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AN UNCONVENTIONAL FINITE DIFFERENCE TECHNIQUE FOR THE SOLUTION OF TWO-POINT THIRD-ORDER BOUNDARY VALUE PROBLEMS IN ODES

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ABSTRACT. In this paper, we propose a numerical method to solve the boundary value problem involving third-order differential equations in ODEs in a closed--open domain, where the boundary conditions are prescribed at the closed and open end points. An application of interpolation and the smooth function method to incorporate the boundary condition at the open end point implied an unconventional finite difference method for the approximate numerical solution of the problem. Hence, the proposed method uses the boundary conditions in an exact and natural way. The method is simple in application and solves the problem directly. We have established the quadratic order of the accuracy and convergence of the proposed method analytically. The numerical results of computational experiments on model test problems, both nonlinear and linear, confirm the competency and the theoretical discussion on the order of convergence of the method. A singular problem is considered, and satisfying numerical results obtained in computational experiments is an extended application of the method.

1. Introduction

The investigation of differential equations and related boundary value problems for their analytical and approximate solutions is a significant field of study due to its relevance in various domains of science and engineering. To describe an observation or problem, we mathematically model the problem that leads to a differential equation. In particular, to study and explain the observations in the theory of beam, fluid dynamics, boundary layer flow, etc.,

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problems are modeled by third-order differential equations [1]. The domain of definition of the study of modelled problem or observation is sometimes closed, open, or closed-open. There are some important examples of third-order differential equations in the closed-open domain reported in literature [3]. The following nonlinear differential equation in the closed-open domain reported in [2] is

$$2u'''(x) + u(x)u''(x) = 0, \quad x \in [0, \infty),$$

subject to boundary conditions

$$u(0) = 0, \quad u'(0) = 0 \quad \text{and} \quad \lim_{x\to\infty} u'(x) = 1$$

and notations $u'(x)$, $u''(x)$ are the first- and second-order derivatives of the function $u(x)$ with respect to x, respectively. Boundary conditions are prescribed at one end, and the other end is open. It is a specific example of third order differential equation and boundary value problem in closed-open domain.

In this paper, we consider the following third-order differential equation and related boundary conditions

$$u'''(x) = f\big(x, u(x)\big), \quad x \in [a, \infty), \tag{1}$$

subject to the boundary conditions

$$u(a) = \alpha, \quad u'(a) = \beta \quad \text{and} \quad u'(\infty) = \gamma,$$

where α, β and γ are real constants and $f(x, u)$ is continuous.

It is always desirable to validate observations or explain the physical problems that are modelled mathematically. Hence, it is essential to find a general analytical solution to the model problem. But in practice, it is difficult to find a general analytical solution to the model Problem (1) without a specific case. However, some research works are reported in the literature on analytical solutions to Problem (1), but these works focus on the existence and uniqueness or multiplicity of the solution to Problem (1) in [4–7] in specific cases only. Some researchers [8–11] have shown interest in finding approximate numerical solutions to Problem (1) using different indirect or direct approaches, such as the transformation method, the chasing method, and the initial value problem method.

One of the indirect classical approaches in [12, 13] for the approximate numerical solution is to convert boundary value problems into a pair of initial value problems and solve the equivalent system. Another method without transforming into an equivalent system, i.e. direct method, computational approaches and techniques have been presented for the approximate numerical solution of the third-order BVPs in [14–20], and references therein. Thus, recent developments clearly indicate the involvement and interest of researchers in the numerical solution of the modelled problems.

In the present work, we assume the existence and uniqueness of the solution of Problem (1) in the closed-open interval/domain. The objective and emphasis of this present work is to develop and propose an alternative direct method for the approximate numerical solution of linear and nonlinear boundary value problems involving third-order differential equations (1).

