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NEUMANN PROBLEM WITH A NONLINEAR $p(x)$-ELLIPTIC EQUATION SOLVED BY TOPOLOGICAL DEGREE METHODS

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ABSTRACT. In this paper, we prove the existence of weak solutions to Neumann boundary value problems for a nonlinear $p(x)$-elliptic equation of the form

$$-\operatorname{div} a(x,u,\nabla u) = b(x)|u|^{p(x)-2}u + \lambda H(x,u,\nabla u).$$

We established the existence result by using the topological degree introduced by Berkovits.

1. Introduction

The objective of this paper is to study the existence of a weak solution to the Neumann boundary value problems associated with the following equation:

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = b(x)|u|^{p(x)-2}u + \lambda H(x,u,\nabla u) & \text{in} \quad \Omega, \\ \quad a(x,u,\nabla u)\cdot \nu = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open domain with Lipschitz boundary $\partial\Omega$, $p\colon \overline{\Omega} \to (2,\infty)$ is a continuous function, λ is a real parameter, $b \in L^\infty(\Omega)$ with $b(x) > 0$ a.e. $x \in \Omega$, and ν is the outer unit normal vector on $\partial\Omega$.

The function $a\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the following assumptions for a.e. in $x \in \Omega$ and all ξ, $\xi' \in \mathbb{R}^N$, $(\xi \neq \xi')$ and $s \in \mathbb{R}$

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(H$_1$) $a(x,s,\xi)\,\xi \geq \alpha\ |\xi|^{p(x)}$,

(H$_2$) $|a(x,s,\xi)| \leq \beta\left(k(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\right)$,

(H$_3$) $\big(a(x,s,\xi) - a(x,s,\xi')\big)(\xi - \xi') > 0$,

for some positive constants α and β, where $k(x)$ is a positive function belonging to $L^{p'(x)}(\Omega)$ ($p'(x)$ is the conjugate exponent of $p(x)$). Moreover,

$$H\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

is a Carathéodory function verified by the following growth condition. For all $t \in \mathbb{R}^N$, $s \in \mathbb{R}$ and a.e. $x \in \Omega$,

(H$_4$) $|H(x,s,t)| \leq \varrho(e(x) + |s|^{p(x)-1} + |t|^{p(x)-1})$,

where ϱ is a positive constant and $e(x)$ is a positive function in $L^{p'(x)}(\Omega)$. In the case when

$$a(x,u,\nabla u) = A(x,\nabla u) \quad \text{and} \quad H(x,u,\nabla u) = f(x,\nabla u) + s(x)|u|^{p-1}u,$$

T. He et al. [21] proved that problem (1) has at least three nontrivial smooth solutions: two of constant sign (one positive and one negative) and a third nodal. S. Liu [25] established some existence results when $a(x,u,\nabla u) = |\nabla u|^{p-2}u$ and $H(x,u,\nabla u) = f(x,\nabla u)$ with Dirichlet boundary conditions. Moreover, A. Abbassi, C. Allalou, and A. Kassidi [3] studied problem (1) using the topological degree method in the case where p is constant (i.e., $p(\cdot) = p$) within the framework of classical Sobolev space. For more important and interesting results, we recommend the readers to see [1,5,15,19,20].

Motivated by the works mentioned above, we will try to study problem (1) and improve the findings in literature [10,21] to provide a physical interpretation of equation (1) and other fascinating applications of fluid mechanics models (see [7,8]), nonlinear diffusion (see [28]), and nonlinear elasticity (see [6,22]).

We aim to study the existence of weak solutions of problem (1) using the topological degree method. We propose this theory for operators of generalized monotone-type introduced by Berkovits in [9]. The expanded degree class of operators is effectively obtained by replacing the compact perturbation with a composition of monotone-type operators, based on the Leray-Schauder degree [23]. The topological degree hypothesis was first introduced by Leray-Schauder [23] in their study of nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Later, Browder [12] used the Galerkin method to construct a topological degree for operators of class (S_+) in reflexive Banach spaces. For more details, we recommend that the reader consult classical works, such as [13,14,18,30].

The plan of the paper is as follows. In Section 2, we present some mathematical preliminaries. In Section 3, we prepare for the proof of the main theorem by introducing several lemmas. Section 4 is devoted to proving the existence of weak solutions for problem (1).

2. Mathematical Preliminaries

To discuss problem (1), we recall some definitions and basic properties of the generalized Lebesgue–Sobolev spaces with variable exponent $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$. Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, we say that a real-valued continuous function $p(.)$ is log-Hölder continuous in Ω if there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{|\log|x-y||} \tag{2}$$

$\forall x, y \in \overline{\Omega}$ with $|x-y| < \frac{1}{2}$. Let us set

$C_+(\overline{\Omega}) = \{\log\text{-Hölder continuous function } p \in C(\overline{\Omega})\colon \text{ such that } 1 < p^- < p^+ < N\}$,
where

$$p^- = \min\{p(x), x \in \overline{\Omega}\} \quad \text{and} \quad p^+ = \max\{p(x), x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u\colon \Omega \to \mathbb{R}; u \text{ is measurable with} \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty \right\},$$

the space $L^{p(x)}(\Omega)$ under the "Luxembourg norm"

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leq 1 \right\},$$

which is a separable and reflexive Banach space. The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [16, 29]).

