TaTra
MouNTaiNS
Mathematical Publications

EXISTENCE RESULTS OF MILD SOLUTIONS FOR SECOND ORDER PERTURBED PSEUDO INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS WITH INFINITE STATE-DEPENDENT DELAY IN FRÉCHET SPACES

Imane Abibssi — Selma Baghli-Bendimerad

Department of Mathematics, Djillali Liabes University, Sidi Bel-Abbes, ALGERIA

Dedicated to Prof. Noureddine GHOUALI

ABSTRACT. This paper is concerned with the existence of mild solutions in Fréchet spaces combined with semigroup theory, using Avramescu's nonlinear alternative for the sum of compact operators and contraction maps, to the second order perturbed pseudo integro-differential evolution equations with state-dependent delay.

1. Introduction

In this paper, we provide in a real Banach space $(E, |\cdot|)$, sufficient conditions for the existence of mild solutions defined on a semi-infinite real interval $J := [0, +\infty)$ for two classes of second order perturbed pseudo integro-differential evolution equations with infinite state-dependent delay. Our investigations will be situated in Fréchet space, we will consider the following problem

$$y''(t) = A(t)y(t) + f\left(t, y_{\rho(t,y_t)}\right) + \int_0^t \mathcal{I}(t,s)g\left(s, y_{\rho(s,y_s)}\right) ds, \quad \text{a.e. } t \in J, \quad (1)$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{2}$$

$$y'(0) = y^\star, \tag{3}$$

where $\mathcal{B}$ is an abstract phase space that is defined later; $f, g \colon J \times \mathcal{B} \to E$, $\mathcal{I} \colon J \times J \to \mathbb{R}$, $\rho \colon J \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions; $y^\star$ is a given constant and $\{A(t)\}_{t \geq 0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates a unique evolution system of operators

$$\{U(t, s)\}_{(t,s) \in J \times J} \quad \text{for } s \leq t.$$

For any continuous function y and any $t \leq 0$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta) \quad \text{for } \theta \leq 0.$$

Next, in Section 4, we study the following problem

$$\frac{d}{dt} \left[y'(t) - Q\left(t, y_{\rho(t, y_t)}\right) \right]$$
$$= A(t)y(t) + f(t, y_{\rho(t, y_t)}) + \int_0^t \mathcal{I}(t, s) g\left(s, y_{\rho(s, y_s)}\right) ds, \quad \text{a.e.} \quad t \in J, \tag{4}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{5}$$

$$y'(0) = y^\star, \tag{6}$$

where $f, g, \mathcal{I}, \rho, y^\star, A(\cdot)$ and ϕ are as in problem (1)–(3) and $Q \colon J \times \mathcal{B} \to E$ is a given function. Finally, in Section 5 we give two examples to illustrate the abstract theory.

Functional differential equations theory emerged as a significant branch of nonlinear analysis. Over several years, differential delay equations and, more generally, functional differential equations have been used in the modeling of scientific phenomena. Indeed, we can find a many applications in control, porous media, electrochemistry, chemistry, physics, mathematical biology, fractal media, statistical mechanics, electromagnetism, etc. Technical applications results and inductance separation performs can be related as in [18, 19]. A comprehensive theory has been developed by Ahmed [5] for evolution equations. Benchohra et al. [1, 16, 20, 37] studied very various integro-differential problems. The complex case where when the delay depends on an unknown function and was studied by Willé [38] in 1994, then in the past years, it has been shown that equations with state-dependent delays also appear in several areas such as classical electrodynamics by Driver et al. [24], in models of blood cell productions by Mackey et al. [34], and population models by Bélair [15]. These equations are called state-dependent equations delay. The outcomes of existence as well as uniqueness and controllability have been defined in the Baghli et al. [6, 9, 12, 12, 21, 22, 35] papers for finite and infinite dependent delays with local and nonlocal conditions for several perturbed and nonperturbed evolution problems in Fréchet and Banach spaces.

There are a lot of studies concerning the second order functional evolution equations; see, for example Balachandran et al. [7, 13, 14], Hernandez et al. [26, 27], Kozak [33] and recently Baghli et al. [2–4, 10, 23]. The fractional order derivative case was investigated also by Baghli and her collaborators [11, 30–32] for finite delay.

The objective of this paper is to give the sufficient conditions for the existence of mild solutions on a semi-infinite interval $J = [0; +\infty)$ for the two classes partial functional and neutral functional to the second order perturbed pseudo integro-differential evolution equations with infinite state-dependent delay (1)–(3) and (4)–(6) using the nonlinear alternative of Avramescu [8] for sum of compact operators and contractions maps in Fréchet spaces, combined with semigroup theory initiated by Ahmed and Pazy [5, 36].

2. Preliminaries

In this section, we introduce some notation, definitions, and theorems that are used in the different steps of this paper.

Let $C(\mathbb{R}^+; E)$ be the space of continuous functions from $\mathbb{R}^+$ into E and $B(E)$ be the space of all bounded linear operators from E into E, with the usual supremum norm

$$\|N\|_{B(E)} = \sup\{|N(y)| \ : \ |y| = 1\}, \qquad N \in B(E).$$

A measurable function $y \colon \mathbb{R}^+ \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [39]).

Let $L^1(\mathbb{R}^+, E)$ denote the Banach space of measurable functions $y \colon \mathbb{R}^+ \to E$ that are Bochner integrable normed by

$$\|y\|_{L^1} = \int\limits_0^{+\infty} |y(t)|\mathrm{d}t.$$

We shall employ an axiomatic definition of the phase space $\mathcal{B}$ presented by Hale and Kato [25] and follow the terminology used by Hino et al. [29]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a semi-normed linear space of functions mapping $\mathbb{R}^-$ into E, and satisfying the following axioms

(A$_1$) If $y \colon (-\infty, b) \to E, b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$, the following conditions hold

   (i) $y_t \in \mathcal{B}$.

   (ii) There exists a positive constant $\mathcal{D}$ such that $|y(t)| \leq \mathcal{D}\|y_t\|_{\mathcal{B}}$.

(iii) There exist two functions $K(\cdot), M(\cdot) \colon \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that

$$\|y_t\|_\mathcal{B} \leq K(t) \sup_{0 \leq s \leq t} |y(s)| + M(t)\|y_0\|_\mathcal{B}.$$

(A_2) For the function y in axiom (A_1), y_t is a $\mathcal{B}-$valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup_{t \in [0,b]} K(t)$ and $M_b = \sup_{t \in [0,b]} M(t)$.

Remark 1.

1. (ii) is equivalent to $|\phi(0)| \leq \mathcal{D}\|\phi\|_\mathcal{B}$ for every $\phi \in \mathcal{B}$.

2. Since $\|\cdot\|_\mathcal{B}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can check $\|\phi - \psi\|_\mathcal{B} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.

3. From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_\mathcal{B} = 0$: We necessarily have $\phi(0) = \psi(0)$.

Here are some examples of phase spaces from the book of Hino [29].

EXAMPLE. Let

BC : denote the space of bounded continuous functions defined from $\mathbb{R}^-$ to E;

BUC : denote the space of bounded uniformly continuous functions defined from $\mathbb{R}^-$ to E;

$C^\infty := \{\phi \in BC \colon \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E\}$;

$C^0 := \{\phi \in BC \colon \lim_{\theta \to -\infty} \phi(\theta) = 0\}$, endowed with the uniform norm

$$\|\phi\| = \sup_{\theta \leq 0} |\phi(\theta)|.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions (A_1)–(A_3). However, BC satisfies the axioms (A_1), (A_3) but the axiom (A_2) is not satisfied.

EXAMPLE. The spaces C_g, UC_g, C_g^∞ and C_g^0. Let g be a positive continuous function on $\mathbb{R}^-$. We define

$C_g := \left\{\phi \in C(\mathbb{R}^-, E) \colon \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } \mathbb{R}^-\right\}$;

$C_g^0 := \left\{\phi \in C_g \colon \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0\right\}$, endowed with the uniform norm

$$\|\phi\| = \sup_{\theta \leq 0} \frac{|\phi(\theta)|}{g(\theta)}.$$

Then we have that the spaces C_g and C_g^0 satisfy conditions (A_3).

We consider the following condition on the function g.

(g_1) For all $a > 0$, $\sup_{0 \le t \le a} \sup_{-\infty < \theta \le -t} \frac{g(t+\theta)}{g(\theta)} < \infty$.

The conditions (A_1) and (A_2) are satisfied if condition (g_1) holds.

EXAMPLE. The space C_γ. For any real constant γ, we define the functional space C_γ by

$$C_\gamma := \left\{ \phi \in C(\mathbb{R}^-, E) : \lim_{\theta \to -\infty} e^{\gamma \theta} \phi(\theta) \text{ exists in } E \right\}$$

endowed with the following norm $\|\phi\| = \sup_{\theta \le 0} e^{\gamma \theta} |\phi(\theta)|$. Then in the space C_γ, the axioms (A_1)–(A_3) are satisfied.

