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EXISTENCE, UNIQUENESS AND SUCCESSIVE APPROXIMATIONS FOR CAPUTO TEMPERED FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This study aims to analyze the global convergence and uniqueness properties of the successive approximations method when applied to Caputo tempered fractional differential equations. We provide rigorous mathematical proofs to establish the convergence of the method, ensuring that the iterative process converges to a unique solution. Furthermore, we investigate the impact of the nonlocal conditions on the uniqueness of the solution obtained through the successive approximations method.

1. Introduction

Fractional calculus has emerged as a powerful mathematical tool for modeling complex phenomena involving memory effects and long-range dependencies. It deals with the expansion of differentiating and integrating a function from integer orders to non-integer orders. To gain a comprehensive understanding of fractional calculus, we recommend consulting monographs such as [2–4, 11, 13, 14, 20, 22, 30], as well as papers such as [6–9, 12, 15, 23, 34, 35].

The Caputo tempered fractional derivative extends the classical Caputo derivative by introducing a tempering function that controls the rate of decay of fractional order. This fractional derivative has been proven to be an effective tool for modeling systems with memory and hereditary properties, making it

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suitable for analyzing real-world phenomena in physics, engineering, and biology, and further elaboration on this topic can be found in [10,16–18,25,26,28,32].

In [17], the authors studied the existence and Ulam stability results for the Cauchy problem of the implicit neutral fractional differential equation involving the Caputo tempered fractional derivative with finite delay

$$ {}_0^C\mathfrak{D}_t^{\zeta,\omega}\big(y(t)-\mathfrak{h}(t,y_t)\big) = \Psi\left(t, y_t,\ {}_0^C\mathfrak{D}_t^{\zeta,\omega}\big(y(t)-\mathfrak{h}(t,y_t)\big)\right); \quad t\in J:=[0,\varkappa], $$

$$ y(t)=\phi(t); \qquad t\in[-r,0], $$

where $0<\zeta<1$, $\omega\geq 0$, $\varkappa, r>0$, ${}_0^C\mathfrak{D}_t^{\zeta,\omega}$ is the Caputo tempered fractional derivative, $\mathfrak{h}\colon J\times E_r\to\mathbb{R}$, $\Psi\colon J\times E_r\times\mathbb{R}\to\mathbb{R}$ are given continuous functions,

$$ \phi\in E_r \quad \text{and} \quad E_r:=C([-r,0],\mathbb{R}). $$

For any $t\in J$, we define $y_t\in E_r$ by $y_t(\theta)=y(t+\theta)$ for $\theta\in[-r,0]$. Their results were based on Krasnoselskii's fixed point theorem in Banach spaces.

The successive approximations method is a powerful numerical technique commonly employed to solve fractional differential equations, see the papers [1, 5]. It involves iteratively refining an initial approximations to converge towards the desired solution.

In [33], the authors discussed the uniform convergence of successive approximations for the $\psi-$Hilfer Cauchy$-$type problem

$$ \mathfrak{D}_{a^+}^{\alpha,\beta;\psi}\rho(t)=g\left(t,\rho(t),\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}\rho(t)\right); \quad 0<\alpha<1, 0\leq\beta\leq 1, 0\leq a<t\leq b. $$

$$ I_{a^+}^{1-\gamma;\psi}\rho(a)=\rho_a, \quad \rho_a\in\mathbb{R}, \qquad \gamma=\alpha+\beta-\alpha\beta, $$

where $\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}$ is the $\psi-$Hilfer fractional derivative, $I_{a^+}^{1-\gamma;\psi}$ is ψ-Riemann-Liouville fractional integral, $g\colon(a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is given function and ρ_a is a constant.

This method has shown great potential for capturing accurate solutions for a wide range of fractional differential equations. See the papers [19–21, 31] for more details. However, its application to fractional differential equations involving the Caputo tempered derivative has received limited attention in the literature.

Motivated by this gap, the present study aims to explore the global convergence of successive approximations and the uniqueness of solutions for the following initial value problem of Caputo tempered fractional differential equations:

$$ \left({}_0^C\mathfrak{D}_t^{\varrho,\varsigma}\xi\right)(t)=\aleph\big(t,\xi(t)\big); \qquad t\in\Bbbk:=[0,\varkappa], \tag{1} $$

$$ \xi(0)=\kappa, \tag{2} $$

where $0<\varrho<1$, $\varsigma\geq 0$, ${}_0^C\mathfrak{D}_t^{\varrho,\varsigma}$ is the Caputo tempered fractional derivative, $\aleph\colon\Bbbk\times\mathbb{R}\to\mathbb{R}$ is a given function, and $\kappa\in\mathbb{R}$.