We also define the variable Sobolev spaces by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

where the norm is

$$\|u\| = \|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Assuming $1 < p^- \leq p^+ < \infty$, the space $W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space. We denote by $W_0^{1,p(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$, i.e.,

$$W_0^{1,p(.)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(.)}(\Omega)},$$

and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 2.1 (Young's Inequality). *Let $p, p' \in C_+(\Omega)$, where p' is the conjugate of p. Then, for all $a, b > 0$, we have*

$$ab \leq \frac{a^{p(x)}}{p(x)} + \frac{b^{p'(x)}}{p'(x)}.$$

Proposition 2.2 (Generalized Hölder Inequality (see [17, 24])).

i) *For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv \mathrm{d}x \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)}.$$

ii) *For all $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ a.e. in Ω, we have $L^{q(x)} \hookrightarrow L^{p(x)}$ and the embedding is continuous.*

The modular of the space $L^{p(x)}(\Omega)$, which is the mapping $\varrho \colon L^{p(x)}(\Omega) \to \mathbb{R}$, is defined by $\varrho(u) = \int_\Omega |u(x)|^{p(x)} \mathrm{d}x$ for all $u \in L^{p(x)}(\Omega)$.

Proposition 2.3 ([16, 29]). *For $u \in L^{p(x)}(\Omega)$ and $\{u_k\}_{k \in N} \subset L^{p(x)}(\Omega)$, the following assertions hold*

$$\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \varrho(u) \leq \|u\|_{p(x)}^{p^+}, \tag{3}$$

$$\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \varrho(u) \leq \|u\|_{p(x)}^{p^-}, \tag{4}$$

$$\lim_{k \to \infty} \|u_k\|_{p(x)} = 0 \quad \Leftrightarrow \lim_{k \to \infty} \varrho(u_k) = 0, \tag{5}$$

$$\lim_{k \to \infty} \|u_k\|_{L^{p(x)}(\Omega)} = \infty \Leftrightarrow \lim_{k \to \infty} \varrho(u_k) = \infty. \tag{6}$$

Proposition 2.4 (See [1]).

i) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then*

$$W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega) \text{ is continuous and compact.}$$

In particular, we have

$$W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega) \text{ is continuous and compact.}$$

ii) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set and $p \in C_+(\overline{\Omega})$. Then, for $u \in W_0^{1,p(x)}(\Omega)$, the $p(x)$-Poincaré inequality*

$$\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)}$$

holds, where the positive constant C depends on $p(x)$ and Ω.

Lemma 2.5. *Let $a \geqslant 0, b \geqslant 0$ and $1 < p^- \leq p^+ < \infty$. Then, we have*

$$(a+b)^{p(x)} \leqslant 2^{p^+ - 1} \big(a^{p(x)} + b^{p(x)} \big).$$

Now, we give some results and properties from the theory of topological degree. We begin by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle\cdot,\cdot\rangle$ between X^* and X, in this order. Given a nonempty subset Ω of X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively.

Definition 2.6. Let Y be another real Banach space. An operator $F\colon \Omega\subset X\to Y$ is said to be

(1) bounded,
if it takes any bounded set into a bounded set.

(2) demicontinuous,
if for any sequence $(u_n)\subset\Omega,\ u_n\to u$ implies $F(u_n)\rightharpoonup F(u)$.

(3) compact,
if it is continuous and the image of any bounded set is relatively compact.

Definition 2.7. A mapping $F\colon\Omega\subset X\to X^*$ is said to be

(1) of type (S_+), if for any sequence
$(u_n)\subset\Omega$ with $u_n\rightharpoonup u$ and $\limsup_{n\to\infty}\langle Fu_n,u_n-u\rangle\le 0$, we have $u_n\to u$.

(2) quasimonotone, if for any sequence
$(u_n)\subset\Omega$ with $u_n\rightharpoonup u$, we have $\limsup_{n\to\infty}\langle Fu_n,u_n-u\rangle\ge 0$.

Definition 2.8. Let $T\colon\Omega_1\subset X\to X^*$ be a bounded operator such that $\Omega\subset\Omega_1$. For any operator $F\colon\Omega\subset X\to X^*$, we say that:

(1) F satisfies condition $(S_+)_T$, if for any sequence
$(u_n)\subset\Omega$ with $u_n\rightharpoonup u,\ y_n:=Tu_n\rightharpoonup y$ and $\limsup_{n\to\infty}\langle Fu_n,\ y_n-y\rangle\le 0$, we have $u_n\to u$.

(2) F has the property $(\mathrm{QM})_T$, if for any sequence
$(u_n)\subset\Omega$ with $u_n\rightharpoonup u,\ y_n:=Tu_n\rightharpoonup y$, we have $\limsup_{n\to\infty}\langle Fu_n,y-y_n\rangle\ge 0$.