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$. We suppose that the family of semi-norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ verifies

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \cdots \quad \text{for every } x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n \quad \text{for all} \quad y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}_{n \in \mathbb{N}}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}_{n \in \mathbb{N}}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote $[x]_n$ the equivalence class of x of the subset X^n and define $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, int(Y^n) and ∂Y^n, respectively, the closure, the interior, and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n.

In what follows, let $\{A(t), t \le 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ that is dense in E and independent of t. The existence of solutions to problem (1)–(3) and (4)–(6) is related to the existence of an evolution operator $U(t, s)$ for the homogeneous problem

$$y''(t) = A(t)y(t), \quad t \in J. \tag{7}$$

This concept of evolution operator has been developed by K o z a k [33].

DEFINITION 2.1. A family U of bounded operators $U(t, s): E \to E, (t, s) \in \Delta$; where $\Delta := \{(t, s) \in J \times J : s \le t\}$, is called an evolution operator of the equation (7) if the following conditions hold.

(D1) For any $x \in E$ the map $(t, s) \mapsto U(t, s)x$ is continuously differentiable and:

(a) for any $t \in J$, $U(t, t) = 0$;

(b) for all $(t, s) \in \Delta$ and for any $x \in E$,

$$\frac{\partial}{\partial t} U(t, s)x|_{t=s} = x \quad \text{and} \quad \frac{\partial}{\partial s} U(t, s)x|_{t=s} = -x.$$

(D2) For all $(t, s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s} U(t, s)x \in D(A(t))$, the map $(t, s) \mapsto U(t, s)x$ is of class C^2, and:

(a) $\frac{\partial^2}{\partial t^2} U(t, s)x = A(t)U(t, s)x$;

(b) $\frac{\partial^2}{\partial s^2} U(t, s)x = U(t, s)A(s)x$;

(c) $\frac{\partial^2}{\partial s \partial t} U(t, s)x|_{t=s} = 0$.

(D3) For all $(t, s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s} U(t, s)x \in D(A(t))$, $\frac{\partial^3}{\partial t^2 \partial s} U(t, s)x$ and $\frac{\partial^3}{\partial s^2 \partial t} U(t, s)x$ exist, and:

(a) $\frac{\partial^3}{\partial t^2 \partial s} U(t, s)x = A(t)\frac{\partial}{\partial s}(t)U(t, s)x$;

(b) $\frac{\partial^3}{\partial s^2 \partial t} U(t, s)x = \frac{\partial}{\partial t} U(t, s)A(s)x$.

Moreover, the map $(t, s) \mapsto A(t)\frac{\partial}{\partial s} U(t, s)x$ is continuous.

For the state-dependent delay notion, let us set

$$\mathcal{R}(\rho^-) = \{\rho(s, \phi) : (s, \phi) \in J \times \mathcal{B}, \rho(s, \phi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Furthermore, we present the following hypothesis

(H$_\phi$) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to \mathbb{R}_+^\star$ such that for every $t \in \mathcal{R}(\rho^-)$

$$\|\phi_t\|_{\mathcal{B}} \leq \mathcal{L}^\phi \|\phi\|_{\mathcal{B}}.$$

Remark 2. Condition (H$_\phi$), is frequently checked by continuous and bounded functions. For more details, see, for instance, [29].

Lemma 2.2 (Hernàndez et al. [28], Lemma 2.4). *If $y : (-\infty; b] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + K_b|y(\theta)|; \theta \in [0, \max\{0, s\}], s \in \mathcal{R}(\rho^-) \cup J$$

where $\mathcal{L}^\phi = \sup_{t \in \mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Proposition 2.3 (Baghli [9]). *By (H$_\phi$), Lemma 2.2 and the property (A1), we have for each $t \in [0, n]$ and $n \in \mathbb{N}$*

$$\|y_{\rho(t, y_t)}\| \leq K_n|y(t)| + (M_n + \mathcal{L}^\phi)\|y_0\|_{\mathcal{B}}.$$

Definition 2.4. A function $f : J \times \mathcal{B} \to E$ is said to be an L_{loc}^1-Carathéodory function if it satisfies:

(i) for each $t \in J$ the function $f(t, \cdot) : \mathcal{B} \to E$ is continuous,

(ii) for each $y \in \mathcal{B}$ the function $f(\cdot, y) \colon J \to E$ is measurable,

(iii) for every positive integer k there exists $\hbar_k \in L^1_{\text{loc}}(J; \mathbb{R}^+)$ such that
$$|f(t, y)| \leq \hbar_k(t)$$
for all $\|y\|_{\mathcal{B}} \leq k$ and almost every $t \in J$.

The following definition is the appropriate concept of contraction in X.

DEFINITION 2.5. A function $f \colon X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $\alpha_n \in (0, 1)$ such that for all $x, y \in X$
$$\|f(x) - f(y)\|_n \leq \alpha_n \, \|x - y\|_n.$$

THEOREM 2.6 (Nonlinear Alternative of Avramescu [8])**.** *Let* X *be a Fréchet space and let* $A, B \colon X \to X$ *be two operators satisfying:*

(1) *A is a compact operator,*

(2) *B is a contraction.*

Then either one of the following statements holds:

(Av1) *The operator $A + B$ has a fixed point.*

(Av2) *The set $\left\{x \in X, x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right)\right\}$ is unbounded for some $\lambda \in (0, 1)$.*

3. Main Results

In this section, we give an existence result for the second order perturbed pseudo integro-differential evolution problem (1)–(3). Firstly, we define the concept of a mild solution for this problem.

DEFINITION 3.1. We say that the function $y(\cdot) \colon \mathbb{R} \to E$ is a mild solution of (1)–(3) if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation
$$y(t) = -\frac{\partial}{\partial s} U(t, 0)\phi(0) + U(t, 0)y^{\star}$$
$$+ \int_0^t U(t, s) f(s, y_{\rho(s, y_s)}) \mathrm{d}s$$
$$+ \int_0^t U(t, s) \int_0^s \mathcal{I}(s, r) g\left(r, y_{\rho(r, y_r)}\right) \mathrm{d}r \mathrm{d}s \quad \text{a.e.} \quad t \in J. \tag{8}$$

It is necessary to introduce the following hypotheses which are assumed thereafter:

(H1) $U(t,s)$ is compact for $t - s > 0$, and for every $(t,s) \in \Delta$, and there exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M},$$

and there exists a constant $\widetilde{M} \geq 0$ such that

$$\left\|\frac{\partial}{\partial s} U(t,s)\right\|_{B(E)} \leq \widetilde{M}.$$

(H2) There exist a function $p \in L^1_{\mathrm{loc}}(J; \mathbb{R}_+)$ and a continuous nondecreasing function
$\psi : \mathbb{R}_+ \to [0, +\infty)$ and such that

$$|f(t,u)| \leq p(t)\ \psi(\|u\|_{\mathcal{B}}),$$

for all $t \in J$ and for each $u \in \mathcal{B}$.

(H3) For all $R > 0$, there exists $l_R \in L^1_{\mathrm{loc}}(J; \mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \|u - v\|_{\mathcal{B}},$$

for all $t \in J$ and each $u,v \in \mathcal{B}$ with $\|u\|_{\mathcal{B}} \leq R$ and $\|v\|_{\mathcal{B}} \leq R$.

(H4) For all $t \in J$, $\mathcal{I}(t,s)$ is measurable on $[0,t]$ and $\mathcal{I}(t) = \sup_{0 \leq s \leq t} |\mathcal{I}(t,s)|$ is bounded on $[0,n]$. Let

$$S_n = \sup_{t \in [0,n]} \mathcal{I}(t).$$

(H5) There exists a function $\eta \in L^1(J; \mathbb{R}_+)$ such that

$$|g(t,u) - g(t,v)| \leq \eta(t)\|u - v\|_{\mathcal{B}},$$

for all $t \in J$ and for each $u,v \in \mathcal{B}$.

Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by

$$\|y\|_n := \sup_{t \in [0,n]} e^{-\tau L_n^*(t)} |y(t)|,$$

where

$$L_n^*(t) = \int_0^t \bar{l}_n\,\mathrm{d}s, \ \bar{l}_n(t) = \widehat{M} n S_n \eta(s) K_n \quad \text{and} \quad \eta$$

is the function from the hypothesis H5). Then $B_{+\infty}$ is a Fréchet space with those family of semi-norms $\|\cdot\|_{n \in \mathbb{N}}$.

