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A NOTE ON $\mathcal{I}$-CLOSURE AND $\mathcal{I}$-HYPERCONNECTED SPACES

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ABSTRACT. This article is a follow-up of the work done in the papers [10] and [56], where, along with various results on $\mathcal{I}$-convergence, several properties of $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions were investigated. In this paper, we investigate some properties of $\mathcal{I}$-boundary points of a set in a topological space. We make some observations on $\mathcal{I}$-convergence in the product space. We consider the ideas of an $\mathcal{I}$-hyperconnected space and an $\mathcal{I}$-submaximal space and derive their basic properties.

1. Introduction

In [29], Kostyrko et al. proposed an interesting generalization of the notion of statistical convergence of a sequence, namely $\mathcal{I}$-convergence. In recent studies, there has been increasing interest in the topic of $\mathcal{I}$-convergence in topological spaces, especially in which $\mathcal{I}$-open sets and the mappings preserving $\mathcal{I}$-convergence are their primary concepts. A number of its features have been investigated and from this concept, analogues of a number of concepts found in General Topology, Mathematical Analysis, and Number Theory have emerged (see [6, 7, 10–13, 20, 24, 28, 31–33, 43, 49, 51, 53]). The present study was inspired by references [10] and [56], which showcased several characteristics of $\mathcal{I}$-open sets and the mappings preserving $\mathcal{I}$-convergence. In this paper, we broaden our understanding of $\mathcal{I}$-convergence in General Topology. We introduce and investigate ideal version of a boundary point, a hyperconnected space and a submaximal space.

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Recall that if any non-empty open subset of a space X is dense in X, that is, if there is non-empty intersection between any two non-empty open subsets of X, then the space X is hyperconnected. If every dense subset of X is open, then X is submaximal. Several characteristics of the submaximal spaces and hyperconnected spaces were provided in a number of articles. See [1, 2, 5, 23, 30, 34, 38–40] and [42] as a few examples. You may view the latest progress on submaximal spaces in [8] and [34].

2. $\mathcal{I}$-closure and $\mathcal{I}$-interior

Assume that $\mathcal{I} \subseteq 2^{\mathbb{N}}$. If $\mathcal{I}$ is additive (if $(A, B) \in \mathcal{I}^2$, then $A \cup B \in \mathcal{I}$), hereditary (if $B \subseteq A \in \mathcal{I}$, then $B \in \mathcal{I}$), containing all singletons and $\mathbb{N} \notin \mathcal{I}$, then it is called an admissible ideal on $\mathbb{N}$. Using Zorn's lemma we can show that in the family of all admissible ideals of $\mathbb{N}$, there exists a maximal ideal (with respect to inclusion). If $\mathcal{I}$ is a maximal ideal of $\mathbb{N}$, then for each $A \subseteq \mathbb{N}$ we have either $A \in \mathcal{I}$ or $\mathbb{N} \backslash A \in \mathcal{I}$. In the following, if not otherwise mentioned, we consider $\mathcal{I}$ to be an admissible ideal on $\mathbb{N}$, and $\tau_{\mathcal{I}}$ the collection of all $\mathcal{I}$-open subsets of a topological space. We refer to [56] for additional terms that are utilized but not included in this paper.
Example.

(a) The class of all finite subsets of $\mathbb{N}$ forms an admissible ideal.

(b) For each subset A of $\mathbb{N}$ the asymptotic density of A, denoted $\delta(A)$, is given by

$$\delta(A) = \lim_{n \to \infty} \frac{1}{n} |\{k \in A \colon k \leq n\}|;$$

the family $\mathcal{I}_\delta = \{A \subseteq \mathbb{N} \colon \delta(A) = 0\}$ is an admissible ideal.

(c) Let $u(A)$ be a uniform density of the set $A \subseteq \mathbb{N}$ (see [11]). The family $\mathcal{I}_u = \{A \subset \mathbb{N} \colon u(A) = 0\}$ is an admissible ideal.