This paper is organized as follows. In the next section, we propose a finite difference method. We discuss the derivation and convergence of the proposed method under the appropriate conditions in Section 3 and Section 4, respectively. The application of the proposed method to the model problems and the numerical results produced to show the efficiency are shown in Section 5. The discussion and conclusion about the performance of the proposed method are presented in Section 6.

2. The Difference Method

Let us consider the following smooth function $x = x(t)$ defined in [21, 22]

$$x(t) = -L\ln(1-t), \quad 0 \le t \le 1, \tag{2}$$

where $L > 0$ is the parameter and $0 \le x(t) \le \infty$. The function $x(t)$ is strictly monotonic and semi-uniform. Let us define equidistant $N{+}1$ nodes $t_i = ih$ in $[0, 1]$ with uniform step length $h = \frac{1}{N}$. Hence, we find

$$t_{n+1} = t_n + \frac{1}{N} = t_n + h.$$

These nodes enable us to generate a single parameter family of semi-uniform nodes $x_i = x(t_i)$ in the interval $[0, \infty]$. We compute the approximate solution of Problem (1) at these nodes. Let us denote u_i and f_i as the numerical approximations of $u(x)$, and $f\big(x, u(x)\big)$ at node $x = x_i$ for $i = 1, 2, \ldots, N-1$, respectively. We will follow the same definition as above in defining the other notation in the present article.

Let us define the following approximations

$$\alpha_1 = x_i - \tfrac{2}{3}h, \qquad \alpha_2 = x_i - \tfrac{1}{3}h, \qquad \alpha_3 = x_i + \tfrac{1}{3}h, \qquad \alpha_4 = x_i + \tfrac{2}{3}h,$$

$$\alpha_5 = x_i - \tfrac{5}{3}h, \qquad \alpha_6 = x_i - \tfrac{4}{3}h, \qquad \alpha_7 = x_i - \tfrac{8}{3}h, \qquad \alpha_8 = x_i - \tfrac{7}{3}h,$$

and

$$A_0 = \tfrac{\alpha_3-\alpha_2}{\alpha_4-\alpha_1}, \qquad A_1 = \tfrac{\alpha_4-\alpha_3}{\alpha_4-\alpha_1}, \qquad A_2 = \tfrac{\alpha_2-\alpha_1}{\alpha_4-\alpha_1}, \qquad A_3 = \tfrac{\alpha_2-\alpha_1}{\alpha_2-\alpha_5},$$

$$A_4 = \tfrac{\alpha_6-\alpha_5}{\alpha_2-\alpha_5}, \qquad A_5 = \tfrac{\alpha_6-\alpha_5}{\alpha_6-\alpha_7}, \qquad A_6 = \tfrac{\alpha_8-\alpha_7}{\alpha_6-\alpha_7}.$$

Let us set

$$\overline{u}_{i-\frac{1}{2}} = \frac{1}{3}\begin{cases} 2(2A_0u_{i-1} + A_1u_i + A_2u_{i+1}), & i=1, \\ 3(A_3u_{i-2} + A_4u_{i-1} + A_1u_i + A_2u_{i+1}), & 2\le i\le N-1, \\ 9A_3u_i + A_4u_{i-1} + A_5u_{i-2} + A_6u_{i-3}, & i=N, \end{cases} \tag{3}$$

and

$$\overline{f}_{i-\frac{1}{2}} = f\big(x_{i-\frac{1}{2}}, \overline{u}_{i-\frac{1}{2}}\big).$$