THEOREM 3.2. *Assume that (H_ϕ) and (H1)–(H5) are satisfied and moreover for all $n > 0$*

$$\int_{\sigma_n}^{+\infty} \frac{\mathrm{d}s}{s + \psi(s)} > K_n \widehat{M} \int_0^n \max\big(p(s); n S_n \eta(s)\big)\mathrm{d}s, \tag{9}$$

with

$$\sigma_n = \left(M_n + \mathcal{L}^\phi + K_n \widetilde{M \mathcal{D}} \right) \|\phi\|_\mathcal{B} + K_n \widehat{M \mathcal{D}} \|y^\star\|_\mathcal{B}$$

$$+ \widehat{M} S_n K_n \int\limits_0^n \int\limits_0^s |g(r,0)| \mathrm{d}r \mathrm{d}s.$$

Then, the second order perturbed pseudo integro-differential evolution problem (1)–(3) has at least one mild solution on $\mathbb{R}$.

P r o o f. We transform problem (1)–(3) into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\[2mm] -\frac{\partial}{\partial s} U(t,0)\phi(0) + U(t,0)y^\star \\[2mm] + \int\limits_0^t U(t,s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s \\[2mm] + \int\limits_0^t U(t,s) \int\limits_0^s \mathcal{I}(s,r) g\left(r, y_{\rho(r,y_r)}\right) \mathrm{d}r \mathrm{d}s, & \text{if } t \geq 0. \end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem (1)–(3) for $\phi \in \mathcal{B}$, we will define the function

$$x(t) = \begin{cases} \phi(t), & \text{if } t \leq 0, \\[2mm] -\frac{\partial}{\partial s} U(t,0)\phi(0) + U(t,0)y^\star, & \text{if } t \geq 0. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t).$$

It is obvious that y satisfies the definition of mild solution (8) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int\limits_0^t U(t,s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s$$

$$+ \int\limits_0^t U(t,s) \int\limits_0^s \mathcal{I}(s,r) g\left(r, z_{\rho(r,z_r+x_r)} + x_{\rho(r,z_r+x_r)}\right) \mathrm{d}r \mathrm{d}s.$$

Let

$$B_{+\infty}^0 = \left\{ z \in B_{+\infty} : z_0 = 0 \in \mathcal{B} \right\}.$$

We define for $t \in J$ the operators $F, G : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t, s) f\left(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\right) \mathrm{d}s$$

and

$$G(z)(t) = \int_0^t U(t, s) \int_0^s \mathcal{I}(s, r) g\left(r, z_{\rho(r, z_r + x_r)} + x_{\rho(r, z_r + x_r)}\right) \mathrm{d}r \mathrm{d}s.$$

Obviously, the operator N has a fixed points is equivalents to $F + G$ has one, so it turns to prove that $F + G$ has a fixed point. The proof will be given in several steps.

Step 1. We show the continuity of F. Let $(z_n)_n$ be a sequence in $B^0_{+\infty}$ such that $z_n \to z \in B^0_{+\infty}$, by the hypothesis (H1), we obtain

$$|F(z_n)(t) - F(z)(t)|$$

$$\leq \int_0^t \|U(t, s)\|_{B(E)} |f\left(s, z_{n\rho(s, z_{ns} + x_s)} + x_{\rho(s, z_{ns} + x_s)}\right)$$

$$- f\left(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\right)| \mathrm{d}s$$

$$\leq \widehat{M} \int_0^t |f\left(s, z_{n\rho(s, z_{ns} + x_s)} + x_{\rho(s, z_{ns} + x_s)}\right)$$

$$- f\left(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\right)| \mathrm{d}s.$$

Since f is continuous, by dominated convergence theorem of Lebesgue, we get

$$\|F(z_n) - F(z)\|_n \to 0 \quad \text{if } n \to +\infty.$$

So, F is continuous.

Step 2. Show that F transforms any bounded of $B^0_{+\infty}$ in a bounded set. For any $d > 0$, there exists a positive constant ϱ such that for all

$$z \in B_d = \left\{z \in B^0_{+\infty} : \|z\|_n \leq d\right\}$$

we get

$$\|F(z)\|_n \leq \varrho.$$

Let $z \in B_d$ By the hypotheses (H1) and (H2), we have for all $t \in [0, n]$,

$$|F(z)(t)| \leq \int_0^t \|U(t, s)\|_{B(E)} |f\left(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\right)| \mathrm{d}s.$$

$$\leq \widehat{M} \int_0^t p(s) \psi\left(\left\|z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\right\|_{\mathcal{B}}\right) \mathrm{d}s.$$

From (H_ϕ), Proposition 2.3 and Assumption (A_1), we have for every $t \in [0, n]$

$$\left\| z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)} \right\|_{\mathcal{B}}$$

$$\leq \|z_{\rho(t, z_t + x_t)}\|_{\mathcal{B}} + \|x_{\rho(t, z_t + x_t)}\|_{\mathcal{B}}$$

$$\leq K_n |z(t)| + (M_n + \mathcal{L}^\phi) \|z_0\|_{\mathcal{B}} + K_n |x(t)| + (M_n + \mathcal{L}^\phi) \|x_0\|_{\mathcal{B}}$$

$$\leq K_n |z(t)| + K_n \left\| \frac{\partial}{\partial s} U(t, 0) \right\|_{B(E)} |\phi(0)|$$

$$+ K_n \|U(t, 0)\|_{B(E)} |y^\star| + (M_n + \mathcal{L}^\phi) \|\phi\|_{\mathcal{B}}$$

$$\leq K_n |z(t)| + K_n \widetilde{M} |\phi(0)| + K_n \widehat{M} |y^\star| + (M_n + \mathcal{L}^\phi) \|\phi\|_{\mathcal{B}}.$$

Using (A_1) (ii), we get

$$\left\| z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)} \right\|_{\mathcal{B}}$$

$$\leq K_n |z(t)| + K_n \widetilde{M} \mathcal{D} \|\phi\|_{\mathcal{B}} + K_n \widehat{M} \mathcal{D} \|y^\star\|_{\mathcal{B}} + (M_n + \mathcal{L}^\phi) |\phi\|_{\mathcal{B}}$$

$$\leq K_n |z(t)| + (M_n + \mathcal{L}^\phi + K_n \widetilde{M} \mathcal{D}) \|\phi\|_{\mathcal{B}} + K_n \widehat{M} \mathcal{D} \|y^\star\|_{\mathcal{B}}.$$

Set

$$c_n := (M_n + \mathcal{L}^\phi + K_n \widetilde{M} \mathcal{D}) \|\phi\|_{\mathcal{B}} + K_n \widehat{M} \mathcal{D} \|y^\star\|_{\mathcal{B}}$$

and $\varsigma_n := K_n d + c_n$. Then

$$\left\| z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)} \right\|_{\mathcal{B}} \leq K_n d + c_n := \varsigma_n. \tag{10}$$

Using the nondecreasing character of ψ, we obtain for each $t \in [0, n]$

$$|F(z)(t)| \leq \widehat{M} \int_0^t p(s) \psi(\varsigma_n) \mathrm{d}s \leq \widehat{M} \psi(\varsigma_n) \int_0^t p(s) \mathrm{d}s$$

$$\leq \widehat{M} \psi(\varsigma_n) \|p\|_{L^1} := \varrho.$$

So, there exists a positive constant ϱ such that

$$\|F(z)\|_n \leq \varrho, \quad \text{hence} \quad F(B_d) \subset B_\varrho.$$

Step 3. F maps bounded sets into equicontinuous sets of $B_{+\infty}^0$. We consider B_d as in Step 2 and show that $F(B_d)$ is equicontinuous. Let

$$\tau_1, \tau_2 \in J \quad \text{with} \quad \tau_2 > \tau_1 \quad \text{and} \quad z \in B_d.$$

Then
$$|F(z)(\tau_2) - F(z)(\tau_1)|$$

$$\leq \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} \left| f\left(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\right)\right|$$

$$+ \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)} \left| f\left(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\right)\right| \mathrm{d}s$$

$$\leq \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} p(s) \psi\left(\|z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\|_{\mathcal{B}}\right) \mathrm{d}s$$

$$+ \widehat{M} \int_{\tau_1}^{\tau_2} p(s) \psi\left(\|z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\|_{\mathcal{B}}\right) \mathrm{d}s.$$

By (10) and using the nondecreasing character of ψ, we get
$$|F(z)(\tau_2) - F(z)(\tau_1)|$$

$$\leq \psi(\varsigma_n) \int_0^{\tau_1} U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} p(s) \mathrm{d}s + \widehat{M} \psi(\varsigma_n) \int_{\tau_1}^{\tau_2} p(s) \mathrm{d}s.$$

Observe that $|F(z)(\tau_2) - F(z)(\tau_1)|$ tends to zero as $\tau_2 - \tau_1 \to 0$ independently of $z \in B_d$. The right-hand side of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$. Since $U(t, s)$ is a strongly continuous operator and the compactness of $U(t, s)$ for $t > s$ implies continuity in the uniform operator topology. As a result of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a pre-compact set in E.