Definition 2.1 ([29]). A sequence $(x_n)_n$ in a topological space X is said to be $\mathcal{I}$-convergent to a point $x \in X$, denoted $x_n \xrightarrow{\mathcal{I}} x$, provided for any neighborhood U of x the set $\{n \colon x_n \notin U\} \in \mathcal{I}$ and the point x is referred to as an $\mathcal{I}$-limit point of the sequence $(x_n)_n$.

In a topological space X, all convergent sequences are obviously $\mathcal{I}$-convergent, but some $\mathcal{I}$-convergent sequences are not convergent. For example, let $X = \mathbb{R}$ have the usual topology. Examine the following sequence:

$$x_n = \begin{cases} 0, & \text{if } n \text{ is a square,} \\ 1, & \text{if } n \text{ is not a square.} \end{cases}$$

Then, although it is not convergent, $x_n \xrightarrow{\mathcal{I}_\delta} 1$.

Definition 2.2. Let A be a subset of a topological space X.

(1) The $\mathcal{I}$-closure of A is the set

$$\mathcal{I}\text{-}\operatorname{Cl}(A) = \{x \in X : \exists\ (x_n)_n \subseteq A \quad \text{such that} \quad x_n \xrightarrow{\mathcal{I}} x \ \text{ in } \ X\}.$$

(2) The $\mathcal{I}$-interior of A is the set $\mathcal{I}\text{-}\operatorname{Int}(A) = A \backslash \mathcal{I}\text{-}\operatorname{Cl}(X \backslash A)$.

(3) A subset F of X is said to be $\mathcal{I}$-closed if $F = \mathcal{I}\text{-}\operatorname{Cl}(F)$. Equivalently, F is $\mathcal{I}$-closed if for each sequence $(x_n)_n$ in F with $x_n \xrightarrow{\mathcal{I}} x \in X$, then $x \in F$.

(4) $U \subseteq X$ is said to be $\mathcal{I}$-open if $X \backslash U$ is $\mathcal{I}$-closed.

Example. There exists a subset A of a topological space X for which $\mathcal{I}\text{-}\operatorname{Cl}(A)$ is not $\mathcal{I}$-closed.

Proof. Consider the Arens space $X = (\mathbb{N} \times \mathbb{N}) \cup \mathbb{N} \cup \{\infty\}$ where points in $\mathbb{N} \times \mathbb{N}$ are isolated, neighborhoods of $n \in \mathbb{N}$ are of the form

$$\{n\} \cup \big(\{n\} \times (\mathbb{N} \backslash F)\big),$$

where F is finite, and neighborhoods of ∞ are of the form

$$\{\infty\} \cup (\mathbb{N} \backslash F_0) \cup \bigcup_{n \in \mathbb{N} \backslash F_0} \{n\} \times (\mathbb{N} \backslash F_n),$$

where $F_0,\ F_1, \dots$ are finite sets. Then,

$$\text{Fin-Cl}(\mathbb{N} \times \mathbb{N}) = X \backslash \{\infty\}, \quad \text{but} \quad \text{Fin-Cl}(X \backslash \{\infty\}) = X.$$

□

Lemma 2.3. *Let $\mathcal{J} \subseteq \mathcal{I}$ be two ideals on $\mathbb{N}$ and X a topological space.*

(1) *If $U \subseteq X$ is $\mathcal{I}$-open, then it is also $\mathcal{J}$-open.*

(2) $\tau \subseteq \tau_{\mathcal{J}} \subseteq \tau_{\mathcal{I}}$.