Let us transform the continuous Problem (1) into the following discrete problem at node $x_{i-\frac{1}{2}}$

$$u'''_{i-\frac{1}{2}} = f_{i-\frac{1}{2}} \tag{4}$$

with the boundary conditions

$$u_0 = \alpha, \quad u'_0 = \beta \qquad \text{and} \qquad u'_N = \gamma.$$

We assume the existence and uniqueness of the solution to the discrete Problem (4). Although the idea of discretization in [19] uses boundary conditions in a natural way. Following the idea of discretization of Problem (1) in [19], we discretize Problem (4) in $(0,1)$ at nodes x_i

$$\begin{aligned} -9u_{i-1}+12u_i-3u_{i+1} &= 6hu'_{i-1}+h^3\big(f_{i-1}-3\overline{f}_{i-\frac{1}{2}}\big), && i=1, \\ u_{i-2}-3u_{i-1}+3u_i-u_{i+1} &= -h^3\overline{f}_{i-\frac{1}{2}}, && 2\le i\le N-1, \\ -u_{i-3}+9u_{i-2}-27u_{i-1}+19u_i &= 12hu'_i-3h^3\overline{f}_{i-\frac{1}{2}}, && i=N. \end{aligned} \tag{5}$$

Thus, we obtained our proposed method; an algebraic system of equations in u_i, $i = 1, 2, \ldots, N$. The system of equations (5) uses $u'_N = u'(\infty)$ not $u'_N = \infty$. The algebraic system of equations is linear if the forcing function $f(x,u)$ is linear, otherwise, it is nonlinear. We considered the solution of a system of equations to be the approximate solution of the problem at the nodes x_i in the considered domain/region.

3. Development of the non-standard finite difference method

In this section, we discuss the development and estimate the local truncation error in the proposed non-standard finite difference method. Let us calculate and simplify A_0, A_1, A_2. Using $x_{i+a} = x_i + ah$ we have

$$A_0 = \frac{1}{2}, \quad A_1 = \frac{1}{4}, \quad \text{and} \quad A_2 = \frac{1}{4}. \tag{6}$$

For $i = 1$ substitute A_0, A_1, A_2 from (6) in (3), we have

$$\overline{u}_{i-\frac{1}{2}} = \frac{1}{6}(4u_{i-1} + u_i + u_{i+1}). \tag{7}$$

Thus, $\frac{1}{6}(4u_{i-1} + u_i + u_{i+1})$ provides the $O(h^2)$ approximation for $u_{i-\frac{1}{2}}$ and

$$\begin{aligned} f\left(x_{i-\frac{1}{2}}, \frac{1}{6}(4u_{i-1} + u_i + u_{i+1})\right) &= f\left(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}} + \frac{7h^2}{24}u''_{i-\frac{1}{2}}\right) \\ &= f\left(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}\right) + LF_i\,, \end{aligned} \tag{8}$$

where LF_i, the remainder terms in the Taylor series expansion of the function $f(x, u)$, are given by

$$LF_i = \frac{7h^2}{24}u''_{i-\frac{1}{2}}\left(\frac{\partial f}{\partial u}\right)_{i-\frac{1}{2}}, \quad i = 1.$$

Hence, $\overline{f}_{i-\frac{1}{2}}$ provides $O(h^2)$ for the function $f_{i-\frac{1}{2}}$.

For $i = 1$, at the discrete points x_{i-1}, x_i and x_{i+1} of the domain $[a, b]$, let us define the following linear combination of $u'''(x)$,

$u(x)$: solution of the Problem (1), and

$u'(x)$: derivative of the solution of Problem (1)

$$a_0u_{i-1} + a_1u_i + a_2u_{i+1} + hb_0u'_{i-1} + h^3c_0u'''_{i-1} - h^3u'''_{i-\frac{1}{2}} = 0. \tag{9}$$

In equation (9), the constants a_0, a_1, a_2, b_0, and c_0 have to be determined under the proper conditions. Let us write the Taylor series expansion of each term in (9) about the point $x_{i-\frac{1}{2}}$. Applying the method of undetermined coefficients, comparing the coefficients of various power of h terms, i.e., $h^p, p = 0, 1, \ldots, 4$ in the Taylor series of (9), e obtain the following system of linear equations:

$$\begin{aligned} a_0 + a_1 + a_2 &= 0, \\ -a_0 + a_1 + 3a_2 + 2b_0 &= 0, \\ a_0 + a_1 + 9a_2 - 4b_0 &= 0, \\ -a_0 + a_1 + 27a_2 + 6b_0 + 48c_0 &= 48, \\ a_0 + a_1 + 81a_2 - 8b_0 - 192c_0 &= 0. \end{aligned} \tag{10}$$