Let $t \in J$ be fixed and let ϵ be a real number such that $0 < \epsilon < t$. For $z \in B_d$ we define
$$F_\epsilon(z)(t) = U(t, t - \epsilon) \int_0^{t - \epsilon} U(t - \epsilon, s) f\left(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\right) \mathrm{d}s.$$

Since $U(t, s)$ is a compact operator, the set
$$Z_\epsilon(t) = \{F_\epsilon(z)(t) : z \in B_d\}$$

is pre-compact in E for every ϵ sufficiently small, $0 < \epsilon < t$. Moreover, using and the nondecreasing character of ψ, we have
$$|F(z)(t) - F_\epsilon(z)(t)| \leq \widehat{M} \psi(\varsigma_n) \int_{t - \epsilon}^t p(s) \mathrm{d}s.$$

Therefore, the set
$$\{F(z)(t) : z \in B_d\}$$

is pre-compact in E. So we deduce from Steps 1, 2 and 3 that F is a continuous compact operator.

Step 4. Now we show that the operator G is a contraction. In fact, let
$$z, \overline{z} \in B_{+\infty}^0.$$

By the hypotheses (H1), (H4) and (H5), we get for all $t \in [0, n]$ and $n \in \mathbb{N}$

$$
\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| \leq & \int_0^t \|U(t,s)\|_{B(E)} \times \int_0^s |\mathcal{I}(s,r)| \\
& \times \left| g\left(r, z_{\rho(r,z_r+x_r)} + x_{\rho(r,z_r+x_r)}\right) \right. \\
& \left. - g\left(r, \overline{z}_{\rho(r,\overline{z}_r+x_r)} + x_{\rho(r,\overline{z}_r+x_r)}\right) \right| \mathrm{d}r \mathrm{d}s \\
\leq & \int_0^t \widehat{M} S_n \int_0^s \eta(s) \|z_{\rho(r,z_r+x_r)} - \overline{z}_{\rho(r,\overline{z}_r+x_r)}\|_{\mathcal{B}} \mathrm{d}s.
\end{aligned}
$$

Using Proposition 2.3, to obtain

$$
\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| \leq & \int_0^t \widehat{M} n S_n \eta(s) K_n |z(s) - \overline{z}(s)| \mathrm{d}s \\
\leq & \int_0^t \left[\overline{l}_n(s) \; e^{\tau L_n^*(s)} \right] \left[e^{-\tau L_n^*(s)} |z(s) - \overline{z}(s)| \right] \mathrm{d}s \\
\leq & \int_0^t \left[\frac{e^{\tau L_n^*(s)}}{\tau} \right]' \mathrm{d}s \; \|z - \overline{z}\|_n \leq \frac{1}{\tau} \; e^{\tau \; L_n^*(t)} \; \|z - \overline{z}\|_n.
\end{aligned}
$$

Therefore,
$$\|G(z) - G(\overline{z})\|_n \leq \frac{1}{\tau} \|z - \overline{z}\|_n.$$

So, the operator G is a contraction for all $n \in \mathbb{N}$.

Step 5. To apply Theorem 2.6, we must check the statement (Av2): i.e., it remains to show that the following set

$$\Gamma = \left\{ z \in B_{+\infty}^0 : z = \lambda \; F(z) + \lambda \; G\left(\frac{z}{\lambda}\right) ; \text{ for some } 0 < \lambda < 1 \right\}$$

is bounded.

Let $z \in \Gamma$. Then

$$
\begin{aligned}
\frac{|z(t)|}{\lambda} \leq & \int_0^t \|U(t,s)\|_{B(E)} |f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)| \mathrm{d}s \\
& + \int_0^t \|U(t,s)\|_{B(E)} \int_0^s |\mathcal{I}(s,r)| \left| g\left(r, \frac{z_{\rho(r,z_r+x_r)} + x_{\rho(r,z_r+x_r)}}{\lambda}\right) \right| \mathrm{d}r \mathrm{d}s.
\end{aligned}
$$

By (H1), (H2), (H4), (H5), Proposition 2.3 and inequality (10), we have for each $t \in [0, n]$

$$\frac{|z(t)|}{\lambda} \leq \widehat{M} \int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_B)\mathrm{d}s$$

$$+ \int_0^t \widehat{M} \int_0^s |\mathcal{I}(s,r)| \left| g\left(r, \frac{z_{\rho(r,z_r+x_r)} + x_{\rho(r,z_r+x_r)}}{\lambda}\right) - g(r,0) + g(r,0) \right| \mathrm{d}r\mathrm{d}s$$

$$\leq \widehat{M} \int_0^t p(s)\psi\left(K_n|z(s)| + c_n\right)\mathrm{d}s$$

$$+ \int_0^t \widehat{M} n S_n \eta(s) \left\| \frac{z_{\rho(r,z_r+x_r)} + x_{\rho(r,z_r+x_r)}}{\lambda} \right\|_B + \widehat{M} S_n \int_0^n \int_0^s |g(r,0)|\mathrm{d}r\mathrm{d}s$$

$$\leq \widehat{M} \int_0^t p(s)\psi\left(K_n|z(s)| + c_n\right)\mathrm{d}s$$

$$+ \widehat{M} n S_n \int_0^t \eta(s) \left(\frac{K_n}{\lambda}|z(s)| + c_n\right)\mathrm{d}s + \widehat{M} S_n \int_0^n \int_0^s |g(r,0)|\mathrm{d}r\mathrm{d}s.$$

We consider the function $u(t) := \sup |z(\theta)|$ defined for $t \in J$ with the fact that $0 < \lambda < 1$. Then we have

$$\frac{K_n}{\lambda}u(t) + c_n \leq c_n + K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s$$

$$+ K_n\widehat{M}nS_n\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s$$

$$+ K_n\widehat{M}S_n\int_0^n\int_0^s |g(r,0)|\mathrm{d}r\mathrm{d}s.$$

Set

$$\sigma_n := c_n + K_n\widehat{M}S_n\int_0^n\int_0^s |g(r,0)|\mathrm{d}r\mathrm{d}s.$$

Then, we have

$$\frac{K_n}{\lambda}u(t) + c_n \leq \sigma_n + K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s$$

$$+ K_n\widehat{M}nS_n\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s.$$

We consider the function $\mu(t)$ defined by

$$\mu(t) = \left\{ \sup_{s \in [0,t]} \frac{K_n|u(s)|}{\lambda} + c_n \right\}, \quad 0 \le t \le +\infty.$$

Let $t^\star \in [0,t]$ be such that $\mu(t^\star) = \frac{K_n|u(t^\star)|}{\lambda} + c_n$, by the previous inequality and the nondecreasing character of ψ, we have $t \in [0,n]$

$$\mu(t) \le \sigma_n + K_n\widehat{M}\int_0^t p(s)\psi\big(\mu(s)\big)\,\mathrm{d}s + K_n\widehat{M}nS_n\int_0^t \eta(s)\mu(s)\mathrm{d}s.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t) \le v(t), \quad \text{for all } t \in [0,n].$$

From the definition of v, we have

$$v(0) = \sigma_n \quad \text{and} \quad v'(t) = K_n\widehat{M}\,p\,(t)\psi\big(v(t)\big) + K_n\widehat{M}nS_n\eta(t)v(t) \quad \text{a.e. } t \in [0,n],$$

$$v'(t) = K_n\widehat{M}\big[p(t)\psi\big(v(t)\big) + nS_n\eta(t)v(t)\big].$$

This implies that for all $t \in [0,n]$ and using Theorem 3.2 we get

$$\int_{\sigma_n}^{v(t)} \frac{\mathrm{d}s}{s + \psi(s)} \le K_n\widehat{M}\int_0^t \max\big(p(s); nS_n\eta(s)\big)\mathrm{d}s$$

$$\le K_n\widehat{M}\int_0^n \max\big(p(s); nS_n\eta(s)\big)\mathrm{d}s$$

$$< \int_{\sigma_n}^{+\infty} \frac{\mathrm{d}s}{s + \psi(s)}.$$

Thus, for every $t \in [0,n]$, there exists a constant Φ_n such that $v(t) \le \Phi_n$ and hence $\mu(t) \le \Phi_n$. Since $\|z\|_n \le \mu(t)$, we have $\|z\|_n \le \Phi_n$. This shows that the set Γ is bounded. Then statement (Av2) in Theorem 2.6 does not hold. The nonlinear alternative of Avramescu implies that (Av1) is satisfied, we deduce that the operator $F + G$ has a fixed point z^*. Then

$$y^*(t) = z^*(t) + x(t), \quad t \in \mathbb{R}$$

is the fixed point of the operator N which is a mild solution to problem (1)–(3). $\square$

3.1. Neutral perturbed pseudo integro-differential evolution equations

In this section, we give an existence result for the problem (4)–(6). Firstly, we define the concept of a mild solution for this problem.