Let $\mathcal{O}$ be the collection of all bounded open sets in X. For any $\Omega\subset X$, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega):=\{F\colon\Omega\to X^*\text{such that }F\text{ is bounded, demicontinuous and of type }(S_+)\},$$

$$\mathcal{F}_{T,B}(\Omega):=\{F\colon\Omega\to X\text{ such that }F\text{ is bounded, demicontinuous and of type }(S_+)_T\},$$

$$\mathcal{F}_T(\Omega):=\{F\colon\Omega\to X\text{ such that }F\text{ is demicontinuous and of type }(S_+)_T\},$$

$$\mathcal{F}_B(X):=\{F\in\mathcal{F}_{T,B}(\overline{E})\text{ such that }E\in\mathcal{O},\mathrm{T}\in\mathcal{F}_1(\overline{\mathrm{E}})\}.$$

For any $F\in\mathcal{F}_T(\overline{E})$, we call $T\in\mathcal{F}_1(\overline{E})$ the essential inner mapping of F.

Lemma 2.9 ([9], Lemma 2.2 and Lemma 2.4)**.** *Let $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S\colon D_S \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D_s$, where E is a bounded open set in a real reflexive Banach space X. Then, the following statements are true:*

(1) *If S is quasimonotone, then $I+SoT \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.*
(2) *If S is of class (S_+), then $SoT \in \mathcal{F}_T(\overline{E})$.*

Definition 2.10. Suppose that E is a bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ is continuous, and let $F, S \in \mathcal{F}_T(\overline{E})$. The affine homotopy $\Lambda\colon [0,1] \times \overline{E} \to X$ defined by

$$\Lambda(t,u) := (1-t)Fu + tSu \quad for \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 1 ([9])**.** The above affine homotopy satisfies condition $(S_+)_T$.

Now, we introduce the Berkovits topological degree for the class $\mathcal{F}_B(X)$ (for more details, see [9]).

Theorem 2.11. *There exists a unique degree function*

$$d\colon \big\{(F,E,h) | E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\big\} \to \mathbb{Z}$$

which satisfies the following properties:

Normalization: *For any $h \in E$, we have $d(I,E,h) = 1$.*

Additivity: *Let $F \in \mathcal{F}_{T,B}(\overline{E})$.* If *$E_1$ and E_2 are two disjoint open subsets of E such that $h \notin F(\overline{E} \backslash (E_1 \cup E_2))$, then we have*

$$d(F,E,h) = d(F,E_1,h) + d(F,E_2,h).$$

Homotopy invariance: *If $\Lambda\colon [0,1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h\colon [0,1] \to X$ is a continuous path in X such that $h(t) \notin \Lambda(t, \partial E)$ for all $t \in [0,1]$, then the value of $d\big(\Lambda(t,\cdot), E, h(t)\big)$ is constant for all $t \in [0,1]$.*

Existence: *If $d(F,E,h) \neq 0$, then the equation $Fu = h$ has a solution in E.*

3. Notions of solutions and technical lemmas

Definition 3.1. $u \in W^{1,p(x)}(\Omega)$ is said to be a weak solution of (1), if

$$\int_\Omega a(x,u,\nabla u)\nabla v \mathrm{d}x = \int_\Omega b(x)|u|^{p(x)-2}uv\mathrm{d}x + \lambda\int_\Omega H(x,u,\nabla u)v\mathrm{d}x, \quad \forall\, v \in W^{1,p(x)}(\Omega).$$

Lemma 3.2 ([4]). *Let*

$$g \in L^{r(x)}(\Omega) \quad \textit{and} \quad g_n \subset L^{r(x)}(\Omega)$$

such that

$$|g_n|_{r(x)} \le C, \ \ 1 < r(x) < \infty.$$

If

$$g_n(x) \to g(x) \ \ \textit{a.e. in } \Omega, \quad \textit{then} \quad g_n \rightharpoonup g \ \ \textit{weakly in } L^{r(x)}(\Omega).$$

Let us consider the nonlinear operator T from $W^{1,p(x)}(\Omega)$ into its dual of the form

$$Tu = -\mathrm{div}\, a(x,u,\nabla u).$$

Then,

$$\langle Tu, v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v\, \mathrm{d}x, \quad \text{for all } v \in W^{1,p(x)}(\Omega).$$

Lemma 3.3. *Assume that (H_1)–(H_3) hold, then T is bounded, coercive, continuous and of type (S_+).*

Proof. First, we prove that the operator T is bounded. Let $u, v \in W^{1,p(x)}(\Omega)$. By using the Hölder's inequality, we have

$$\begin{aligned} |\langle Tu, v\rangle| &= \left|\int_\Omega a(x,u,\nabla u)\nabla v\, \mathrm{d}x\right| \\ &\le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|a(x,u,\nabla u)\|_{p'(x)}\|\nabla v\|_{p(x)}. \end{aligned}$$

On the other hand, from (H_2) we can easily show that

$$\|a(x,u,\nabla u)\|_{p'(x)} \quad \text{is bounded for all } u \text{ in } W^{1,p(x)}(\Omega).$$