Next, we discuss the global convergence of successive approximations and the uniqueness of solutions for the following nonlocal problem of Caputo tempered fractional differential equations:

$$\left({}_0^C\mathfrak{D}_t^{\varrho,\varsigma}\xi\right)(t) = \aleph\big(t, \xi(t)\big); \qquad t \in \Bbbk := [0, \varkappa], \tag{3}$$

$$\xi(0) + \bar{\wp}(\xi) = \kappa, \tag{4}$$

where $0 < \varrho < 1$, $\varsigma \geq 0$, ${}_0^C\mathfrak{D}_t^{\varrho,\varsigma}$ is the Caputo tempered fractional derivative, $\aleph\colon \Bbbk \times \mathbb{R} \to \mathbb{R}$ is a given function, $\bar{\wp}\colon C(\Bbbk, \mathbb{R}) \to \mathbb{R}$ is a continuous function, and $\kappa \in \mathbb{R}$.

Most earlier studies, such as those in [16–18] and [25, 26] have predominantly relied on fixed-point theorems to establish the existence of solutions in this setting. While the fixed point approach provides a powerful theoretical framework, it often requires stronger assumptions on the nonlinearity or the initial conditions. The current work offers an alternative by applying the successive approximations method to Caputo tempered fractional differential equations, providing a constructive approach under potentially less restrictive conditions.

This paper presents several novel aspects, which can be outlined as follows:

- This research employs the Caputo tempered fractional derivative to study both local and nonlocal problems, offering a more realistic representation of memory effects and exponential decay. This approach generalizes classical fractional models and enhances the modeling of complex dynamic behaviors.
- We apply the method of successive approximations to study Caputo tempered fractional differential equations, an area that has been less explored compared to approaches based on fixed point theorems.
- We establish the global convergence of the successive approximations and the uniqueness of the solution, offering a constructive alternative under potentially less restrictive conditions.

This paper is arranged as follows: Section 2 introduces some preliminaries, definitions, lemmas, and auxiliary results on the tempered fractional derivative. Section 3 discusses the global convergence of successive approximations and the uniqueness of solutions for a class of Caputo tempered fractional derivative. We provide a theorem on the global convergence of successive approximations for a unique solution of our problems. Finally, we present two examples to show the validity of our results.

2. Preliminaries

The Banach space of continuous functions from $\Bbbk$ to $\mathbb{R}$ is represented as $C(\Bbbk, \mathbb{R})$, where the norm is given by

$$\|\aleph\|_\infty = \sup_{t\in\Bbbk} |\aleph(t)|.$$

The space of absolutely continuous functions from $\Bbbk$ to $\mathbb{R}$ is denoted by $AC(\Bbbk)$. Furthermore, for any $n \in \mathbb{N}^*$, we represent the space defined by $AC^n(\Bbbk)$ as follows:

$$AC^n(\Bbbk) := \left\{\aleph\colon \Bbbk \to \mathbb{R}\colon \frac{d^n}{dt^n}\aleph(t) \in AC(\Bbbk)\right\}.$$

Consider the space $X_b^p(\varkappa_1, \varkappa_2)$ of those complex-valued Lebesgue measurable functions $\aleph$ on $[\varkappa_1, \varkappa_2]$ for which $\|\aleph\|_{X_b^p} < \infty$ with:

$$\|\aleph\|_{X_b^p} = \left(\int_{\varkappa_1}^{\varkappa_2} |t^b \aleph(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \quad (1 \le p < \infty, b \in \mathbb{R}).$$

Definition 2.1 ([27, 29, 32])**.** Suppose that the real function $\aleph$ is piecewise continuous on $[\varkappa_1, \varkappa_2]$ and $\aleph \in X_b^p(\varkappa_1, \varkappa_2)$, $\varsigma > 0$. Then, the Riemann-Liouville tempered fractional integral of order ϱ is defined by

$$_{\varkappa_1}I_t^{\varrho,\varsigma}\aleph(t) = e^{-\varsigma t}\ _{\varkappa_1}I_t^{\varrho}\left(e^{\varsigma t}\aleph(t)\right) = \frac{1}{\Gamma(\varrho)}\int_{\varkappa_1}^{t} \frac{e^{-\varsigma(t-s)}\aleph(s)}{(t-s)^{1-\varrho}} ds, \tag{5}$$

where $_{\varkappa_1}I_t^{\varrho}$ denotes the Riemann-Liouville fractional integral [24], defined by

$$_{\varkappa_1}I_t^{\varrho}\aleph(t) = \frac{1}{\Gamma(\varrho)}\int_{\varkappa_1}^{t} \frac{\aleph(s)}{(t-s)^{1-\varrho}} ds. \tag{6}$$

Obviously, the tempered fractional integral (5) reduces to the Riemann-Liouville fractional integral (6) if $\varsigma = 0$.