Proof. Follows from [56, Remark 3.2 and Lemma 3.3]. □

Lemma 2.4 ([10])**.** *For a subset A of a topological space X, the following assertions hold:*

(1) *A is $\mathcal{I}$-closed if and only if $A = \bigcap\{F \subseteq X : F$ is $\mathcal{I}$-closed and $A \subseteq F\}$.*

(2) *A is $\mathcal{I}$-open if and only if $A = \bigcup\{U \subseteq X : U$ is $\mathcal{I}$-open and $U \subseteq A\}$.*

Lemma 2.5 ([56])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. For a subset A of a topological space X, the following are equivalent:*

(1) *A is $\mathcal{I}$-open;*

(2) *$\{n : x_n \in A\} \notin \mathcal{I}$ for each sequence $(x_n)_n$ in X with $x_n \xrightarrow{\mathcal{I}} x \in A$;*

(3) *$|\{n : x_n \in A\}| = \omega$ for each sequence $(x_n)_n$ in X with $x_n \xrightarrow{\mathcal{I}} x \in A$.*

Theorem 2.6. *Let X be a topological space and $A,\ B \in \mathcal{P}(X)$. The following assertions are valid:*

(1) $\mathcal{I}\text{-}\mathrm{Cl}(X) = X$ *and* $\mathcal{I}\text{-}\mathrm{Cl}(\emptyset) = \emptyset$*;*

(2) $A \subseteq \mathcal{I}\text{-}\mathrm{Cl}(A)$*;*

(3) *If* $A \subseteq B$*, then* $\mathcal{I}\text{-}\mathrm{Cl}(A) \subseteq \mathcal{I}\text{-}\mathrm{Cl}(B)$*;*

(4) *If a point* $x \in \mathcal{I}\text{-}\mathrm{Cl}(A)$*, then* $A \cap U \neq \emptyset$ *for each* $\mathcal{I}$*-open subset* U *of* X *containing* x *but the converse is not true. It can be seen by the aforementioned example of Arens space;*

(5) $\mathcal{I}\text{-}\mathrm{Cl}(A) \cup \mathcal{I}\text{-}\mathrm{Cl}(B) = \mathcal{I}\text{-}\mathrm{Cl}(A \cup B)$ *provided* $\mathcal{I}$ *is a maximal ideal on* $\mathbb{N}$*;*

(6) $\mathcal{I}\text{-}\mathrm{Cl}(A \cap B) \subseteq \mathcal{I}\text{-}\mathrm{Cl}(A) \cap \mathcal{I}\text{-}\mathrm{Cl}(B)$*.*

P r o o f. We will prove items (4) and (5). The rest items are obvious.

(4) Assume that $x \in \mathcal{I}\text{-}\mathrm{Cl}(A)$. Let $(x_n)_n$ be a sequence in A with $x_n \xrightarrow{\mathcal{I}} x$ in X. By Lemma 2.5 for any $\mathcal{I}$-open subset U of X containing x, $|\{n\colon x_n \in U\}| = \omega$. Therefore, $A \cap U \neq \emptyset$.

(5) For, see [57, Proposition 3.9]. $\square$

Theorem 2.7. *Let A and B be any two subsets of a topological space X. The following assertions are valid:*

(1) $\mathcal{I}\text{-}\mathrm{Int}(X) = X$ *and* $\mathcal{I}\text{-}\mathrm{Int}(\emptyset) = \emptyset$*;*

(2) $\mathcal{I}\text{-}\mathrm{Int}(A) \subseteq A$*;*

(3) *If* $A \subseteq B$*, then* $\mathcal{I}\text{-}\mathrm{Int}(A) \subseteq \mathcal{I}\text{-}\mathrm{Int}(B)$*;*

(4) *A point* $x \in \mathcal{I}\text{-}\mathrm{Int}(A)$ *if and only if there exists an* $\mathcal{I}$*-open subset* U *of* X *containing* x *such that* $U \subseteq A$*;*

(5) $\mathcal{I}\text{-}\mathrm{Int}\big(\mathcal{I}\text{-}\mathrm{Int}(A)\big) = \mathcal{I}\text{-}\mathrm{Int}(A)$*;*

(6) $\mathcal{I}\text{-}\mathrm{Int}(A) \cup \mathcal{I}\text{-}\mathrm{Int}(B) \subseteq \mathcal{I}\text{-}\mathrm{Int}(A \cup B)$*;*