Indeed,

$$\begin{aligned}
\|a(x,u,\nabla u)\|_{p'(x)} &\leq \left[\int\limits_{\Omega} \mid a(x,v,\nabla v)\mid^{p'(x)} \mathrm{d}x\right]^{\frac{1}{\gamma}} \\
&\leq \left[\int\limits_{\Omega} \left(\beta\left(k(x)+|u|^{p(x)-1}+|\nabla u|^{p(x)-1}\right)\right)^{p'(x)} \mathrm{d}x\right]^{\frac{1}{\gamma}} \\
&\leq \left[\beta^{\,p'^{+}} 2^{2(p'^{+}-1)} \int\limits_{\Omega} \left(|k(x)|^{p'(x)}+|u|^{p(x)}+|\nabla u|^{p(x)}\right) \mathrm{d}x\right]^{\frac{1}{\gamma}} \\
&\leq \; C_1 \left(\int\limits_{\Omega} |k(x)|^{p'(x)} \mathrm{d}x\right)^{\frac{1}{\gamma}} \\
&\quad + C_2 \left(\int\limits_{\Omega} \left(|u|^{p(x)}+|\nabla u|^{p(x)}\right) \mathrm{d}x\right)^{\frac{1}{\gamma}} \\
&\leq C_1 \|k\|_{p'(x)}^{\frac{\delta'}{\gamma}} + C_2 \|u\|^{\frac{\delta}{\gamma}} \\
&\leq C_m + C_m \|u\|^{\frac{\delta}{\gamma}} \\
&\leq C_m \left(1+\|u\|^{\frac{\delta}{\gamma}}\right),
\end{aligned}$$

where

$$\gamma = \begin{cases} p'^{+} & \text{if} \quad \|a(x,u,\nabla u)\|_{p'(x)} \leq 1, \\ p'^{-} & \text{if} \quad \|a(x,u,\nabla u)\|_{p'(x)} \geq 1, \end{cases}$$

$$\delta' = \begin{cases} p'^{+} & \text{if} \quad \|k\|_{p'(x)} \leq 1, \\ p'^{-} & \text{if} \quad \|k\|_{p'(x)} \geq 1 \end{cases} \qquad \text{and,} \quad \delta = \begin{cases} p^{+} & \text{if} \quad \|u\| \leq 1, \\ p^{-} & \text{if} \quad \|u\| \geq 1, \end{cases}$$

$$C_m = \max\left(C_1 ||k||_{p'(x)}^{\frac{\delta'}{\gamma}}, C_2\right).$$

Hence,

$$|\langle Tu, v\rangle| \leq c\,\|v\|\,,$$

which implies that the operator T is bounded.

Secondly, we prove that T is coercive. Let $v \in W^{1,p(x)}(\Omega)$. We get from (H_1) that

$$\begin{aligned}
\frac{\langle Tv, v\rangle}{\|v\|} &= \frac{\int_\Omega a(x,v,\nabla v)\nabla v \mathrm{d}x}{\|v\|} \\
&\geq \alpha \frac{\int_\Omega |\nabla v|^{p(x)}\mathrm{d}x + \int_\Omega |v|^{p(x)}\mathrm{d}x - \int_\Omega |v|^{p(x)}\mathrm{d}x}{\|v\|} \\
&\geq \alpha \frac{||v||^{\gamma'} - ||v||^{\gamma}_{p(x)}}{\|v\|} \\
&\geq \alpha \|v\|^{\gamma'-1} - \alpha \frac{\|v\|^{\gamma}_{p(x)}}{\|v\|} \\
&\geq \alpha \|v\|^{\gamma'-1} - C. \quad (\text{Due to} W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)),
\end{aligned}$$

where

$$\gamma' = \begin{cases} p^+ & \text{if} \quad ||v|| \leq 1, \\ p^- & \text{if} \quad ||v|| \geq 1, \end{cases}$$

and

$$\gamma = \begin{cases} p^+ & \text{if} \quad ||v||_{p(x)} \leq 1, \\ p^- & \text{if} \quad ||v||_{p(x)} \geq 1. \end{cases}$$

Therefore,

$$\frac{\langle Tv, v\rangle}{\|v\|} \to \infty \quad \text{as} \quad \|v\| \to \infty.$$

Hence, T is coercive.

Now, we show that T is continuous. Let $u_n \to u$ in $W^{1,p(x)}(\Omega)$.
Then, $\nabla u_n \to \nabla u$ in $\left(L^{p(x)}(\Omega)\right)^N$. Hence, there exist a subsequence (u_k) of (u_n) and measurable functions h in $L^{p(x)}(\Omega)$ and g in $\left(L^{p(x)}(\Omega)\right)^N$ such that

$$u_k(x) \to u(x) \quad \text{and} \quad \nabla u_k(x) \to \nabla u(x),$$

$$|u_k(x)| \leq h(x) \quad \text{and} \quad |\nabla u_k(x)| \leq |g(x)|,$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Since a satisfies the Carathéodory condition, we obtain

$$a(x, u_k, \nabla u_k) \to a(x, u, \nabla u) \qquad \text{a.e.} \;\; x \in \Omega. \tag{7}$$

According to (H_2), we have

$$|a(x, u_k, \nabla u_k)| \leq \beta \left(k(x) + |h(x)|^{p(x)-1} + |g(x)|^{p(x)-1}\right),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$. Since

$$\left(k(x) + |h(x)|^{p(x)-1} + |g(x)|^{p(x)-1}\right) \in L^{p'(x)}(\Omega),$$

and by using (7), we have

$$\int_\Omega |a(x, u_k, \nabla u_k) - a(x, u, \nabla u)|^{p'(x)} \mathrm{d}x \longrightarrow 0.$$

The dominated convergence theorem implies that

$$a(x, u_k, \nabla u_k) \to a(x, u, \nabla u) \ \text{ in } \ \left(L^{p'(x)}(\Omega)\right)^N.$$

Thus, the entire sequence $a(x, u_n, \nabla u_n)$ converges to $a(x, u, \nabla u)$ in $\left(L^{p'(x)}(\Omega)\right)^N$. Therefore, for all $v \in W^{1,p(x)}(\Omega)$, we have $\langle Tu_n, v\rangle \to \langle Tu, v\rangle$, which implies that the operator T is continuous.