Definition 2.2 ([27, 29])**.** For $n - 1 < \varrho < n$; $n \in \mathbb{N}^+$, $\varsigma \ge 0$. The Riemann--Liouville tempered fractional derivative is defined by

$$_{\varkappa_1}\mathfrak{D}_t^{\varrho,\varsigma}\aleph(t) = e^{-\varsigma t}{}_{\varkappa_1}\mathfrak{D}_t^{\varrho}\left(e^{\varsigma t}\aleph(t)\right) = \frac{e^{-\varsigma t}}{\Gamma(n-\varrho)}\frac{d^n}{dt^n}\int_{\varkappa_1}^{t} \frac{e^{\varsigma s}\aleph(s)}{(t-s)^{\varrho-n+1}} ds,$$

where $_{\varkappa_1}\mathfrak{D}_t^{\varrho}$ denotes the Riemann-Liouville fractional derivative [24], given by

$$_{\varkappa_1}\mathfrak{D}_t^{\varrho}\aleph(t) = \frac{d^n}{dt^n}\left(_{\varkappa_1}I_t^{n-\varrho}\aleph(t)\right) = \frac{1}{\Gamma(n-\varrho)}\frac{d^n}{dt^n}\int_{\varkappa_1}^{t} \frac{\aleph(s)}{(t-s)^{\varrho-n+1}} ds.$$

Definition 2.3 ([27, 32])**.** For $n-1<\varrho<n$; $n\in\mathbb{N}^+$, $\varsigma\geq 0$. The Caputo tempered fractional derivative is defined as

$$ {}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho,\varsigma}_t\aleph(t)=e^{-\varsigma t}\,{}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho}_t\left(e^{\varsigma t}\aleph(t)\right)=\frac{e^{-\varsigma t}}{\Gamma(n-\varrho)}\int\limits_{\varkappa_1}^{t}\frac{1}{(t-s)^{\varrho-n+1}}\frac{d^n}{ds^n}\left(e^{\varsigma s}\aleph(s)\right)ds, $$

where ${}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho,\varsigma}_t$ denotes the Caputo fractional derivative [24], given by

$$ {}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho}_t\aleph(t)=\frac{1}{\Gamma(n-\varrho)}\int\limits_{\varkappa_1}^{t}\frac{1}{(t-s)^{\varrho-n+1}}\frac{d^n}{ds^n}\aleph(s)ds. $$

Lemma 2.4 ([27])**.** *For a constant θ,*

$$ {}_{\varkappa_1}\mathfrak{D}^{\varrho,\varsigma}_t(\theta)=\theta e^{-\varsigma t}{}_{\varkappa_1}\mathfrak{D}^{\varrho}_t e^{\varsigma t},\qquad {}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho,\varsigma}_t(\theta)=\theta e^{-\varsigma t}\,{}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho}_t e^{\varsigma t}. $$

Obviously,

$$ {}_{\varkappa_1}\mathfrak{D}^{\varrho,\varsigma}_t(\theta)\neq{}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho,\varsigma}_t(\theta). $$

And, ${}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho,\varsigma}_t(\theta)$ is no longer equal to zero, being different from ${}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho}_t(\theta)\equiv 0$.

Lemma 2.5 ([27, 32])**.** *Let $\aleph(t)\in AC^n[\varkappa_1,\varkappa_2]$, $\varsigma\geq 0$ and $n-1<\varrho<n$. The composite properties of the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral are given as follows:*

$$ {}_{\varkappa_1}I^{\varrho,\varsigma}_t\left[{}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho,\varsigma}_t\aleph(t)\right]=\aleph(t)-\sum_{\varepsilon=0}^{n-1}e^{-\varsigma t}\frac{(t-\varkappa_1)^{\varepsilon}}{\varepsilon!}\left[\left.\frac{d^{\varepsilon}\left(e^{\varsigma t}\aleph(t)\right)}{dt^{\varepsilon}}\right|_{t=\varkappa_1}\right], $$

and

$$ {}^{C}_{\varkappa_1}\mathfrak{D}^{\varrho,\varsigma}_t\left[{}_{\varkappa_1}I^{\varrho,\varsigma}_t\aleph(t)\right]=\aleph(t),\quad \textit{for}\quad \varrho\in(0,1). $$

Lemma 2.6. *Let $h\colon\Bbbk\to\mathbb{R}$ be a continuous function and $0<\varrho<1$. Then the initial value problem*

$$ \left({}^{C}_{0}\mathfrak{D}^{\varrho,\varsigma}_t\xi\right)(t)=h(t);\quad t\in\Bbbk:=[0,\varkappa], \tag{7} $$

$$ \xi(0)=\kappa,\quad \kappa\in\mathbb{R}, \tag{8} $$

has a unique solution given by

$$ \xi(t)=\kappa e^{-\varsigma t}+\frac{1}{\Gamma(\varrho)}\int\limits_{0}^{t}e^{-\varsigma(t-s)}(t-s)^{\varrho-1}h(s)ds. \tag{9} $$