Applying Gaussian elimination: a direct method to solving the system of linear equations (10), we have

$$a_0 = 3, \quad a_1 = -4, \quad a_2 = 1, \quad b_0 = 2, \quad \text{and} \quad c_0 = \frac{1}{3}. \tag{11}$$

Substituting the value of these from (11) into (9) and simplifying the expression, we obtain

$$u'''_{i-\frac{1}{2}} = \frac{1}{3h^3}(9u_{i-1} - 12u_i + 3u_{i+1} + 6hu'_{i-1} + h^3f_{i-1} + R_i), \quad i = 1, \tag{12}$$

where R_i is the leading term that we obtained after the fifth term in the Taylor series expression of (9), and it is defined as

$$R_i = \frac{13h^5}{40} u^{(5)}_{i-\frac{1}{2}}, \quad i = 1. \tag{13}$$

Using (12) in (4) and simplifying the expression, we get

$$-9u_{i-1} + 12u_i - 3u_{i+1} - 6hu'_{i-1} = h^3\left(f_{i-1} - 3\overline{f}_{i-\frac{1}{2}}\right) + R_i, \quad i = 1. \tag{14}$$

Following a similar procedure as above for $i = 1$, we obtain the following equations in (5) as

$$u_{i-\frac{1}{2}} = \frac{1}{12}\begin{cases} 3(u_{i-2} + u_{i-1} + u_i + u_{i+1}), & 2 \le i \le N-1, \\ 9u_i + u_{i-1} + u_{i-2} + u_{i-3}, & i = N. \end{cases} \tag{15}$$

$$LF_i = \frac{h^2}{24} u''_{i-\frac{1}{2}} \left(\frac{\partial f}{\partial u}\right)_{i-\frac{1}{2}} \begin{cases} 15, & 2 \le i \le N-1, \\ 11, & i = N. \end{cases} \tag{16}$$

$$u'''_{i-\frac{1}{2}} = \frac{1}{3h^3}\begin{cases} 3(-u_{i-2} + 3u_{i-1} - 3u_i + u_{i+1}) + R_i, & 2 \le i \le N-1, \\ u_{i-3} - 9u_{i-2} + 27u_{i-1} - 19u_i + 12hu'_i + R_i, & i = N, \end{cases} \tag{17}$$

where

$$R_i = \frac{h^5}{40} u^{(5)}_{i-\frac{1}{2}} \begin{cases} 15, & 2 \le i \le N-1, \\ -9, & i = N. \end{cases} \tag{18}$$

Using (17) in (4) and simplifying the expression, we obtained

$$\begin{aligned} u_{i-2} - 3u_{i-1} + 3u_i - u_{i+1} &= -h^3\overline{f}_{i-\frac{1}{2}} + R_i, && 2 \le i \le N-1, \\ -u_{i-3} + 9u_{i-2} - 27u_{i-1} + 19u_i &= 12hu'_i - 3h^3\overline{f}_{i-\frac{1}{2}} + R_i, && i = N. \end{aligned} \tag{19}$$

The term R_i in (14) and (19) is called the remainder term. Let us truncate the remainder terms R_i in (14) and (19), hence, we obtained our proposed Method (5). For the solution of the system of equations (5), we have used the iterative method of Gauss-Seidel and Newton-Raphson for the linear and nonlinear system of equations, respectively.

To estimate the possible error in the proposed Method (5), let us define a local remainder term in the proposed Method (5). This local remainder term is known as the computational truncation error in the proposed Method (5). With the term LF_i from (8), and the term LF_i from (16), we have obtained $O(h^2)$ approximations for the forcing function $f_{i-\frac{1}{2}}$. The term R_i from (13) and (18) is of $O(h^5)$. Hence, there are remainder terms of $O(h^5)$ in the proposed Method (5).