DEFINITION 3.3. We say that the function $y(\cdot)\colon \mathbb{R} \to E$ is a mild solution of the problem (4)–(6) if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation

$$y(t) = -\frac{\partial}{\partial s}U(t,0)\phi(0) + U(t,0)[y^\star - Q(0,\phi)]$$

$$-\int_0^t \frac{\partial}{\partial s}U(t,s)Q(s,y_{\rho(s,y_s)})\,\mathrm{d}s + \int_0^t U(t,s)f(s,y_{\rho(s,y_s)})\,\mathrm{d}s$$

$$+\int_0^t U(t,s)\int_0^s \mathcal{I}(s,r)g(r,y_{\rho(r,y_r)})\,\mathrm{d}r\,\mathrm{d}s \qquad \text{for all } t \in J. \qquad (11)$$

We consider the hypotheses (H_ϕ), (H1)–(H5) and will need the following assumptions:

(H6) There exists a constant $\overline{M_0} > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M_0} \qquad \text{for all } t \in J.$$

(H7) There exists a constant $L > 0$ such that

$$|A(t)Q(t,\phi)| \leq L(\|\phi\|_{\mathcal{B}} + 1) \qquad \text{for all } t \in J \text{ and all } \phi \in \mathcal{B}.$$

(H8) There exists a constant $L_* > 0$ such that

$$|A(s)Q(s,\phi) - A(\overline{s})Q(\overline{s},\overline{\phi})| \leq L_* \left(|s - \overline{s}| + (\|\phi - \overline{\phi}\|_{\mathcal{B}})\right)$$

for all $s, \overline{s} \in J$ and every $\phi, \overline{\phi} \in \mathcal{B}$.

THEOREM 3.4. *Assume that (H_ϕ), (H1)–(H8) are satisfied and moreover for all $n \geq 0$*

$$\int_{\widetilde{\sigma}_n}^{+\infty} \frac{\mathrm{d}s}{s + \psi(s)} > K_n \int_0^n \max\left(\widehat{M}p(s), \widetilde{M}\,\overline{M_0}L + \widehat{M}nS_n\eta(s)\right)\mathrm{d}s, \qquad (12)$$

with $\widetilde{\sigma}_n := K_n\chi_n + c_n$, *and*

$$c_n = (K_n\widetilde{M}\mathcal{D} + M_n + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + K_n\widehat{M}\mathcal{D}\|y^\star\|_{\mathcal{B}}$$

and

$$\chi_n = \overline{M_0}L(\widehat{M} + \widetilde{M}n) + \widehat{M}\overline{M_0}L\|\phi\|_{\mathcal{B}} + \widehat{M}S_n\int_0^n\int_0^s |g(r,0)|\mathrm{d}r\mathrm{d}s.$$

Then, the second order neutral perturbed pseudo integro-differential evolution problem (4)–(6) has at least one mild solution on $\mathbb{R}$.

P r o o f. We consider the operator $\widetilde{N} : B_{+\infty} \to B_{+\infty}$ defined by:

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t), & \text{if } t \le 0; \\ -\frac{\partial}{\partial s} U(t,0)\phi(0) + U(t,0)[y^{\star} - Q(0,\phi)] \\ \quad - \int\limits_0^t \frac{\partial}{\partial s} U(t,s)Q\big(s, y_{\rho(s,y_s)}\big)\mathrm{d}s \\ \quad + \int\limits_0^t U(t,s)f\big(s, y_{\rho(s,y_s)}\big)\mathrm{d}s \\ \quad + \int\limits_0^t U(t,s)\int\limits_0^s \mathcal{I}(s,r)g(r, y_{\rho(r,y_r)})\mathrm{d}r\mathrm{d}s & \text{if } t \ge 0. \end{cases}$$

Then fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (4)–(6). For $\phi \in \mathcal{B}$, we consider the function $x(\cdot) : \mathbb{R} \to E$ defined by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \le 0; \\ -\frac{\partial}{\partial s} U(t,0)\phi(0) + U(t,0)y^{\star}, & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi(0)$ for each function $z \in B_{+\infty}$ set $y(t) = z(t) + x(t)$. It is obvious that y satisfies (11) if and only if $z_0 = 0$ and

$$z(t) = -U(t,0)Q(0,\phi) - \int\limits_0^t \frac{\partial}{\partial s} U(t,s)Q\big(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\big)\,\mathrm{d}s$$

$$+ \int\limits_0^t U(t,s)f\big(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\big)\mathrm{d}s$$

$$+ \int\limits_0^t U(t,s)\int\limits_0^s \mathcal{I}(s,r)g\big(r, z_{\rho(r,z_r+x_r)} + x_{\rho(r,z_r+x_r)}\big)\,\mathrm{d}r\mathrm{d}s.$$

Let $B_{+\infty}^0 = \{z \in B_{+\infty} : z_0 = 0\}$. Define the operator $F, \widetilde{G} : B_{+\infty}^0 \longrightarrow B_{+\infty}^0$ by

$$F(z)(t) = \int\limits_0^t U(t,s)f\big(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\big)\mathrm{d}s.$$

and

$$\widetilde{G}(z)(t) = -U(t,0)Q(0,\phi) - \int\limits_0^t \frac{\partial}{\partial s} U(t,s)Q\big(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\big)\,\mathrm{d}s$$

$$+ \int\limits_0^t U(t,s)\int\limits_0^s \mathcal{I}(s,r)g\big(r, z_{\rho(r,z_r+x_r)} + x_{\rho(r,z_r+x_r)}\big)\,\mathrm{d}r\mathrm{d}s.$$

Obviously, the operator $\widetilde{N}$ having fixed points is equivalent to $F + \widetilde{G}$ having one, so it turns out that $F + \widetilde{G}$ has a fixed point. We have shown that the operator F is continuous and compact as in Section 3.

It remains to show that the operator $\widetilde{G}$ is a contraction. Let $z, \overline{z} \in B_0^{+\infty}$. By (H1),(H4)–(H6) and (H8) we have for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$
\begin{aligned}
|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| &\leq \int_0^t \left\| \frac{\partial}{\partial s} U(t,s) \right\|_{B(E)} \|A^{-1}(s)\|_{B(E)} \\
&\quad \times \left| A(s)Q\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \right. \\
&\quad \left. - A(s)Q\left(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\right) \right| \mathrm{d}s \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)} \int_0^s |\mathcal{I}(s,r)| \\
&\quad \times \left| g\left(, z_{\rho(r,z_r+x_r)} + x_{\rho(r,z_r+x_r)}\right) \right. \\
&\quad \left. - g\left(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\right) \right| \mathrm{d}r\mathrm{d}s \\
&\leq \int_0^t \widetilde{M}\,\overline{M_0} L_* \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \mathrm{d}s \\
&\quad + \int_0^t \widehat{M} n S_n \eta(s) \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \mathrm{d}s.
\end{aligned}
$$

By (10), we have

$$
\begin{aligned}
|\widetilde{G}(z)(t)\widetilde{G}(\overline{z})(t)| &\leq \int_0^t \widetilde{M}\ \overline{M_0} L_* K_n |z(s) - \overline{z}(s)| \mathrm{d}s + \int_0^t \widehat{M} n S_n \eta(s) K_n |z(s) - \overline{z}(s)| \mathrm{d}s \\
&\leq \int_0^t K_n \left[\widetilde{M}\ \overline{M_0} L_* + \widehat{M} n S_n \eta(s) \right] |z(s) - \overline{z}(s)| \mathrm{d}s.
\end{aligned}
$$

Set $\overline{l_n}(t) = K_n \left[\widetilde{M}\ \overline{M_0} L_* + \widehat{M} n S_n \eta(t) \right]$ for the family of semi-norms $\{ \| \cdot \|_n \}_{n \in \mathbb{N}}$, then

$$
\begin{aligned}
|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| &\leq \int_0^t \left(\overline{l_n}(s) e^{\tau L_n^*(s)} \right) \left(e^{-\tau L_n^*(s)} |z(s) - \overline{z}(s)| \right) \mathrm{d}s \\
&\leq \int_0^t \left[\frac{e^{\tau L_n^*(s)}}{\tau} \right]' \mathrm{d}s \|z - \overline{z}\|_n \leq \frac{1}{\tau} e^{\tau L_n^*(t)} \|z - \overline{z}\|_n.
\end{aligned}
$$

Therefore,

$$
\|\widetilde{G}(z) - \widetilde{G}(\overline{z})\|_n \leq \frac{1}{\tau} \|z - \overline{z}\|_n.
$$

Assume that $\tau > 1$. Then the operator $\widetilde{G}$ is a contraction for all $n \in \mathbb{N}$. For applying Theorem 2.6, we must check the statement (Av2): i.e., it remains to show that the following set

$$\widetilde{\Gamma} = \left\{ z \in B_{+\infty}^0 \colon z = \lambda \, F(z) + \lambda \widetilde{G}\left(\frac{z}{\lambda}\right) \ \text{ for some } \ 0 < \lambda < 1 \right\}.$$

is bounded.