It remains to prove that the operator T is of type (S_+).
Let $(u_n)_n$ be a sequence in $W^{1,p(x)}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } \ W^{1,p(x)}(\Omega), \\ \limsup_{n\to\infty} \langle Tu_n, u_n - u\rangle \leq 0. \end{cases} \tag{8}$$

We will show that $u_n \to u$ in $W^{1,p(x)}(\Omega)$.

The first part of (8) yields $M_1 > 0$ such that

$$\|\nabla u_n\|_{p(x)} \leq M_1 \quad \text{for all } \ n \geq 1.$$

In view of (H_3) and (8), we get

$$\lim_{n\to\infty} \langle Tu_n,\ u_n - u\rangle = \lim_{n\to\infty} \langle Tu_n - Tu,\ u_n - u\rangle = 0. \tag{9}$$

Hence,

$$D_n \to 0 \quad \text{in } L^1(\Omega),$$

where

$$D_n = \big[a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\big]\big(\nabla u_n - \nabla u\big).$$

By using $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, we can find a subsequence, still denoted by u_n, such that

$$\begin{cases} u_n \to u \quad \text{a.e. in } \Omega, \\ D_n \to 0 \quad \text{a.e. in } \Omega. \end{cases}$$

Then, there exists a subset B of Ω, such that $\text{mes}(B) = 0$, and for

$$x \in \Omega \backslash B, \qquad |u(x)| < \infty, \qquad |\nabla u(x)| < \infty,$$
$$|k(x)| < \infty, \qquad u_n(x) \to u(x), \qquad D_n(x) \to 0.$$

Let us define

$$\xi_n = \nabla u_n, \;\; \xi = \nabla u.$$

We have

$$\begin{aligned} D_n(x) &= \big[a(x,u_n,\xi_n) - a(x,u_n,\xi)\big].\big(\xi_n - \xi\big) \\ &= \; a(x,u_n,\xi_n)\xi_n + a(x,u_n,\xi)\xi - a(x,u_n,\xi_n)\xi - a(x,u_n,\xi)\xi_n \\ &\geq \; \alpha|\xi_n|^{p(x)} + \alpha|\xi|^{p(x)} - \beta\big(k(x) + |u_n|^{p(x)-1} + |\xi_n|^{p(x)-1}\big)|\xi| \\ &\quad - \beta\big(k(x) + |u_n|^{p(x)-1} + |\xi|^{p(x)-1}\big)|\xi_n| \\ &\geq \; \alpha|\xi_n|^{p(x)} - C_x[1 + |\xi_n|^{p(x)-1} + |\xi_n|], \end{aligned}$$

where C_x is a function that depends on x and not on n.

Since $u_n \to u$ a.e. in Ω, we have $|u_n(x)| \leq M_x$, where M_x is some positive constant. Then, by a standard argument, we will have that $|\xi_n|$ is uniformly bounded compared to n. From this, we deduce that

$$D_n(x) \geq |\xi_n|^{p(x)} \left(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p(x)-1}} \right).$$

If $|\xi_n| \to \infty$ (for a subsequence), then $D_n(x) \to \infty$, which is absurd.

Let now ξ^* be an adherent point of ξ_n. We have $|\xi^*| < \infty$, and by continuity of the operator a compared to the last two variables, we will have:

$$\big[a(x,u_n,\xi^*) - a(x,u_n,\xi)\big]\big(\xi^* - \xi\big) = 0.$$

According to (H_3), we obtain

$$\xi^* = \xi,$$

the unicity of adherent point implies

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega.$$

On the other hand, since the sequence $\big(a(x,u_n,\nabla u_n)\big)$ is bounded in

$$\big(L^{p'(x)}(\Omega)\big)^N,$$

and

$$a(x,u_n,\nabla u_n)\to a(x,u,\nabla u)\quad \text{a.e. in }\ \Omega,$$

in view of Lemma 3.2, we have

$$a(x,u_n,\nabla u_n)\rightharpoonup a(x,u,\nabla u)\quad \text{in } \big(L^{p'(x)}(\Omega)\big)^N \quad \text{a.e. in }\ \Omega. \tag{10}$$

We set

$$\overline{y}_n=a(x,u_n,\nabla u_n)\nabla u_n \quad\text{and}\quad \overline{y}=a(x,u,\nabla u)\nabla u.$$