However, we are incorporating these terms as LF_i and R_i into computations. Thus, the computational truncation error is defined as

$$T_i = \begin{cases} -3h^3 LF_i + R_i, & i = 1, \\ -h^3 LF_i + R_i, & 2 \leq N-1, \\ -3h^3 LF_i + R_i, & i = N. \end{cases} \tag{20}$$

Thus, the local truncation error in the proposed Method (5) is of order at least $O(h^5)$. Hence, the proposed Method (5) has an order of accuracy of at least $O(h^2)$.

4. Convergence Analysis

It is important to establish that the solution of the finite difference Method (5) is an approximate solution of Problem (1). Hence, we shall discuss how the solution of the proposed finite difference Method (5) converges to the approximate numerical solution of Problem (1) under the appropriate conditions. Let us consider the following linear test problem and boundary conditions, which we have obtained after applying (2) to Problem (1) and the prescribed boundary conditions:

$$u'''(x) = f(x,u), \quad 0 < t < 1 \tag{21}$$

subject to the boundary conditions

$$u(0) = \alpha, \quad u'(0) = \beta \quad \text{and} \quad u'(1) = \gamma.$$

Let $\boldsymbol{u}$ and $\boldsymbol{U}$ be an approximate and exact solution of Problem (21) and

$$\begin{aligned} \boldsymbol{u} &= [u_0, u_1, \ldots, u_N]^T, \\ \boldsymbol{U} &= [U_0, U_1, \ldots, U_N]^T. \end{aligned}$$

Let us write the proposed finite difference Method (5) for these approximate solutions in the absence of truncation error terms and the exact solution of the problem in the presence of truncation error terms in the matrix equation form

$$\boldsymbol{J}\boldsymbol{u} = h^3 \boldsymbol{f} + \text{boundary conditions}, \tag{22}$$

$$\boldsymbol{J}\boldsymbol{U} = h^3 \boldsymbol{F} + \boldsymbol{T} + \text{boundary conditions}, \tag{23}$$

where

$$\boldsymbol{f} = \left[-3f\left(x_{\frac{1}{2}}, -u_{\frac{1}{2}}\right), -f\left(x_{\frac{3}{2}}, u_{\frac{3}{2}}\right), \cdots, -f\left(x_{N-\frac{3}{2}}, u_{N-\frac{3}{2}}\right), -3f\left(x_{N-\frac{1}{2}}, u_{N-\frac{1}{2}}\right)\right]^T,$$

$$\boldsymbol{F} = \left[-3f\left(x_{\frac{1}{2}}, U_{\frac{1}{2}}\right), -f\left(x_{\frac{3}{2}}, U_{\frac{3}{2}}\right), \ \cdots, -f\left(x_{N-\frac{3}{2}}, u_{N-\frac{3}{2}}\right), -3f\left(x_{N-\frac{1}{2}}, U_{N-\frac{1}{2}}\right)\right]^T.$$

Let us linearize the forcing function $f(x,u)$, and we have

$$f\big(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}\big) - f\big(x_{i-\frac{1}{2}}, U_{i-\frac{1}{2}}\big) = \big(u_{i-\frac{1}{2}} - U_{i-\frac{1}{2}}\big)G_{i-\frac{1}{2}}, \tag{24}$$

where

$$G_{i-\frac{1}{2}} = \left(\frac{\partial f}{\partial U}\right)_{i-\frac{1}{2}}, \qquad i = 1, 2, \ldots, N.$$