P r o o f. Applying ${}_0I_t^{\varrho,\varsigma}(\cdot)$ to both sides of (7), and by using Lemma 2.5 and if $t \in \Bbbk$, we get

$$\xi(t) - \xi(0)e^{-\varsigma t} = \frac{1}{\Gamma(\varrho)} \int_0^t e^{-\varsigma(t-s)}(t-s)^{\varrho-1} h(s)ds.$$

From the initial condition (8), we get

$$\xi(t) = \kappa e^{-\varsigma t} + \frac{1}{\Gamma(\varrho)} \int_0^t e^{-\varsigma(t-s)}(t-s)^{\varrho-1} h(s)ds.$$

Conversely, we can easily show by Definition 2.3, Lemma 2.4 and Lemma 2.5 that if ξ verifies (9), then it satisfies the problem (7)–(8). □

As a consequence of Lemma 2.6, we can have the following result.

Lemma 2.7. *Let $\aleph\colon \Bbbk \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the problem* (1)–(2) *is equivalent to the following integral equation*

$$\xi(t) = \kappa e^{-\varsigma t} + \frac{1}{\Gamma(\varrho)} \int_0^t e^{-\varsigma(t-s)}(t-s)^{\varrho-1} \aleph\big(s, \xi(s)\big)ds. \tag{10}$$

Lemma 2.8. *Let $h\colon \Bbbk \to \mathbb{R}$ be a continuous function and $0 < \varrho < 1$. Then the nonlocal problem*

$$\left({}_0^C\mathfrak{D}_t^{\varrho,\varsigma}\xi\right)(t) = h(t); \qquad t \in \Bbbk := [0, \varkappa], \tag{11}$$

$$\xi(0) + \bar{\wp}(\xi) = \kappa, \qquad \kappa \in \mathbb{R}, \tag{12}$$

has a unique solution defined by

$$\xi(t) = \big(\kappa - \bar{\wp}(\xi)\big)e^{-\varsigma t} + \frac{1}{\Gamma(\varrho)} \int_0^t e^{-\varsigma(t-s)}(t-s)^{\varrho-1} h(s)ds. \tag{13}$$

P r o o f. We can prove as in Lemma 2.6 that the problem (11)–(12) has a unique solution identified in equation (13). □

Lemma 2.9. *Let $\aleph\colon \Bbbk \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the problem* (3)–(4) *is equivalent to the following integral equation*

$$\xi(t) = \big(\kappa - \bar{\wp}(\xi)\big)e^{-\varsigma t} + \frac{1}{\Gamma(\varrho)} \int_0^t e^{-\varsigma(t-s)}(t-s)^{\varrho-1} \aleph\big(s, \xi(s)\big)ds. \tag{14}$$

3. Successive Approximations and Uniqueness Results

3.1. The Cauchy Problem

In this section, we present the main result for the global convergence of successive approximations to the unique solution of the problem (1)–(2).

Definition 3.1. By a solution of problem (1)–(2), we mean a function $\xi \in C(\Bbbk, \mathbb{R})$ that satisfies the equation (1) on $\Bbbk$ and the initial condition (2).

Define the successive approximations of the problem (1)–(2) as follows:

$$\xi_0(t) = \kappa e^{-\varsigma t}; \quad t \in \Bbbk,$$

$$\xi_{n+1}(t) = \kappa e^{-\varsigma t} + \frac{1}{\Gamma(\varrho)} \int_0^t e^{-\varsigma(t-s)} (t-s)^{\varrho-1} \aleph\big(s, \xi_n(s)\big) ds; \quad t \in \Bbbk.$$

Set $\Bbbk_\Lambda = [0, \Lambda\varkappa]$; for any $\Lambda \in [0, 1]$.

The hypotheses are as follows:

(Cd_1): The function $\aleph : \Bbbk \times \mathbb{R} \to \mathbb{R}$ is continuous.

(Cd_2): There exist a constant $\delta > 0$ and a continuous function

$$\varphi : \Bbbk \times [0, \delta] \to \mathbb{R}_+$$

such that $\varphi(t, \cdot)$ is non-decreasing for all $t \in \Bbbk$, and the inequality

$$|\aleph(t, \xi) - \aleph(t, \overline{\xi})| \leq \varphi(t, |\xi - \overline{\xi}|), \tag{15}$$

holds for all $t \in \Bbbk$ and $\xi, \overline{\xi} \in \mathbb{R}$ with $|\xi - \overline{\xi}| \leq \delta$.

(Cd_3): $\vartheta \equiv 0$ is the only function in $C(\Bbbk_\gamma, [0, \delta])$, where

$$\vartheta(t) \leq \frac{1}{\Gamma(\varrho)} \int_0^{\gamma\varkappa} (t-s)^{\varrho-1} \varphi\big(s, \vartheta(s)\big) ds, \tag{16}$$

with $\Lambda \leq \gamma \leq 1$.