We can write

$$\overline{y}_n\to\overline{y}\quad \text{in } L^1(\Omega).$$

From (H_1), we have

$$\alpha\,|\nabla u_n|^{p(x)}\le a(x,u_n,\nabla u_n)\nabla u_n.$$

Let

$$z_n=|\nabla u_n|^{p(x)},\quad z=|\nabla u|^{p(x)},\quad y_n=\frac{\overline{y}_n}{\alpha},\quad y=\frac{\overline{y}}{\alpha}.$$

Thanks to Fatou's lemma,

$$\int_\Omega 2\,y\,\mathrm{d}x\le\liminf_{n\to\infty}\int_\Omega y+y_n-|z_n-z|\mathrm{d}x,\quad \text{i.e.,}\ 0\le-\limsup_{n\to\infty}\int_\Omega |z_n-z|\,\mathrm{d}x.$$

Then,

$$0\le\liminf_{n\to\infty}\int_\Omega |z_n-z|\,\mathrm{d}x\le\limsup_{n\to\infty}\int_\Omega |z_n-z|\,\mathrm{d}x\le 0$$

implies

$$\nabla u_n\longrightarrow\nabla u\quad\text{in}\quad \big(L^{p(x)}(\Omega)\big)^N. \tag{11}$$

From the compact embedding of $W^{1,p(x)}(\Omega)$ into $L^{p(x)}(\Omega)$, we obtain, by the assumptions, that

$$\|u_n\|_{p(x)}\to\|u\|_{p(x)}\quad \text{as }\ n\to\infty,$$

whence $\|u_n\|\longrightarrow\|u\|$ (see (11)). Since $W^{1,p(x)}(\Omega)$ satisfies the Kadec-Klee property [27, Remark 2.47(a),(c)], the facts that $u_n\rightharpoonup u$ and $\|u_n\|\longrightarrow\|u\|$ ensure that

$$u_n\longrightarrow u\quad\text{in }\ W^{1,p(x)}(\Omega)\quad as\ \ n\to\infty,$$

which completes the proof. $\square$

Lemma 3.4. *Suppose that hypothesis (H_1) holds. Then, the operator*

$$S : W^{1,p(x)}(\Omega) \to \big(W^{1,p(x)}(\Omega)\big)^*$$

defined by

$$\langle Su,\ v\rangle = -\int\limits_{\Omega} \big(b(x)|u|^{p(x)-2}u + \lambda H(x,u,\nabla u)\big)v\,\mathrm{d}x, \qquad \forall u,\ v \in W^{1,p(x)}(\Omega)$$

is compact.

P r o o f. The proof was divided into three steps:

Step 1. Let $\phi : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be the operator setting by

$$\phi\, u(x) := -b(x)|u(x)|^{p(x)-2}u(x) \quad \text{for} \quad u \in W^{1,p(x)}(\Omega) \quad \text{and} \quad x \in \Omega.$$

It is obvious that ϕ is continuous.

Next, we show that ϕ is bounded. For all $u \in W^{1,p(x)}(\Omega)$, by using the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, we have

$$\begin{aligned}
\|\phi\, u\|^{\gamma}_{p'(x)} &= \int\limits_{\Omega} |-b(x)|u|^{p(x)-2}u|^{p'(x)}\mathrm{d}x \\
&\le \|b^{p'^{+}}\|_{\infty} \int\limits_{\Omega} |u|^{(p(x)-1)p'(x)}\mathrm{d}x \\
&\le \|b^{p'^{+}}\|_{\infty} \int\limits_{\Omega} |u|^{p(x)}\mathrm{d}x \\
&\le C\,\|u\|^{\gamma'},
\end{aligned}$$

where

$$\gamma = \begin{cases} p^{+} & \text{if} \quad \|\phi\, u\|_{p'(x)} \le 1, \\ p^{-} & \text{if} \quad \|\phi\, u\|_{p'(x)} \ge 1, \end{cases}$$

and

$$\gamma' = \begin{cases} p^{+} & \text{if} \quad ||u|| \le 1, \\ p^{-} & \text{if} \quad ||u|| \ge 1. \end{cases}$$

This implies that ϕ is bounded on $W^{1,p(x)}(\Omega)$.

Step 2. We show that the operator ψ defined from $W^{1,p(x)}(\Omega)$ into $L^{p'(x)}(\Omega)$ by

$$\psi u(x) := -\lambda H(x,u,\nabla u) \quad \text{for} \quad u \in W^{1,p(x)}(\Omega) \quad \text{and} \quad x \in \Omega$$

is bounded and continuous.