Let us define the following error term in approximate and exact solution of Problem (1)

$$\epsilon_i = u_i - U_i, \qquad i = 1, 2, \ldots, N. \tag{25}$$

Using (3) and (25) in (24), we will obtain

$$f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}) - f(x_{i-\frac{1}{2}}, U_{i-\frac{1}{2}})$$
$$= \begin{cases} \frac{1}{6}(\epsilon_i + \epsilon_{i+1})G_{i-\frac{1}{2}}, & i = 1, \\ (\epsilon_{i-1} + \epsilon_i + \epsilon_{i+1})G_{i-\frac{1}{2}}, & i = 2, \\ (\epsilon_{i-2} + \epsilon_{i-1} + \epsilon_i + \epsilon_{i+1})G_{i-\frac{1}{2}}, & 3 \le i \le N-1, \\ \frac{1}{12}(\epsilon_{i-3} + \epsilon_{i-2} + \epsilon_{i-1} + 9\epsilon_i)G_{i-\frac{1}{2}}, & i = N. \end{cases} \tag{26}$$

Subtracting (23) from (22) and applying (26), we have

$$(\boldsymbol{J} + \boldsymbol{S})\boldsymbol{E} = -\boldsymbol{T}, \tag{27}$$

where

$$\boldsymbol{J} = \begin{pmatrix} 12 & -3 & & & & 0 \\ -3 & 3 & -1 & & & \\ 1 & -3 & 3 & -3 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -3 & 3 & -1 \\ 0 & & -1 & 9 & -27 & 19 \end{pmatrix}_{N\times N},$$

$$\boldsymbol{S} = \frac{h^3}{12}\begin{pmatrix} 2G_{\frac{1}{2}} & 2G_{\frac{1}{2}} & & & & 0 \\ 4G_{\frac{3}{2}} & 4G_{\frac{3}{2}} & 4G_{\frac{3}{2}} & & & \\ 4G_{\frac{5}{2}} & 4G_{\frac{5}{2}} & 4G_{\frac{5}{2}} & 4G_{\frac{5}{2}} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 4G_{N-\frac{3}{2}} & 4G_{N-\frac{3}{2}} & 4G_{N-\frac{3}{2}} & 4GN-\frac{3}{2} \\ 0 & & G_{N-\frac{1}{2}} & G_{N-\frac{1}{2}} & G_{N-\frac{1}{2}} & 9G_{N-\frac{1}{2}} \end{pmatrix}_{N\times N},$$

$$\boldsymbol{E} = [\epsilon_1, \epsilon_2, \cdots, \epsilon_N]^T,$$

and matrix $\boldsymbol{T} = (T_i)_{N\times 1}$,

$$T_i = \frac{h^5}{40}\begin{cases} 13u^{(5)}_{i-\frac{1}{2}} - 35u''_{i-\frac{1}{2}}\left(\frac{\partial f}{\partial u}\right)_{i-\frac{1}{2}}, & i = 1, \\ 15u^{(5)}_{i-\frac{1}{2}} - 25u''_{i-\frac{1}{2}}\left(\frac{\partial f}{\partial u}\right)_{i-\frac{1}{2}}, & 2 \le i \le N-1, \\ -9u^{(5)}_{i-\frac{1}{2}} - 55u''_{i-\frac{1}{2}}\left(\frac{\partial f}{\partial u}\right)_{i-\frac{1}{2}}, & i = N. \end{cases} \tag{28}$$

The matrix $\boldsymbol{J}$ is invertible [23, 24], and let us assume matrix $\boldsymbol{B} = (b_{l,m})_{N\times N}$ be the explicit inverse of matrix $\boldsymbol{J}$. Thus, the matrix $\boldsymbol{B}$ is given below,

$$b_{l,m} = \begin{cases} \frac{2(2(N-m)+1)}{24N}, & 1 = l = m, \\ \frac{6(2(N-m)+1)l^2}{24N}, & 1 \le l \le m \le N-2, \\ \frac{19l^2}{24N}, & l \le m = N-1, \\ \frac{l^2}{24N}, & l \le m = N, \\ \frac{2(2N-l)l}{24N}, & 1 = m \le l, \\ \frac{3N(N^2-(N-2m)(N-2m+2))-6(2m-1)(N-l)^2}{24}, & 2 \le m \le N-2 \le l, \\ \frac{3N(N^2-(N-2m)(N-2m+2))+N^2}{24N}, & N-1 = m \le l = N. \end{cases} \tag{29}$$