(7) $\mathcal{I}\text{-}\mathrm{Int}(A \cap B) = \mathcal{I}\text{-}\mathrm{Int}(A) \cap \mathcal{I}\text{-}\mathrm{Int}(B)$ *provided* $\mathcal{I}$ *is a maximal ideal on* $\mathbb{N}$*.*

Theorem 2.8. *Let $\mathcal{I} \subseteq \mathcal{J}$ be two ideals of $\mathbb{N}$. For any subset A of a topological space X, the following are valid:*

(1) $\mathcal{I}\text{-}\mathrm{Cl}(A) \subseteq \mathcal{J}\text{-}\mathrm{Cl}(A)$*;*

(2) $\mathcal{J}\text{-}\mathrm{Int}(A) \subseteq \mathcal{I}\text{-}\mathrm{Int}(A)$*.*

P r o o f.

(1) Let $(x_n)_n$ be a sequence in A with $x_n \xrightarrow{\mathcal{I}} x$ in X. Then for any neighborhood U of x, the set $\{n\colon x_n \notin U\} \in \mathcal{I}$. Since $\mathcal{I} \subseteq \mathcal{J}$, so $\{n\colon x_n \notin U\} \in \mathcal{J}$. This implies that $x \in \mathcal{J}\text{-}\mathrm{Cl}(A)$; $\mathcal{I}\text{-}\mathrm{Cl}(A) \subseteq \mathcal{J}\text{-}\mathrm{Cl}(A)$.

(2) Let $x \in \mathcal{J}\text{-}\operatorname{Int}(A)$. Then by applying Lemma 2.5, $\{n\colon x_n \in \mathcal{J}\text{-}\operatorname{Int}(A)\} \notin \mathcal{J}$ for each sequence $(x_n)_n$ in X with $x_n \xrightarrow{\mathcal{J}} x$ in X. This implies that $\{n\colon x_n \in \mathcal{J}\text{-}\operatorname{Int}(A)\} \notin \mathcal{I}$ for each sequence $(x_n)_n$ in X with $x_n \xrightarrow{\mathcal{I}} x$ in X. Therefore, $x \in \mathcal{I}\text{-}\operatorname{Int}(A)$, i.e., $\mathcal{J}\text{-}\operatorname{Int}(A) \subseteq \mathcal{I}\text{-}\operatorname{Int}(A)$. □

Corollary 2.9. *Let $\{\mathcal{I}_\alpha\colon \alpha \in \Lambda\}$ be a collection of ideals of $\mathbb{N}$ such that $\bigcup_{\alpha\in\Lambda} \mathcal{I}_\alpha$ is also an ideal on $\mathbb{N}$. For a subset A of a topological space X, the following are valid:*

(1) $(\bigcap_{\alpha\in\Lambda} \mathcal{I}_\alpha)\text{-}\operatorname{Cl}(A) \subseteq \bigcap_{\alpha\in\Lambda}\big(\mathcal{I}_\alpha\text{-}\operatorname{Cl}(A)\big)$;

(2) $\bigcup_{\alpha\in\Lambda}\big(\mathcal{I}_\alpha\text{-}\operatorname{Cl}(A)\big) \subseteq (\bigcup_{\alpha\in\Lambda} \mathcal{I}_\alpha)\text{-}\operatorname{Cl}(A)$;

(3) $\bigcap_{\alpha\in\Lambda}\big(\mathcal{I}_\alpha\text{-}\operatorname{Int}(A)\big) \subseteq (\bigcap_{\alpha\in\Lambda} \mathcal{I}_\alpha)\text{-}\operatorname{Int}(A)$;

(4) $(\bigcup_{\alpha\in\Lambda} \mathcal{I}_\alpha)\text{-}\operatorname{Int}(A) \subseteq \bigcup_{\alpha\in\Lambda}\big(\mathcal{I}_\alpha\text{-}\operatorname{Int}(A)\big)$.