Let $z \in \widetilde{\Gamma}$. By (H1), (H2), (H4)–(H8), we have for each $t \in [0, n]$.

$$
\begin{aligned}
\frac{|z(t)|}{\lambda} &\leq \widehat{M}\|A^{-1}(0)\|_{B(E)}|A(0)Q(0,\phi)| \\
&\quad + \int_0^t \left\| \frac{\partial}{\partial s}U(t,s)\right\|_{B(E)} \|A^{-1}(s)\|_{B(E)} \\
&\qquad \times |A(s)Q\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)|\mathrm{d}s \\
&\quad + \widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\mathrm{d}s \\
&\quad + \widehat{M}\int_0^t\int_0^s |\mathcal{I}(s,r)|\left|g\left(s, \frac{z_{\rho(s,z_s+x_s)}}{\lambda} + x_{\rho(s,z_s+x_s)}\right) - g(r,0)\right|\mathrm{d}r\mathrm{d}s \\
&\quad + \widehat{M}S_n\int_0^n\int_0^s |g(r,0)|\mathrm{d}r\mathrm{d}s \\
&\leq \widehat{M}\overline{M_0}L(\|\phi\|_{\mathcal{B}} + 1) + \widetilde{M}\,\overline{M_0}L\int_0^t \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + 1 \ \mathrm{d}s \\
&\quad + \widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\mathrm{d}s + \widehat{M}S_n\int_0^n\int_0^s |g(r,0)|\mathrm{d}r\mathrm{d}s \\
&\quad + \widehat{M}nS_n\int_0^t \eta(s)\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}\right\|\mathrm{d}s \\
&\leq \overline{M_0}L(\widehat{M} + \widetilde{M}n) + \widehat{M}\overline{M_0}L\|\phi\|_{\mathcal{B}} + \widehat{M}S_n\int_0^n\int_0^s |g(r,0)|\mathrm{d}r\mathrm{d}s \\
&\quad + \widetilde{M}\,\overline{M_0}L\int_0^t \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\mathrm{d}s \\
&\quad + \widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\mathrm{d}s \\
&\quad + \widehat{M}nS_n\int_0^t \eta(s)\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}\right\|\mathrm{d}s.
\end{aligned}
$$

Using Proposition 2.3, and inequality (10), we get

$$\left\| \frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)} \right\|_{\mathcal{B}}$$

$$\leq \frac{1}{\lambda}\|z_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}$$

$$\leq \frac{K_n}{\lambda}|z(s)| + \frac{M_n+\mathcal{L}^\phi}{\lambda}\|z_0\|_{\mathcal{B}} + K_n|x_s|$$

$$\quad + (M_n+\mathcal{L}^\phi)\|x_0\|_{\mathcal{B}}$$

$$\leq \frac{K_n}{\lambda}|z(s)| + K_n\left[\left\|\frac{\partial}{\partial s}U(s,0)\right\|_{B(E)}|\phi(0)| + \|U(s,0)\|_{B(E)}|y^\star|\right]$$

$$\quad + (M_n+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}$$

$$\leq \frac{K_n}{\lambda}|z(s)| + (K_n\widetilde{M}\mathcal{D}+M_n+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + K_n\widehat{M}\mathcal{D}\|y^\star\|_{\mathcal{B}}.$$

Hence,

$$\left\| \frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)} \right\|_{\mathcal{B}} \leq \frac{K_n}{\lambda}|z(s)| + c_n. \tag{13}$$

Use the function $u(\cdot)$ and the nondecreasing character of ψ to get

$$\frac{u(t)}{\lambda} \leq \overline{M_0}L(\widehat{M}+\widetilde{M}n) + \widehat{M}\,\overline{M_0}L\|\phi\|_{\mathcal{B}} + \widehat{M}S_n\int_0^n\int_0^s|g(r,0)|\mathrm{d}r\mathrm{d}s$$

$$\quad + \widetilde{M}\,\overline{M_0}L\int_0^t K_nu(s) + c_n\mathrm{d}s + \widehat{M}\int_0^t p(s)\psi(K_nu(s)+c_n)\mathrm{d}s$$

$$\quad + \widehat{M}nS_n\int_0^t\eta(s)\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s.$$

Set

$$\chi_n = \overline{M_0}L(\widehat{M}+\widetilde{M}n) + \widehat{M}\,\overline{M_0}L\|\phi\|_{\mathcal{B}} + \widehat{M}S_n\int_0^n\int_0^s|g(r,0)|\mathrm{d}r\mathrm{d}s.$$

Then

$$\frac{u(t)}{\lambda} \leq \chi_n + \widetilde{M}\,\overline{M_0}L\int_0^t\frac{K_n}{\lambda}u(s) + c_n\mathrm{d}s + \widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s$$

$$\quad + \widehat{M}nS_n\int_0^t\eta(s)\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s.$$

So,

$$\frac{K_n}{\lambda}u(t) + c_n \leq c_n + K_n\chi_n + K_n\widetilde{M}\,\overline{M_0}L\int_0^t \frac{K_n}{\lambda}u(s) + c_n\mathrm{d}s$$

$$+ K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s$$

$$+ K_n\widehat{M}nS_n\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s.$$

Set $\widetilde{\sigma_n} := K_n\chi_n + c_n$. Then

$$\frac{K_n}{\lambda}u(t) + c_n \leq \widetilde{\sigma_n} + K_n\int_0^t \left(\widetilde{M}\{\overline{M_0}\}L + \widehat{M}nS_n\eta(s)\right)\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s$$

$$+ K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s.$$

We consider the function μ defined by:

$$\mu(t) = \sup\left\{\frac{K_n}{\lambda}u(t) + c_n : 0 \leq s \leq t\right\}, \qquad 0 \leq t \leq +\infty.$$

Let $t^* \in [0, t]$ be such that

$$\mu(t) = \frac{K_n}{\lambda}u(t^*) + c_n$$

by the previous inequality, we have for $t \in [0, n]$

$$\mu(t) \leq \widetilde{\sigma_n} + K_n\int_0^t \left(\widetilde{M}\,\overline{M_0}L + \widehat{M}nS_n\eta(s)\right)\mu(s)\mathrm{d}s$$

$$+ K_n\int_0^t \widehat{M}p(s)\psi\big(\mu(s)\big)\mathrm{d}s.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq v(t) \qquad \text{for all } t \in [0, n].$$

From the definition of v, we get $v(0) = \widetilde{\sigma_n}$ and

$$v'(t) = K_n\left(\widetilde{M}\,\overline{M_0}L + \widehat{M}nS_n\eta(t)v(t) + \widehat{M}p(t)\psi\big(v(t)\big)\right).$$

$$\int\limits_{\widetilde{\sigma_n}}^{v(t)} \frac{\mathrm{d}s}{s + \psi(s)} \le K_n \int\limits_0^t \max\left(\widehat{M}p(s), \widetilde{M}\ \overline{M_0}L + \widehat{M}nS_n\eta(s)\right)\mathrm{d}s$$

$$\le K_n \int\limits_0^n \max\left(\widehat{M}p(s), \widetilde{M}\ \overline{M_0}L + \widehat{M}nS_n\eta(s)\right)\mathrm{d}s < \int\limits_{\widetilde{\sigma_n}}^{+\infty} \frac{\mathrm{d}s}{s + \psi(s)}.$$

Thus, for every $t \in [0, n]$, there exists a constant $\widetilde{\Phi}_n$ such that $v(t) \le \widetilde{\Phi}_n$ and hence $\mu(t) \le \widetilde{\Phi}_n$. Since $\|z\|_n \le \mu(t)$, we have $\|z\|_n \le \widetilde{\Phi}_n$. This shows that the set $\widetilde{\Gamma}$ is bounded. Then statement (Av2) in Theorem 2.6 does not hold. The nonlinear alternative of Avramescu implies that (Av1) is satisfied, we deduce that the operator $F + \widetilde{G}$ has a fixed point z^*. Then

$$y^*(t) = z^*(t) + x(t), \qquad t \in \mathbb{R}$$

is the fixed point of the operator $\widetilde{N}$ which is a mild solution to problem (4)–(6).

□

4. Examples

To illustrate the previous results, we give two examples in this section.