Let $u \in W^{1,p(x)}(\Omega)$. By using the growth condition (H_1), we obtain

$$
\begin{aligned}
\|\psi\, u\|^{\theta}_{p'(x)} &\leq \int_\Omega |\lambda H(x,u,\nabla u)|^{p'(x)} \mathrm{d}x \\
&\leq \int_\Omega \lambda^{p'(x)} \varrho^{p'(x)} \left| e(x) + |u|^{p(x)-1} + |\nabla u|^{p(x)-1} \right|^{p'(x)} \mathrm{d}x \\
&\leq \int_\Omega (\varrho\lambda)^{p'^{+}} 2^{p'^{+}-1} \Big(|e(x)|^{p'(x)} \\
&\qquad\qquad + \big(|u|^{p(x)-1} + |\nabla u|^{p(x)-1}\big)^{p'(x)} \Big) \mathrm{d}x \\
&\leq \int_\Omega (\varrho\lambda)^{p'^{+}} 2^{p'^{+}-1} \Big(|e(x)|^{p'(x)} \\
&\qquad + 2^{p'^{+}-1} \big(|u|^{(p(x)-1)p'(x)} + |\nabla u|^{(p(x)-1)p'(x)}\big) \Big) \mathrm{d}x \\
&\leq \int_\Omega (\varrho\lambda)^{p'^{+}} 2^{2(p'^{+}-1)} \big(|e(x)|^{p'(x)} + |u|^{p(x)} + |\nabla u|^{p(x)} \big) \mathrm{d}x \\
&\leq C \int_\Omega |e(x)|^{p'(x)} \mathrm{d}x + C \int_\Omega |u|^{p(x)} + |\nabla u|^{p(x)} \, \mathrm{d}x \\
&\leq C\, \|e\|^{\theta_1}_{p'(x)} + C \|u\|^{\theta_2} \\
&\leq \; C_{\max}(|u|^{\theta_2} + 1),
\end{aligned} \tag{12}
$$

where

$$C_{\max} = \max\big((\varrho\lambda)^{p'^{+}} 2^{2(p'^{+}-1)} \|e\|^{\theta_1}_{p'(x)}, (\varrho\lambda)^{p'^{+}} 2^{2(p'^{+}-1)} \big).$$

Since $e(x)$ is a positive function in $L^{p'(x)}(\Omega)$,

$$
\theta = \begin{cases} p'^{+} & \text{if} \quad \|\psi\, u\|_{p'(x)} \leq 1, \\ p'^{-} & \text{if} \quad \|\psi\, u\|_{p'(x)} \geq 1, \end{cases}
\qquad
\theta_1 = \begin{cases} p'^{+} & \text{if} \quad \|e\|_{p'(x)} \leq 1, \\ p'^{-} & \text{if} \quad \|e\|_{p'(x)} \geq 1, \end{cases}
$$

and

$$
\theta_2 = \begin{cases} p^{+} & \text{if} \quad \|u\| \leq 1, \\ p^{-} & \text{if} \quad \|u\| \geq 1. \end{cases} \tag{13}
$$

Therefore, ψ is bounded on $W^{1,p(x)}(\Omega)$.

Next, we show that ψ is continuous. Let $u_n \to u$ in $W^{1,p(x)}(\Omega)$. Then,

$$u_n \to u \text{ in } L^{p(x)}(\Omega) \quad \text{and} \quad \nabla u_n \to \nabla u \text{ in } \big(L^{p(x)}(\Omega)\big)^N.$$

Thus, there exists a subsequence, still denoted by (u_n), and measurable functions

$$\varphi \text{ in } L^{p(x)}(\Omega) \qquad \text{and} \qquad \sigma, \text{ in } \big(L^{p(x)}(\Omega)\big)^N$$

such that

$$u_n(x) \to u(x) \qquad \text{and} \qquad \nabla u_n(x) \to \nabla u(x),$$

$$|u_n(x)| \leq \varphi(x) \qquad \text{and} \qquad |\nabla u_n(x)| \leq |\sigma(x)|,$$

for a.e. $x \in \Omega$ and all $n \in \mathbb{N}$. Since H satisfies the Carathéodory condition, we obtain

$$H\big(x, u_n(x), \nabla u_n(x)\big) \to H\big(x, u(x), \nabla u(x)\big) \quad \text{a.e. } x \in \Omega. \tag{14}$$

Thanks to (H_1), we obtain

$$\big|H\big(x, u_n(x), \nabla u_n(x)\big)\big| \leq \varrho\big(e(x) + |\varphi(x)|^{p(x)-1} + |\sigma(x)|^{p(x)-1}\big)$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Since

$$e(x) + |\varphi(x)|^{p(x)-1} + |\sigma(x)|^{p(x)-1} \in L^{p'(x)}(\Omega),$$

and from (14), we get

$$\int_\Omega \big|H\big(x, u_k(x), \nabla u_k(x)\big) - H\big(x, u(x), \nabla u(x)\big)\big|^{p'(x)} \mathrm{d}x \longrightarrow 0,$$

and by using the dominated convergence theorem, we have

$$\psi u_k \to \psi u \quad \text{in} \quad L^{p'(x)}(\Omega).$$

Thus, the entire sequence (ψu_n) converges to ψu in $L^{p'(x)}(\Omega)$ and then, ψ is continuous.

Step 3. As the embedding $I \colon W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* \colon L^{p'(x)}(\Omega) \to \big(W^{1,p(x)}(\Omega)\big)^*$ is also compact. Therefore, the compositions

$$I^* o\phi \quad \text{and} \quad I^* o\psi$$

from

$$W^{1,p(x)}(\Omega) \quad \text{into} \quad \big(W^{1,p(x)}(\Omega)\big)^*$$

are compact. We deduce that

$$S = I^* o\phi + I^* o\psi$$

is compact, which completes the present proof. □

4. Main results

In this section, we study the existence of weak solutions for the nonlinear problem (1) based on the recent Berkovits topological degree introduced in Section 2.