Theorem 3.2. *Assume that the hypotheses $(Cd_1) - (Cd_3)$ hold. Then the successive approximations $\{\xi_n; n \in \mathbb{N}\}$ are well defined and converge to the unique solution of problem (1)–(2) uniformly on $\Bbbk$.*

Proof. From (Cd_1), the successive approximations are well defined.

There exists a constant $\delta_1 > 0$ such that

$$\|\xi_n\|_\infty \leq \delta_1.$$

For each $t_1, t_2 \in \Bbbk$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|\xi_n(t_2) - \xi_n(t_1)| \\
&\leq \left|\kappa e^{-\varsigma t_2} - \kappa e^{-\varsigma t_1}\right| + \frac{1}{\Gamma(\varrho)} \left| \int_0^{t_2} e^{-\varsigma(t_2-s)}(t_2-s)^{\varrho-1} \aleph\big(s, \xi_{n-1}(s)\big) ds \right. \\
&\qquad\qquad \left. - \int_0^{t_1} e^{-\varsigma(t_1-s)}(t_1-s)^{\varrho-1} \aleph\big(s, \xi_{n-1}(s)\big) ds \right| \\
&\leq \kappa \left|e^{-\varsigma t_2} - e^{-\varsigma t_1}\right| \\
&\quad + \frac{1}{\Gamma(\varrho)} \int_0^{t_1} \left|e^{-\varsigma(t_2-s)}(t_2-s)^{\varrho-1} - e^{-\varsigma(t_1-s)}(t_1-s)^{\varrho-1}\right| \left|\aleph\big(s, \xi_{n-1}(s)\big)\right| ds \\
&\quad + \frac{1}{\Gamma(\varrho)} \int_{t_1}^{t_2} e^{-\varsigma(t_2-s)}(t_2-s)^{\varrho-1} \left|\aleph\big(s, \xi_{n-1}(s)\big)\right| ds.
\end{aligned}$$

Then

$$\begin{aligned}
&|\xi_n(t_2) - \xi_n(t_1)| \leq \kappa \left|e^{-\varsigma t_2} - e^{-\varsigma t_1}\right| + \sup_{(t,\xi)\in \Bbbk\times[0,\delta_1]} |\aleph(t,\xi)| \\
&\quad \times \left[\int_0^{t_1} \frac{\left|e^{-\varsigma(t_2-s)}(t_2-s)^{\varrho-1} - e^{-\varsigma(t_1-s)}(t_1-s)^{\varrho-1}\right|}{\Gamma(\varrho)} ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\varrho-1}}{\Gamma(\varrho)} ds \right] \\
&\leq \kappa \left|e^{-\varsigma t_2} - e^{-\varsigma t_1}\right| + \sup_{(t,\xi)\in \Bbbk\times[0,\delta_1]} |\aleph(t,\xi)| \\
&\quad \times \left[\int_0^{t_1} \frac{\left|e^{-\varsigma(t_2-s)}(t_2-s)^{\varrho-1} - e^{-\varsigma(t_1-s)}(t_1-s)^{\varrho-1}\right|}{\Gamma(\varrho)} ds + \frac{(t_2-t_1)^{\varrho}}{\Gamma(\varrho+1)} \right] \\
&\quad \longrightarrow 0, \quad \text{as} \quad t_1 \to t_2.
\end{aligned}$$

Thus

$$\|\xi_n(t_2) - \xi_n(t_1)\|_\infty \longrightarrow 0, \quad \text{as} \quad t_1 \to t_2.$$

Consequently, the sequence $\{\xi_n(t); n \in \mathbb{N}\}$ is equicontinuous on $\Bbbk$.

According the theorem of Arzéla-Ascoli, the sequence $\{\xi_n(t); n \in \mathbb{N}\}$ is relatively compact. Then, the sequence $\{\xi_n(t); n \in \mathbb{N}\}$ converges.

Let

$$\eta := \sup\left\{\Lambda \in [0,1] : \{\xi_n(t); n \in \mathbb{N}\} \quad \text{converges uniformly on } \Bbbk_\Lambda\right\}.$$

If $\eta = 1$, then we have the global convergence of successive approximations. Suppose that $\eta < 1$, then the sequence $\{\xi_n(t); n \in \mathbb{N}\}$ converges uniformly on $\Bbbk_\eta$. As this sequence is equicontinuous, and converges uniformly to a continuous function $\tilde{\xi}(t)$. If we prove that there exists $\gamma \in (\eta, 1]$ such that $\{\xi_n(t); n \in \mathbb{N}\}$ converges uniformly on $\Bbbk_\gamma$, this will eventually leads to a contradiction.