4.1. Example 1

Consider the partial second order perturbed pseudo integro-differential evolution equations

$$\begin{cases} \frac{\partial^2}{\partial t^2}v(t,\xi) = \frac{\partial^2}{\partial \xi^2}v(t,\xi) + a_0(t,\xi)v(t,\xi) \\ + \int\limits_{-\infty}^0 a_1(s-t)v\left[s - \rho_1(t)\rho_2\left(\int\limits_0^{2\pi}a_2(\theta)|v(t,\theta)|^2 d\theta\right),\xi\right]\mathrm{d}s \\ + \int\limits_{-\infty}^t \eta(t,s)\int\limits_{-\infty}^0 a_3(s-\tau)v\left[s - \rho_1(\tau)\rho_2\left(\int\limits_0^{2\pi}a_2(\theta)|v(\tau,\theta)|^2 d\theta\right),\xi\right]d\tau\mathrm{d}s, \\ \hspace{6cm} t \ge 0,\ \xi \in [0, 2\pi], \\ v(t,0) = v(t,2\pi) = 0, \hspace{3.8cm} t \ge 0, \\ v(\theta,\xi) = v_0(\theta,\xi), \hspace{1.2cm} \frac{\partial}{\partial t}v(0,\xi) = \varphi(\xi) - \infty < \theta \le 0,\ \ \xi \in [0, 2\pi], \end{cases} \tag{14}$$

where $a_0 : \mathbb{R}^+ \times [0, 2\pi] \to \mathbb{R}$ is a continuous function,

$$a_1, a_3 : \mathbb{R}_- \to \mathbb{R} \quad \text{and} \quad a_2 : [0, 2\pi] \to \mathbb{R},$$

$$\rho_i : \mathbb{R}^+ \to \mathbb{R} \quad \text{for} \quad i = 1, 2.$$

Let

$$E = L^2(\mathbb{R}, \mathbb{C})$$

the space of 2π-periodic square-integrable functions from $\mathbb{R}$ into $\mathbb{C}$, and let $H^2(\mathbb{R}, \mathbb{C})$ denote the Sobolev space of 2π-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that

$$x'' \in L^2(\mathbb{R}, \mathbb{C}).$$

We consider the operator

$$A_1 y(\xi) = y''(\xi) \quad \text{with domain} \quad D(A_1) = H^2(\mathbb{R}, \mathbb{C}).$$

In addition, we take

$$A^2(t)y(s) = a(t)y'(s) \quad \text{defined on} \quad H^1(\mathbb{R}, \mathbb{C}),$$

and consider the closed linear operator

$$A(t) = A_1 + A_2(t),$$

which generates an evolution operator U defined by

$$U(t, s) = \sum_{n \in \mathbb{Z}} z_n(t, s) < x, w_n > w_n,$$

where z_n is a solution to the scalar initial value problem

$$\begin{cases} z''(t) = -n^2 z(t) + i\, n\, a(t)\, z(t), \\ z(s) = 0, \qquad\qquad z'(s) = z_1. \end{cases} \tag{15}$$

THEOREM 4.1. *Let $\mathcal{B} = BUC(\mathbb{R}_-, E)$ and $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds. Suppose that the functions*

$$a_1, a_3 : \mathbb{R}_- \to \mathbb{R}, \quad a_2 : [0, 2\pi] \to \mathbb{R}$$

and

$$\rho_i : \mathbb{R}_- \to \mathbb{R} \quad \text{for } i = 1, 2$$

are continuous. Then there exists a mild solution of (14) on $\mathbb{R}$.

P r o o f. Define $f, g, I, \rho, \phi, \frac{\partial v}{\partial t}(0, \xi)$ by

$$f(t, \psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s, \xi)\mathrm{d}s, \qquad t \geq 0,\ \xi \in [0, 2\pi],$$

$$\rho(t, \psi)(\xi) = t - \rho_1(t)\rho_2 \left(\int_{0}^{\pi} a_2(\theta)|\psi(0, \xi)|^2 \mathrm{d}\theta \right), \qquad t \geq 0,\ \xi \in [0, 2\pi],$$

$$g(t, \psi) = \int_{-\infty}^{0} a_3(s)\psi(s, \xi)\mathrm{d}s, \qquad I(t, \psi)(\xi) = \int_{-\infty}^{t} \eta(s)\psi(s, \xi)\mathrm{d}s,$$

$$\phi(t)(\xi) = v_0(t, \xi), \qquad t \leq 0,\ \xi \in [0, 2\pi],$$

and

$$\frac{\partial}{\partial t} v(0, \xi) = \frac{\partial}{\partial t} y(0, \xi).$$

Then, (14) can be written in the abstract form (1)–(2) with A and $f, g, I, \rho, \phi,$ defined above. Now, the existence of a mild solution on $\mathbb{R}$ can be concluded from an application of Theorem 3.2. □

4.2. Example 2

Consider the neutral second order perturbed pseudo integro-differential evolution equation

$$\begin{cases}
\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} v(t, \xi) - \int\limits_{-\infty}^{0} a_4(s-t) v\left(s - \rho_1(t)\rho_2\left(\int\limits_0^\pi a_2(\theta)|v(t, \theta)|^2 d\theta\right), \xi\right) ds\right] \\
\quad = \frac{\partial^2 v}{\partial \xi^2}(t, \xi) + a_0(t, \xi) v(t, \xi) \\
\qquad + \int\limits_{-\infty}^{0} a_1(s-t) v\left[s - \rho_1(t)\rho_2\left(\int\limits_0^{2\pi} a_2(\theta)|v(t, \theta)|^2 d\theta\right), \xi\right] ds \\
\qquad + \int\limits_{-\infty}^{t} \eta(t, s) \int\limits_{-\infty}^{0} a_3(s-\tau) v\left[s - \rho_1(\tau)\rho_2\left(\int\limits_0^{2\pi} a_2(\theta)|v(\tau, \theta)|^2 d\theta\right), \xi\right] d\tau ds, \\
\hspace{8cm} t \geq 0, \ \xi \in [0, 2\pi] \\
v(t, 0) = v(t, 2\pi) = 0, \hspace{5.5cm} t \geq 0, \\
v(\theta, \xi) = v_0(\theta, \xi), \quad \frac{\partial}{\partial t} v(0, \xi) = \varphi(\xi) \hspace{1.2cm} -\infty < \theta \leq 0, \ \xi \in [0, 2\pi],
\end{cases} \tag{16}$$

where $a_4 \colon \mathbb{R}_- \to \mathbb{R}$ is a continuous function.

THEOREM 4.2. *Let*

$$\mathcal{B} = BUC(\mathbb{R}_-, E) \quad and \quad \phi \in \mathcal{B}.$$

Assume that the condition (H_ϕ) holds. Suppose that the functions

$$a_1, a_3, a_4 \colon \mathbb{R}_- \to \mathbb{R}, \hspace{1.5cm} a_2 \colon [0, 2\pi] \to \mathbb{R}$$

and

$$\rho_i \colon \mathbb{R}_- \to \mathbb{R} \quad for \quad i = 1, 2$$

are continuous. Then there exists a mild solution of (16) on $\mathbb{R}$.

P r o o f. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)\mathrm{d}s, \qquad t \geq 0, \;\; \xi \in [0,\pi],$$

$$\rho(t,\psi)(\xi) = t - \rho_1(t)\rho_2\left(\int_{0}^{\pi} a_2(\theta)|\psi(0,\xi)|^2\mathrm{d}\theta\right), \qquad t \geq 0, \;\; \xi \in [0,\pi],$$

$$Q(t,\psi) = \int_{-\infty}^{0} a_4(s)\psi(s,\xi)\mathrm{d}s,$$

$$g(t,\psi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)\mathrm{d}s,$$

$$I(t,\psi)(\xi) = \int_{-\infty}^{t} \eta(s)\psi(s,\xi)\mathrm{d}s,$$

$$\phi(t)(\xi) = v_0(t,\xi), \qquad t \leq 0, \;\; \xi \in [0,2\pi],$$

and

$$\frac{\partial}{\partial t}v(0,\xi) = \frac{\partial}{\partial t}y(0,\xi), \qquad t \leq 0, \;\; \xi \in [0,2\pi]$$

are well defined functions, which permit to transform system (16) into the abstract system (4)–(6). Now, the existence of mild solutions can be deduced from a direct application of Theorem 3.4. □

5. Conclusion

In conclusion, this paper establishes new existence results for mild solutions of second-order partial functional and neutral functional perturbed pseudo integro-differential evolution equations with state-dependent delay in Fréchet spaces. By combining semigroup theory with Avramescu's nonlinear alternative for the sum of compact operators and contraction mappings, a comprehensive framework is developed to ensure the existence of solutions under suitable sufficient conditions. The proposed methodology may serve as a foundation for further studies on stability, controllability, and optimal control of delayed pseudo integro-differential systems.

Conflict of Interest.
The authors declare that they have no conflict of interest.

REFERENCES

[1] ABADA, N.—AGARWAL, R. P.—BENCHOHRA, M.—HAMMOUCHE H.: *Existence results for nondensely defined impulsive semilinear functional differential equations with state-dependent delay*, Asian-Eur. J. Math. **1** (2008), no. 4, 449–468.

[2] ABIBSSI, I.—BAGHLI-BENDIMERAD, S.: *On nonlocal second order perturbed pseudo integro-differential evolution equations with infinite state-dependent delay*, Math. Appl.: Functional Analysis Applications **52** (2024), no. 2, 271–298.