Let matrix $\boldsymbol{R} = (SR_l)_{N\times 1}$, denote the sum of rows of the matrix $\boldsymbol{B} = (b_{l,m})_{N\times N}$, i.e.,

$$SR_l = \sum_{m=1}^{N} b_{l,m}, \quad l = 1, 2, \ldots, N. \tag{30}$$

Thus,

$$SR_l = \begin{cases} \frac{2Nl(3Nl-2l^2-2)}{24N}, & l \le N-2, \\ \frac{4N(3N^2-12N+10)l-6N(N-24)l^2-4N(N^3-6N^2+11N-6)}{24N}, & l = N-1, \\ \frac{4N(3N^2-12N+10)l-(6N^2-24N+19)l^2-N^2(4N^2-24N+19)}{24N}, & l = N. \end{cases} \tag{31}$$

It is clear that SR_l is the maximum at $l = N$. Hence, we have obtained

$$\max_{1\le l\le N}(SR_l) = \frac{N(N^2-1)}{12}. \tag{32}$$

Thus, we have, for large N,

$$\|\boldsymbol{B}\| \le \frac{1}{12h^3}. \tag{33}$$

Let $G_{i-\frac{1}{2}} \geq 0, i = 1, 2, \ldots, N$ and

$$G = \max_{1\leq i\leq N} G_{i-\frac{1}{2}}, \quad M = \max_{t\in[0,1]} \left|u^{(5)}(t)\right|, \quad M_1 = \max_{t\in[0,1]} \left|u^{''}(t)\right|.$$

Let

$$M_T = \max(13M - 35M_1G,\ 15M - 25M_1G,\ -9M - 55M_1G).$$

Assume $G < 9$ and $\|\boldsymbol{S}\|\|\boldsymbol{B}\| < 1$. Hence, matrix $(\boldsymbol{I} + \boldsymbol{BS})$ is invertible [25, 26], and moreover,

$$\|(\boldsymbol{I} + \boldsymbol{BS})^{-1}\| < \frac{1}{1 - \|\boldsymbol{S}\|\|\boldsymbol{B}\|}. \tag{34}$$

From (27) and (34) we have

$$\|\boldsymbol{E}\| \leq \|\boldsymbol{B}\|(1 + \|\boldsymbol{S}\|\|\boldsymbol{B}\|)\|\boldsymbol{T}\|. \tag{35}$$

From (28), (33) and (35) we have

$$\|\boldsymbol{E}\| \leq \frac{1}{240} h^2 M_T. \tag{36}$$

From (36) it follows that $\|\boldsymbol{E}\| \to 0$ as $h \to 0$. Thus, we have established the convergence of the proposed Method (5) and the order of convergence of the proposed Method (5) is at least $O(h^2)$.

5. Numerical Results

In this section, we verify the theoretical development and computational efficiency of the proposed finite difference Method (5). We have considered both linear and nonlinear model test problems. We have presented the results of the numerical experiment in Tables. In each tabulated numerical result, we have shown MAE, the maximum absolute error, in the approximate solution $u(x)$ of Problem (1) for different values of N and uniform step size h. We have used the following formula in the computation of MAE,

$$\text{MAE} = \max_{1\leq i\leq N} |U(x_i) - u_i|.$$

All computations were performed on a Windows 2007 Home Basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330 M, 2.20 Ghz PC. The solutions are computed in N nodes, and the computation continues until the maximum difference between two successive iterations is less than 10^{-10} or the number of iterations reached 10^4.