Theorem 2.10. *Let $\mathcal{I}$, $\mathcal{J}$ be two ideals of $\mathbb{N}$ and $(x_n)_n$ a sequence in a topological space X. Then:*

(1) *$x_n \xrightarrow{\mathcal{I}\cup\mathcal{J}} x$ in X if and only if either $x_n \xrightarrow{\mathcal{I}} x$ in X or $x_n \xrightarrow{\mathcal{J}} x$ in X provided $\mathcal{I}\cup\mathcal{J}$ is an ideal on $\mathbb{N}$;*

(2) *$x_n \xrightarrow{\mathcal{I}\cap\mathcal{J}} x$ in X if and only if $x_n \xrightarrow{\mathcal{I}} x$ in X and $x_n \xrightarrow{\mathcal{J}} x$ in X.*

Proof.
(1) Suppose that $x_n \xrightarrow{\mathcal{I}\cup\mathcal{J}} x$ in X. If possible, assume that neither $(x_n)_n$ is $\mathcal{I}$-convergent to the point x nor it is $\mathcal{J}$-convergent to the point x in X. Let U and V be two neighborhoods of x such that neither $\{n\colon x_n \notin U\} \in \mathcal{I}$ nor $\{m\colon x_m \notin V\} \in \mathcal{J}$. Then, there exists a neighborhood W of x such that $\{n\colon x_n \notin W\} \notin \mathcal{I}\cup\mathcal{J}$, i.e., $(x_n)_n$ is not $\mathcal{I}\cup\mathcal{J}$-convergent to the point x, which is a contradiction. Therefore, either $x_n \xrightarrow{\mathcal{I}} x$ in X or $x_n \xrightarrow{\mathcal{J}} x$ in X. The converse part is obvious.

(2) The direct part is obvious. Conversely, suppose that $x_n \xrightarrow{\mathcal{I}} x$ and $x_n \xrightarrow{\mathcal{J}} x$ in X. Let U be a neighborhood of x. Then, $\{n\colon x_n \notin U\} \in \mathcal{I}$ and $\{n\colon x_n \notin U\} \in \mathcal{J}$, and hence, $\{n\colon x_n \notin U\} \in \mathcal{I}\cap\mathcal{J}$ showing that $x_n \xrightarrow{\mathcal{I}\cap\mathcal{J}} x$ in X. □

Theorem 2.11. *Let $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_n$ be ideals of $\mathbb{N}$ and $(x_n)_n$ a sequence in a topological space X. Then,*

(1) *$x_n \xrightarrow{\bigcup_{i=1}^n \mathcal{I}_i} x$ in X if and only if for some $i \in \mathbb{N}$ with $1 \le i \le n$, $x_n \xrightarrow{\mathcal{I}_i} x$ in X provided $\bigcup_{i=1}^n \mathcal{I}_i$ is an ideal on $\mathbb{N}$;*

(2) *$x_n \xrightarrow{\bigcap_{i=1}^n \mathcal{I}_i} x$ in X if and only if $x_n \xrightarrow{\mathcal{I}_i} x$ in X for each $i \in \mathbb{N}$ with $1 \le i \le n$.*

Theorem 2.12. *Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n$ be ideals of $\mathbb{N}$. For a subset A of a topological space X, the following hold:*

(1) $(\bigcap_{i=1}^n \mathcal{I}_i)\text{-}\mathrm{Cl}(A) = \bigcap_{i=1}^n (\mathcal{I}_i\text{-}\mathrm{Cl}(A))$;

(2) $\bigcup_{i=1}^n (\mathcal{I}_i\text{-}\mathrm{Cl}(A)) = (\bigcup_{i=1}^n \mathcal{I}_i)\text{-}\mathrm{Cl}(A)$ *provided* $\bigcup_{i=1}^n \mathcal{I}_i$ *is an ideal on* $\mathbb{N}$;

(3) $\bigcap_{i=1}^n (\mathcal{I}_i\text{-}\mathrm{Int}(A)) = (\bigcap_{i=1}^n \mathcal{I}_i)\text{-}\mathrm{Int}(A)$;

(4) $(\bigcup_{i=1}^n \mathcal{I}_i)\text{-}\mathrm{Int}(A) = \bigcup_{i=1}^n (\mathcal{I}_i\text{-}\mathrm{Int}(A))$ *provided* $\bigcup_{i=1}^n \mathcal{I}_i$ *is an ideal on* $\mathbb{N}$.