Put $\xi(t) = \tilde{\xi}(t)$; for $t \in \Bbbk_\gamma$. By (Cd_2), there exist $\delta > 0$ and a continuous function $\varphi\colon \Bbbk \times [0,\delta] \to \mathbb{R}_+$ ensuring (15). Also, there exist $\gamma \in [\eta, 1]$ and $n_0 \in \mathbb{N}$, where for all $t \in \Bbbk_\gamma$ and $n, m > n_0$, we have

$$|\xi_n(t) - \xi_m(t)| \le \delta.$$

For any $t \in \Bbbk_\gamma$, put

$$\vartheta^{(n,m)}(t) = |\xi_n(t) - \xi_m(t)|,$$

$$\vartheta_\varepsilon(t) = \sup_{n,m \geqslant \varepsilon} \vartheta^{(n,m)}(t).$$

Then the sequence $\{\vartheta_\varepsilon(t), \varepsilon \in \mathbb{N}^*\}$ is non-increasing, one sees that it is convergent to a function $\vartheta(t)$ for each $t \in \Bbbk_\gamma$. From the equicontinuity of $\{\vartheta_\varepsilon(t), \varepsilon \in \mathbb{N}^*\}$, it follows that $\lim_{\varepsilon \to +\infty} \vartheta_\varepsilon(t) = \vartheta(t)$ uniformly on $\Bbbk_\gamma$. Additionally, for $t \in \Bbbk_\gamma$ and $n, m \ge \varepsilon$, we have

$$\begin{aligned}
\vartheta^{(n,m)}(t) &= |\xi_n(t) - \xi_m(t)| \\
&\le \frac{1}{\Gamma(\varrho)} \int_0^t e^{-\varsigma(t-s)} (t-s)^{\varrho-1} \left|\aleph\big(s, \xi_{n-1}(s)\big) - \aleph\big(s, \xi_{m-1}(s)\big)\right| ds \\
&\le \frac{1}{\Gamma(\varrho)} \int_0^{\gamma\varkappa} (t-s)^{\varrho-1} \left|\aleph\big(s, \xi_{n-1}(s)\big) - \aleph\big(s, \xi_{m-1}(s)\big)\right| ds.
\end{aligned}$$

Thus, by (15) we get

$$\begin{aligned}
\vartheta^{(n,m)}(t) &\le \frac{1}{\Gamma(\varrho)} \int_0^{\gamma\varkappa} (t-s)^{\varrho-1} \varphi(s, |\xi_{n-1}(s) - \xi_{m-1}(s)|) ds \\
&\le \frac{1}{\Gamma(\varrho)} \int_0^{\gamma\varkappa} (t-s)^{\varrho-1} \varphi\big(s, \vartheta^{(n-1,m-1)}(s)\big) ds.
\end{aligned}$$

Hence

$$\vartheta_\varepsilon(t) \le \frac{1}{\Gamma(\varrho)} \int_0^{\gamma\varkappa} (t-s)^{\varrho-1} \varphi\big(s, \vartheta_{\varepsilon-1}(s)\big) ds.$$

By the Lebesgue dominated convergence theorem, we have

$$\vartheta(t) \le \frac{1}{\Gamma(\varrho)} \int_0^{\gamma\varkappa} (t-s)^{\varrho-1} \varphi\big(s, \vartheta(s)\big) ds.$$

Moreover, by (Cd_1) and (Cd_3) we have $\vartheta \equiv 0$ on $\Bbbk_\gamma$. This yields that

$$\lim_{\varepsilon \to +\infty} \vartheta_\varepsilon(t) = 0,$$

uniformly on $\Bbbk_\gamma$. Thus $\{\xi_\varepsilon(t)\}_{\varepsilon=1}^{\infty}$ is a Cauchy sequence on $\Bbbk_\gamma$. Consequently, $\{\xi_\varepsilon(t)\}_{\varepsilon=1}^{\infty}$ is uniformly convergent on $\Bbbk_\gamma$ that gives us the contradiction.

Thus $\{\xi_\varepsilon(t)\}_{\varepsilon=1}^{\infty}$ converges uniformly on $\Bbbk$ to a continuous function $\xi_*(t)$. By the Lebesgue dominated convergence theorem, we have for each $t \in \Bbbk$

$$\begin{aligned}\lim_{\varepsilon\to+\infty} \frac{1}{\Gamma(\varrho)} \int_0^t (t-s)^{\varrho-1} \aleph\big(s, \xi_\varepsilon(s)\big) ds \\ = \frac{1}{\Gamma(\varrho)} \int_0^t (t-s)^{\varrho-1} \aleph\big(s, \xi_*(s)\big) ds.\end{aligned}$$

This means that ξ_* is a solution of the problem (1)–(2).