Problem 1. The nonlinear model problem is given by

$$u'''(x) - u^2(x)u'(x) = 0,$$

subject to the boundary conditions

$$u(0) = \frac{\sqrt{6}}{A}, \quad u'(0) = -\frac{\sqrt{6}}{A^2} \quad \text{and} \quad u'(\infty) = 0.$$

The analytical solution to the problem is

$$u(x) = \frac{\sqrt{6}}{A + x}.$$

The MAE computed by Method (5) for different values of parameter A and N are presented in Table 1.

Problem 2. The linear model problem is given by

$$u'''(x) + u(x) = f(x),$$

subject to the boundary conditions

$$u(0) = 0, \quad u'(0) = 1 \quad \text{and} \quad u'(\infty) = 0.$$

The forcing function $f(x)$ is calculated so that the analytical solution of the problem is

$$u(x) = x(1 - x)\exp(-x).$$

The MAE computed by Method (5) for different values of N are presented in Table 2.

Problem 3. The singular model problem is given by

$$u'''(x) - \frac{1}{x^2 - 3x + 2}u(x) = f(x), \quad 0 < x < 1$$

subject to the boundary conditions

$$u(0) = -\sin(1.0), \quad u'(0) = \sin(1.0) + \cos(1.0) \quad \text{and} \quad u'(1) = \exp(-1.0).$$

The forcing function $f(x)$ is calculated so that the analytical solution of the problem is

$$u(x) = \sin(x - 1)\exp(-x).$$

The MAE computed by Method (5) for different values of N are presented in Table 3.

TABLE 1. The maximum absolute error and $L = 1.0$ (Problem 1).

N	MAE		
	$A = 1$	$A = 2$	$A = 3$
2^{12}	.55900437e -6	.11711094e -6	.10616356e -6
2^{13}	.12273467e -6	.10976322e -6	.99377416e -7
2^{14}	.11360801e -6	.10241065e -6	.93996804e -7
2^{15}	.10557364e -6	.95981688e -7	.88098863e -7
2^{16}	.98667595e -7	.90287780e -7	.83376868e -7
2^{17}	.92685535e -7	.85328158e -7	.79289661e -7

TABLE 2. Maximum absolute error and $L = 1.0$ (Problem 2).

	N				
	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}
MAE	.56595817e -6	.16392238e -6	.15118362e -6	.12881460e -6	.10491959e -6

TABLE 3. Maximum absolute error (Problem 3).

	N				
	2^{6}	2^{7}	2^{8}	2^{9}	2^{10}
MAE	.15394823e -4	.38219711e -6	.23096289e -7	.88124330e -9	.49016146e -10

We have considered model boundary value problems, both linear and nonlinear, involving the third-order differential equations in ODEs to test the computational efficiency of the proposed method. We observed in the numerical results obtained for the problems considered in the numerical experiment for different values of N presented in the tables that the maximum absolute error in the solution decreases with a decrease in h. Thus, the numerical results approve the theoretical development in terms of the degree of correctness of the method proposed by us in the article. So, it is evident from the tabulated results that Method (5) is accurate, efficient, and convergent.

6. Conclusion

The objective of the paper was to develop an algorithm for a good approximation to the numerical solution of the third-order differential equation and the corresponding boundary value problem. Hence, we have presented an unconventional finite difference method for the approximate numerical solution of boundary value problems in ODEs involving third-order differential equations in a semi-open domain. We faced the real issue of dealing with one boundary condition at the open end, i.e., at ∞, it was easily addressed by defining a smooth function and interpolations. In the application of transformation techniques, the proposed Method (5) uses the boundary condition at the open end ∞ exact and natural, not an approximate boundary condition. The method leads to a system of equations, and the solutions to the problems are obtained by solving a system of equations. The results of computational experiments approve the effective application and order of correctness of the method presented in the article. In the computational experiment, a singular problem was considered as an extended application of the method, and we observed satisfactory performance of the method. It is possible to improve the proposed unconventional method in terms of accuracy or idea. These directions are the subject of advanced studies.

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