Proof.
(1) The inclusion $(\bigcap_{i=1}^n \mathcal{I}_i)\text{-}\mathrm{Cl}(A) \subseteq \bigcap_{i=1}^n (\mathcal{I}_i\text{-}\mathrm{Cl}(A))$ follows from Theorem 2.8. Now, let x be an element of $\bigcap_{i=1}^n (\mathcal{I}_i\text{-}\mathrm{Cl}(A))$. Then, $x \in \mathcal{I}_i\text{-}\mathrm{Cl}(A)$ for each $i \in \{1, 2, \ldots, n\}$. Let $(x_n)_n$ be a sequence in A with $x_n \xrightarrow{\mathcal{I}_i} x$ in X. Then for any neighborhood U of x, $\{n\colon x_n \notin U\} \in \mathcal{I}_i$ for each $i \in \{1, 2, \ldots, n\}$. This means that for any neighborhood U of x, $\{n\colon x_n \notin U\} \in \bigcap_{i=1}^n \mathcal{I}_i$. Therefore,

$$x_n \xrightarrow{\bigcap_{i=1}^n \mathcal{I}_i} x, \quad \text{i.e.,} \quad x \in \left(\bigcap_{i=1}^n \mathcal{I}_i\right)\text{-}\mathrm{Cl}(A).$$

Hence, the assertion follows.
(2) The inclusion $\bigcup_{i=1}^n (\mathcal{I}_i\text{-}\mathrm{Cl}(A)) \subseteq (\bigcup_{i=1}^n \mathcal{I}_i)\text{-}\mathrm{Cl}(A)$ is clear. Let x be an element of $(\bigcup_{i=1}^n \mathcal{I}_i)\text{-}\mathrm{Cl}(A)$ and $(x_n)_n$ a sequence in A with $x_n \xrightarrow{\bigcup_{i=1}^n \mathcal{I}_i} x$ in X. By Theorem 2.11, there is some $k \in \mathbb{N}$ with $1 \le k \le n$ such that $x_n \xrightarrow{\mathcal{I}_k} x$ in X. So, $x \in \mathcal{I}_k\text{-}\mathrm{Cl}(A)$ and thereby the assertion follows.

Similarly, (3) and (4) can be proven. $\square$

3. $\mathcal{I}$-boundary points

Definition 3.1. Let A be a subset of a topological space X. A point x of X is said to be $\mathcal{I}$-boundary point of A if $x \in \mathcal{I}\text{-}\mathrm{Cl}(A) \cap \mathcal{I}\text{-}\mathrm{Cl}(X \backslash A)$.

Given a subset A of a topological space X, denote by $\mathcal{I}\text{-}\mathrm{br}(A)$ the set of all $\mathcal{I}$-boundary points of A.

Example. Let $X = \mathbb{R}$ be the set of reals and τ the co-countable topology on $\mathbb{R}$. Then $\mathcal{I}\text{-}\mathrm{br}(\mathbb{Q}) = \emptyset$, but $\mathrm{br}(\mathbb{Q}) \neq \emptyset$, where $\mathrm{br}(A)$ denotes the set of boundary points of A.

Example. There exists a topological space X and $A \subseteq X$ for which $\mathcal{I}\text{-}\mathrm{br}(A)$ is not $\mathcal{I}$-closed.