[3] ABIBSSI, I.—BAGHLI-BENDIMERAD, S.: *On second order perturbed evolution equations with infinite state-dependent delay*, Appl. Math. **52** (2025), no. 1, 77–99.

[4] ABIBSSI, I.—BOUTEFFAL, Z.—BAGHLI-BENDIMERAD, S.: *Mild solutions for second-order partial functional and neutral functional perturbed evolution equations with state-dependent delays and nonlocal conditions in Fréchet spaces*, El Khayyam (2025), accepted.

[5] AHMED, N. U.: *Semigroup Theory with Applications to Systems and Control*. Pitman Res. Notes Math. Ser. Vol. 246. Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1991.

[6] AOUAD, DJ.—BAGHLI-BENDIMERAD, S.: *Mild solutions for perturbed evolution equations with infinite state-dependent delay*, Electron. J. Qual. Theory Differ. Equ. **59** (2013), 1–24.

[7] ARTHI, G.—BALACHANDRAN, K.: *Controllability of second-order impulsive evolution systems with infinite delay*, Nonlinear Anal. Hybrid Syst. **11** (2014), 139–153.

[8] AVRAMESCU, C.: *Some remarks on a fixed point theorem of Krasnoselskii*, Electron. J. Qual. Theory Differ. Equ. **5** (2003), 1–15.

[9] BAGHLI-BENDIMERAD, S.: *Global mild solution for functional evolution inclusions with state-dependent delay*, J. Adv. Res. Dynam. Control Syst. **5** (2013), no. 4, 1–19.

[10] BAGHLI-BENDIMERAD, S.—ABIBSSI, I.: *The existence of mild solutions for nonlocal perturbed evolution equations with infinite state-dependent delay in Fréchet spaces*, Arch. Control Sci. **35 LXXI** (2025), no. 2, 321–349.

[11] BAGHLI-BENDIMERAD, S.—BOUDEFLA, CH.: *Controllability of Mild Solutions for Fractional Neutral Evolution Equations with State-Dependent Delay*, Arch. Control Sci. (2025), accepted.

[12] BAGHLI, S.—BENCHOHRA, M.: *Multivalued evolution equations with infinite delay in Fréchet spaces*, Electron. J. Qual. Theory Differ. Equ. **2008** (2008), no. 33, 24 pp.

[13] BALACHANDRAN K.–ANTHONI, S. M.: *Existence of solutions of second order neutral functional differential equations*, Tamkang J. Math. **30** (1999), 299–309.

[14] BALACHANDRAN, K.—PARK, D. G.—ANTHONI, S. M.: *Existence of solutions of abstract nonlinear second-order neutral functional integro-differential equations*, Comput. Math. Appl. **46** (2003), 1313–1324.

[15] BÉLAIR, J.: *Population models with state-dependent delays*, Lect. Notes Pure Appl. Math. **131** (1990), 165–176.

[16] BENCHOHRA, M.—N'GUÉRÉKATA, G. M.—SALIM, A.: *Advanced Topics on Semilinear Evolution Equations* In: Series on Concrete and Applicable Mathematics. Vol. 25, Worls Scientific, 2025.

[17] BENCHOHRA, M.—SALIM, A.—ZHOU, Y.: *Integro-Differential equations: Analysis, Stability and Controllability*, De Gruyter, Berlin, Boston, 2024.

[18] BENDIMERAD, S. E.—AYAD, A.—BETTACHE, A.—BAGHLI S. —AYAD, A. N. E. I.: *A novel formula for calculating repulsive forces in horizontally aligned nonferrous cylindrical particles within an Eddy current separator*, Powder Technology, **120300** (2024), 10 pp. Doi: 10.1016/j.powtec.2024.120300.

[19] BENDIMERAD, S. E.—BAGHLI, S.—AYAD, A.—TILMATINE, A.: *Mutual inductance of two coaxial rectangular PCB coils incorporating magnetic layer*, Journal of Electrical Engineering: The Journal of Slovak University of Technology, **72** (2021), no. 5, 287–296. Doi: 10.2478/jee-2021-0041.

[20] BENSALEM, A.—SALIM, A.—BENCHOHRA, M.: *Ulam-Hyers-Rassias stability of neutral functional integro-differential evolution equations with non-instantaneous impulses on an unbounded interval*, Qual. Theory Dyn. Syst. **22** (2023), Art. no. 88.

[21] BOUDEFLA, CH.—BAGHLI-BENDIMERAD, S.: *Nonlocal controllability of mild solutions for evolution equations with state-dependent delay in Fréchet spaces*, Int. J. Appl. Math. **37** (2024), no. 1, 53–71.

[22] BOUDEFLA CH.—BAGHLI-BENDIMERAD, S.: *Nonlocal controllability of mild solutions for neutral evolution equations with state-dependent delay in Fréchet spaces*, Arch. Control Sci. **34 LXX** (2024), no. 1, 117–148.

[23] BOUDEFLA, CH.—SAHRAOUI, F.—BAGHLI-BENDIMERAD, S.: *Controllability of mild solutions for second-order neutral evolution equations with state-dependent delay*, Complex Anal. Oper. Theory **18** (2024), Art. no. 80, 24 pp.

[24] DRIVER, R. D.—NORRIS, M. J.: *Note on uniqueness for a one-dimensional two-body problem of classical electrodynamics*, Ann. Phys. **42** (1967), 347–351.

[25] HALE, J.—KATO, J.: *Phase space for retarded equations with infinite delay*, Funkcialaj Ekvacioj **21** (1978), 11–41.

[26] HERNÀNDEZ, E.: *A remark on second order differential equations with nonlocal conditions*, Cadernos Mat. **4** (2003), 299–309.

[27] HERNÀNDEZ, E.—MCKIBBEN, M. A.: *Some comments on: Existence of solutions of abstractnonlinear second-order neutral functional integro-differential equations*, Comput. Math. Appl. **50** (2005), 655–669.

[28] HERNÀNDEZ, E.—SAKTHIVEL, R,—AKI, S. T.: *Existence results for impulsive evolution differential equations with state-dependent delay*, Electron. J. Differential Equations **28** (2008), 1–11.

[29] HINO, Y.—MURAKAMI, S.—NAITO, T.: *Functional differential equations with infinite delay*, Lecture Notes in Math. Vol. 1473, Springer-Verlag, Berlin, 1991.

[30] LACHACHI-MERAD, N.—BAGHLI-BENDIMERAD, S.—BENCHOHRA, M.: *Unique Mild Solutions for Caputo's Fractional Perturbed Evolution Equations with State-Dependent Delay*, Evol. Equ. Control Theory, **13** (2024), no. 1, 160–172.

[31] LACHACHI-MERAD, N.—BAGHLI-BENDIMERAD, S.—BENCHOHRA, M. — AOUAD, DJ.: *Unique mild solution for fractional partial and neutral evolution equations with state-dependent delay*, Rocky Mountain J. Math., **55** (2025), no. 5, 1351–1359.

[32] LACHACHI-MERAD, N.—BAGHLI-BENDIMERAD, S.—BENCHOHRA, M. — KARAPINAR, E.: *Nonlocal partial fractional evolution equations with state-dependent delay*, Proyecciones, **42** (2023), no. 5, 1191–1210.

[33] KOZAK, M.: *A fundamental solution of a second-order differential equation in a Banach space*, Univ. Iagel. Acta Math. **32** (1995), 275–289.

[34] MACKEY, M. C.—MILTON, J.: *Feedback delays and the origin of blood cell dynamics*, Comm. Theor. Biol. **1** (1990), 299–327.

[35] MEBARKI, A.—BAGHLI-BENDIMERAD, S.: *Neutral multi-valued integro-differential evolution equations with infinite state-dependent delay*, Turkish J. Math. **44** (2020), no. 6, 2312–2329.

[36] PAZY, A.: *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, New York, 1983.

[37] SALIM, A.—MESRI, F.—BENCHOHRA, M.—TUNÇ, C.: *Controllability of second order semilinear random differential equations in Fréchet spaces*, Mediterr. J. Math. **20**, (2023), Art. no. 84.

[38] WILLÉ, D. R.—BAKER, C. T. H: *Stepsize control and continuity consistency for state-dependent delay-differential equations*, J. Comput. Appl. Math. **53** (1994), no. 2, 163–170.

[39] YOSIDA, K.: *Functional Analysis.* 6th edition. Springer-Verlag, Berlin, 1980.

Imane Abibssi
Selma Baghli-Bendimerad
Department of Mathematics
Faculty of Exact Sciences
University of Djillali Liabes
P.O. Box 89
22000–Sidi Bel-Abbes
ALGERIA
E-mail: imane.abibssi@univ-sba.dz
        selmabaghli@gmail.com
        selma.baghli@univ-sba.dz