To know the local phase portrait at q_2, we apply [6, Theorem 2.19] we know that q_2 is a semi-hyperbolic saddle-node. Then statement (i) holds. □

Proof of Proposition 3.2. The differential system (2) in the local chart U_2 takes the form:

$$\begin{aligned} \dot{u} &= \tfrac{1}{2}\big((u-2v)(u+v) + a(u+v)(2+uv) + 2bu(-1-uv+v^2)\big), \\ \dot{v} &= \tfrac{1}{2}v\big((u+v)(-1+av) - 2b(1+uv-v^2)\big). \end{aligned} \tag{18}$$

The origin is a singular point of this system with eigenvalues $(-b)$ and $(a-b)$. So, the origin is an unstable node if $b \in (-\infty, 0)$, a saddle if $b \in (0, a)$, and a stable node if $b \in (a, +\infty)$. Then statement (i) holds.

- If $a \neq 0$ and $b = 0$ the differential system (18) becomes

$$\dot{u} = (u+v)\big(u-2v+a(2+uv)\big), \qquad \dot{v} = \tfrac{1}{2}v(u+v)(-1+av). \tag{19}$$

This system has $u + v = 0$ as a line of singularities. Then, we take the change of variable $(u + v)\mathrm{d}t = \mathrm{d}s$, and we get

$$\dot{u} = u - 2v + a(2 + uv), \qquad \dot{v} = \tfrac{1}{2}v(-1 + av). \tag{20}$$

It is clear that the origin is not a singularity for this system. Then statement (ii) holds.

- If $a = b$ and $b \neq 0$ the differential system (18) becomes

$$\begin{aligned} \dot{u} &= \tfrac{1}{2}\Big(u^2 - uv - 2v^2 - 2bu(1 + uv - v^2) + b\big(2v + u^2v + u(2 + v^2)\big)\Big), \\ \dot{v} &= \tfrac{1}{2}v\big((u+v)(bv-1) - 2b(1 + uv - v^2)\big). \end{aligned} \tag{21}$$

The origin is a semi-hyperbolic singular point of this system with eigenvalues $-b$ and 0. Applying [6, Theorem (2.19)] we get that the origin is a semi-hyperbolic saddle-node. Then the statement (iii) holds.

- If $a = 0$ and $b = 0$, the differential system (18) becomes

$$\dot{u} = \frac{1}{2}(u^2 - uv - 2v^2), \qquad \dot{v} = \frac{1}{2}(-u - v)v. \tag{22}$$

The origin is a linearly zero singular point. We need to do the blow-up to describe the local behaviour at this point. We perform the directional blow-up

$$(u, v) \longrightarrow (u, w), \text{ with } w = \frac{v}{u}$$

and we get the following system:

$$\dot{u} = -\frac{1}{2}u^2(-1 - w + 2w^2), \qquad \dot{w} = uw(w^2 - 1). \tag{23}$$

We eliminate the common factor u between $\dot{u}$ and $\dot{w}$ by doing the rescaling of the independent variable $u\,\mathrm{d}t = \mathrm{d}s$ then we get

$$\dot{u} = -\frac{1}{2}u(-1 - w + 2w^2), \qquad \dot{w} = w(w^2 - 1). \tag{24}$$

Going back through the change of variables $w = \frac{v}{u}$ and $u\,\mathrm{d}t = \mathrm{d}s$, we conclude that the local phase portrait of the origin consists of four hyperbolic sectors. Then the statement (iv) holds. □

According to Proposition 2.2, Proposition 3.1, and Proposition 3.2, and using the invariant straight lines for each case and taking into account the direction of the vector field of the system (2) on the axes, we obtain the global phase portraits of the system (2) in the Poincaré disc described in Figures 1 and 2.

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