Lastly, we prove the uniqueness of solutions of the problem (1)–(2). Let ξ_1 and ξ_2 be two solutions of the problem (1)–(2). As previously, put

$$\bar{\eta} := \sup\left\{\rho \in [0,1] : \xi_1(t) = \xi_2(t) \quad \text{for } \ t \in \Bbbk_\rho\right\},$$

and assume that $\bar{\eta} < 1$. There exist $\delta > 0$ and a comparison function $\varphi : \Bbbk \times [0,\delta] \longrightarrow \mathbb{R}_+$ verifying (15). We take $\gamma \in (\rho, 1)$ where for $t \in \Bbbk_\gamma$

$$|\xi_1(t) - \xi_2(t)| \leq \delta.$$

Then for all $t \in \Bbbk_\gamma$, we get

$$\begin{aligned}|\xi_1(t) - \xi_2(t)| &\leq \frac{1}{\Gamma(\varrho)} \int_0^{\gamma\varkappa} (t-s)^{\varrho-1} \left|\aleph\big(s, \xi_1(s)\big) - \aleph\big(s, \xi_2(s)\big)\right| ds \\ &\leq \frac{1}{\Gamma(\varrho)} \int_0^{\gamma\varkappa} (t-s)^{\varrho-1} \varphi(s, |\xi_1(s) - \xi_2(s)|) ds.\end{aligned}$$

By (Cd_1) and (Cd_3), we obtain $\xi_1 - \xi_2 \equiv 0$ on $\Bbbk_\gamma$. This results $\xi_1 = \xi_2$ on $\Bbbk_\gamma$, which makes a contradiction. Thus, $\bar{\eta} = 1$ and the solution of (1)–(2) is unique on $\Bbbk$. □

3.2. The Nonlocal Problem

In this section, we discuss the global convergence of successive approximations to the unique solution of the problem (3)–(4).

Definition 3.3. By a solution of problem (3)–(4), we mean a function $\xi \in C(\Bbbk, \mathbb{R})$ that satisfies the equation (3) on $\Bbbk$ and the initial nonlocal condition (4).

Define the successive approximations of the problem (3)–(4) as follows:

$$\xi_0(t) = \kappa - \bar{\wp}(\xi_0); \quad t \in \Bbbk,$$

$$\xi_{n+1}(t) = \big(\kappa - \bar{\wp}(\xi_n)\big)e^{-\varsigma t} + \frac{1}{\Gamma(\varrho)}\int_0^t e^{-\varsigma(t-s)}(t-s)^{\varrho-1}\aleph\big(s, \xi_n(s)\big)ds; \quad t \in \Bbbk.$$

Set $\Bbbk_\epsilon = [0, \epsilon\varkappa]$; for any $\epsilon \in [0,1]$. The hypotheses:

$(\mathbf{A}_1)$: The function $\aleph\colon \Bbbk \times \mathbb{R} \to \mathbb{R}$ is continuous.

$(\mathbf{A}_2)$: There exist $\nu > 0$ and a continuous function $\phi\colon \Bbbk \times [0,\nu] \to \mathbb{R}_+$, where $\phi(t,\cdot)$ is non-decreasing for all $t \in \Bbbk$, and

$$|\aleph(t,\xi) - \aleph(t,\bar{\xi})| \le \phi(t, |\xi - \bar{\xi}|),$$

holds for all $t \in \Bbbk$ and $\xi, \bar{\xi} \in \mathbb{R}$ with $|\xi - \bar{\xi}| \le \nu$.

$(\mathbf{A}_3)$: There exists a constant $0 < L < 1$ such that

$$|\bar{\wp}(\xi) - \bar{\wp}(\bar{\xi})| \le L\|\xi - \bar{\xi}\|_\infty, \quad \text{for any } \ \xi, \bar{\xi} \in C(\Bbbk, \mathbb{R}).$$

$(\mathbf{A}_4)$: $\upsilon \equiv 0$ is the only function in $C(\Bbbk_\gamma, [0,\nu])$ which the integral inequality

$$\upsilon(t) \le \frac{(1-L)^{-1}}{\Gamma(\varrho)}\int_0^{\gamma\varkappa}(t-s)^{\varrho-1}\phi\big(s, \upsilon(s)\big)ds, \quad \text{with } \ \epsilon \le \gamma \le 1.$$

Theorem 3.4. *Assume that the hypotheses* (A_1)–(A_4) hold. Then the successive approximations $\{\xi_n; n \in \mathbb{N}\}$ are well defined and converge to the unique solution of problem (3)–(4) uniformly on $\Bbbk$.

Proof.
We can prove as in Theorem 3.2 that successive approximations $\{\xi_n; n \in \mathbb{N}\}$ converge uniformly on $\Bbbk$ to the unique solution of problem (3)–(4). □

4. Examples

Example. Consider the following problem which is an example of our problem (1)–(2)

$$\left({}^C_0\mathfrak{D}_t^{\frac{1}{2},2}\xi\right)(t) = \frac{(t+1)\sqrt{t}}{1+|\xi(t)|}; \quad t \in [0,1], \tag{17}$$

$$\xi(0) = \frac{3}{2}, \tag{18}$$

where $\aleph\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\aleph(t,\xi) = \frac{(t+1)\sqrt{t}}{1+|\xi|}.$$

Clearly, the function $\aleph$ is continuous.