Theorem 4.1. *Assume that the assumptions* (H_1)–(H_4) *hold. Then, problem* (1) *has a weak solution*

$$u \ \ in \ \ W^{1,p(x)}(\Omega).$$

Proof. Let $T,\ S$ be two operators from $W^{1,p(x)}(\Omega)$ into its dual as defined in Lemma 3.3 and in Lemma3.4, respectively. Then,

$$u \in W^{1,p(x)}(\Omega)$$

is a weak solution of problem (1) if and only if

$$Tu = -Su. \tag{15}$$

According to Lemma 3.4, the operator S is bounded, continuous, and quasi-monotone. On the other hand, in light of the properties of the operator T given in Lemma 3.3, and since the operator T is strictly monotone, moreover, by using the Minty-Browder Theorem (see [30], Theorem 26 A), the inverse operator

$$G := T^{-1} \colon \left(W^{1,p(x)}(\Omega)\right)^* \to W^{1,p(x)}(\Omega)$$

is bounded, continuous and of type (S_+).

Hence, equation (15) is equivalent to the abstract Hammerstein equation

$$u = Gv \quad \text{and} \quad v + SoGv = 0. \tag{16}$$

To solve equations (16), we will use the Berkovits topological degree introduced in Section 2. For that, we first prove that the set

$$B := \left\{v \in \left(W^{1,p(x)}(\Omega)\right)^* \;\middle|\; v + tSoGv = 0 \quad \text{for some} \quad t \in [0,1]\right\}$$

is bounded. Indeed, let $v \in B$, and take $u := Gv$.

Thanks to (H$_1$), (H$_4$), (12), the Young's inequality and the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, we have

$$\begin{aligned}
\|Gv\|^{\theta'} &\leq \int_\Omega |u|^{p(x)}\,\mathrm{d}x + \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \\
&\leq \int_\Omega |u|^{p(x)}\,\mathrm{d}x + \frac{1}{\alpha}\int_\Omega a(x,u,\nabla u)\nabla u \mathrm{d}x \\
&\leq \int_\Omega |u|^{p(x)}\,\mathrm{d}x + \frac{1}{\alpha}\langle Tu, u\rangle = \int_\Omega |u|^{p(x)}\,\mathrm{d}x + \frac{1}{\alpha}\langle v, Gv\rangle \\
&\leq \int_\Omega |u|^{p(x)}\,\mathrm{d}x + \frac{t}{\alpha}|\langle S\circ Gv, Gv\rangle| \\
&\leq \int_\Omega |u|^{p(x)}\,\mathrm{d}x + \frac{t}{\alpha}\int_\Omega b(x)|u|^{p(x)}\mathrm{d}x + \frac{t}{\alpha}\int_\Omega |\lambda H(x,u,\nabla u)|u\mathrm{d}x \\
&\leq \int_\Omega |u|^{p(x)}\,\mathrm{d}x + \frac{t\|b\|_\infty}{\alpha}\int_\Omega |u|^{p(x)}\mathrm{d}x \\
&\quad + \frac{t}{\alpha}C_{p'}\int_\Omega |H(x,u,\nabla u)|^{p'(x)}\mathrm{d}x + \frac{t}{\alpha}C_p\int_\Omega |u|^{p(x)}\mathrm{d}x \\
&\leq C_1\int_\Omega |u|^{p(x)}\mathrm{d}x + C_2(\|u\|^{\theta_2}+1) \\
&\leq Cst\big(\|Gv\|^{\theta'} + \|Gv\|^{\theta_2}+1\big),
\end{aligned}$$

where

$$\theta' = \begin{cases} p^+ & \text{if } \|Gv\|\leq 1, \\ p^- & \text{if } \|Gv\|\geq 1, \end{cases} \qquad \text{and} \quad \theta_2 \text{ is defined in (13).}$$

This implies that $\big\{Gv \setminus v\in B\big\}$ is bounded.

Since the operator S is bounded, it is obvious from (16) that the set B is bounded in $\big(W^{1,p(x)}(\Omega)\big)^*$. Therefore, we can choose a positive constant R such that

$$\|v\|_{(W^{1,p(x)}(\Omega))^*} < R \quad \text{for all} \quad v\in B.$$

It follows that

$$v + tSoGv \neq 0 \quad \text{for all} \quad v\in\partial B_R(0) \quad \text{and all} \quad t\in[0,1].$$

By Lemma 2.9, we get

$$I + SoG \in \mathcal{F}_T(\overline{B_R(0)}) \quad \text{and} \quad I = ToG \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators $I,\ S$ and G are bounded, $I + SoG$ is also bounded. We conclude that

$$I + SoG \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \quad \text{and} \quad I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider an affine homotopy

$$\Lambda\colon [0,1] \times \overline{B_R(0)} \to W^{-1,p'(x)}(\Omega)$$

given by

$$\Lambda(t,v) := v + tSoGv \quad \text{for} \quad (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying Theorem 2.11, we have

$$d\big(I + SoG, B_R(0), 0\big) = d\big(I, B_R(0), 0\big) = 1,$$

and thus, there exists a point $v \in B_R(0)$ such that

$$v + SoGv = 0,$$

which implies that $u = Gv$ is a weak solution of (1). This completes the proof. $\square$

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