For each $\xi, \bar{\xi} \in \mathbb{R}$ and $t \in [0,1]$, we get

$$|\aleph(t,\xi) - \aleph(t,\bar{\xi})| \leq \left((t+1)\sqrt{t}\right)|\xi - \bar{\xi}|.$$

Then the hypothesis (Cd_2) is satisfied with

$$\varphi(t,\xi) = (t+1)\xi\sqrt{t}.$$

The successive approximations $\{\xi_n; n \in \mathbb{N}\}$ of the problem (17)–(18) is defined by

$$\xi_0(t) = \frac{3}{2}e^{-2t}, \quad t \in [0,1],$$

$$\xi_{n+1}(t) = \frac{3}{2}e^{-2t} + \frac{1}{\sqrt{\pi}}\int_0^t e^{-2(t-s)}(t-s)^{-\frac{1}{2}}\frac{(s+1)\sqrt{s}}{1+|\xi_n(s)|}ds; \quad t \in [0,1].$$

Then the hypothesis (Cd_3) is satisfied with

$$\vartheta(t) \leq \frac{1}{\sqrt{\pi}}\int_0^\gamma (t-s)^{-\frac{1}{2}}(s+1)\sqrt{s}\vartheta(s)ds.$$

Since the conditions of Theorem 3.2 are verified, successive approximations

$$\{\xi_n, n \in \mathbb{N}\}$$

converge uniformly on $[0,1]$ to the unique solution of problem (17)–(18).

Example. Consider the problem with nonlocal condition

$$\left({}_0^C\mathfrak{D}_t^{\frac{1}{2},5}\xi\right)(t) = \frac{e^t}{1+2|\xi(t)|}; \quad t \in [0,2], \tag{19}$$

$$\xi(0) + \bar{\wp}(\xi) = 3, \tag{20}$$

where

$$\bar{\wp}(\xi) = \frac{\|\xi\|_\infty}{2+\|\xi\|_\infty}, \quad \text{for any} \quad \xi \in C([0,2],\mathbb{R}),$$

and $\aleph\colon [0,2] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\aleph(t,\xi) = \frac{e^t}{1+2|\xi|}.$$

Clearly, the function $\aleph$ is continuous.

For each $\xi, \bar{\xi} \in \mathbb{R}$ and $t \in [0,2]$, we get

$$|\aleph(t,\xi) - \aleph(t,\bar{\xi})| \le 2e^t|\xi - \bar{\xi}|.$$

Then the hypothesis (A_2) is satisfied with

$$\phi(t,\xi) = 2e^t\xi.$$

And for each $\xi, \bar{\xi} \in C([0,2],\mathbb{R})$, we have

$$|\bar{\wp}(\xi) - \bar{\wp}(\bar{\xi})| \le \frac{1}{2}\|\xi - \bar{\xi}\|_\infty.$$

Then the hypothesis (A_3) is satisfied by $L = \frac{1}{2}$.

The successive approximations $\{\xi_n, n \in \mathbb{N}\}$ of the problem (19)–(20) is defined by

$$\xi_0(t) = 3 - \frac{\|\xi_0\|_\infty}{2 + \|\xi_0\|_\infty}, \quad t \in [0,2],$$

$$\xi_{n+1}(t) = \big(3 - \bar{\wp}(\xi_n)\big)e^{-5t} + \frac{1}{\sqrt{\pi}}\int_0^t e^{-5(t-s)}(t-s)^{-\frac{1}{2}}\frac{e^s}{1+2|\xi_n(s)|}ds; \quad t \in [0,2].$$

Then the hypothesis (A_4) is satisfied with

$$\upsilon(t) \le \frac{4}{\sqrt{\pi}}\int_0^{2\gamma}(t-s)^{-\frac{1}{2}}e^s\upsilon(s)ds.$$

Since the conditions of Theorem 3.4 are verified, successive approximations $\{\xi_n, n \in \mathbb{N}\}$ converge uniformly on $[0,2]$ to the unique solution of problem (19)–(20).

5. Conclusion

In this paper, we have made a significant contribution to the study of local and nonlocal fractional differential equations involving the Caputo tempered fractional derivative. Our approach is based on the successive approximations method. We have shown that this method ensures both the global convergence of the approximations and the uniqueness of the solution. These findings highlight the effectiveness of the method for this class of equations. In future work, we plan to focus on applying additional numerical methods to study the existence and uniqueness results for different types of fractional differential equations and inclusions, under different conditions involving the Caputo tempered fractional derivative in different spaces.

`Competing interests.`
It is declared that the authors have no competing interests.

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