Proof. For Arens space, the set of Fin-boundary points of the set $\mathbb{N} \times \mathbb{N}$ is $\mathbb{N}$ which is not a Fin-closed. $\square$

Theorem 3.2. *For a subset A of a topological space X, the following assertions are valid:*

(1) $\mathcal{I}\text{-}\operatorname{Cl}(A) = A \cup \mathcal{I}\text{-br}(A)$;

(2) $\mathcal{I}\text{-}\operatorname{Int}(A) = A \backslash \mathcal{I}\text{-br}(A)$;

(3) $\mathcal{I}\text{-br}(A) = \mathcal{I}\text{-}\operatorname{Cl}(A) \cap \mathcal{I}\text{-}\operatorname{Cl}(X \backslash A)$;

(4) $\mathcal{I}\text{-br}(X \backslash A) = \mathcal{I}\text{-br}(A)$;

(5) $\mathcal{I}\text{-br}(A) = \mathcal{I}\text{-}\operatorname{Cl}(A) \backslash \mathcal{I}\text{-}\operatorname{Int}(A)$;

(6) *if $x \in \mathcal{I}\text{-br}(A)$, then $x \in \operatorname{br}(A)$;*

(7) *A is $\mathcal{I}$-open if and only if $A \cap \mathcal{I}\text{-br}(A) = \emptyset$;*

(8) *A is $\mathcal{I}$-closed if and only if $\mathcal{I}\text{-br}(A) \subseteq A$;*

(9) *A is $\mathcal{I}$-open and $\mathcal{I}$-closed if and only if $\mathcal{I}\text{-br}(A) = \emptyset$.*

Proof.
(1) By definition, $\mathcal{I}\text{-br}(A) \subseteq \mathcal{I}\text{-}\operatorname{Cl}(A)$. Therefore, $A \cup \mathcal{I}\text{-br}(A) \subseteq \mathcal{I}\text{-}\operatorname{Cl}(A)$. For the reverse inclusion, let $x \in \mathcal{I}\text{-}\operatorname{Cl}(A)$. If $x \in A$, then there is nothing to prove. Next, let $x \in X \backslash A$. Then, $x \in \mathcal{I}\text{-}\operatorname{Cl}(X \backslash A)$ and as a result of this, $x \in \mathcal{I}\text{-}\operatorname{Cl}(A) \cap \mathcal{I}\text{-}\operatorname{Cl}(X \backslash A)$. Therefore, $x \in \mathcal{I}\text{-br}(A)$. This implies that $\mathcal{I}\text{-}\operatorname{Cl}(A) \subseteq A \cup \mathcal{I}\text{-br}(A)$. Hence the assertion follows:

(2) Straightforward.

(3) It is obvious by Definition 3.1.

(4) Follows from assertion (3).

(5) First, we show that $\mathcal{I}\text{-}\operatorname{Cl}(A) \backslash \mathcal{I}\text{-}\operatorname{Int}(A) \subseteq \mathcal{I}\text{-br}(A)$. For, assume that $x \in \mathcal{I}\text{-}\operatorname{Cl}(A) \backslash \mathcal{I}\text{-}\operatorname{Int}(A)$. Then, $x \in \mathcal{I}\text{-}\operatorname{Cl}(A)$ but $x \notin \mathcal{I}\text{-}\operatorname{Int}(A)$. Since $x \in \mathcal{I}\text{-}\operatorname{Cl}(A)$, it must be in either A or $\mathcal{I}\text{-br}(A)$. If $x \in \mathcal{I}\text{-br}(A)$, then there is nothing to prove. Suppose that $x \in A$. Since $x \notin \mathcal{I}\text{-}\operatorname{Int}(A)$, by definition, $x \in \mathcal{I}\text{-}\operatorname{Cl}(X \backslash A)$. This gives that $x \in \mathcal{I}\text{-}\operatorname{Cl}(A) \cap \mathcal{I}\text{-}\operatorname{Cl}(X \backslash A)$. Thus, $x \in \mathcal{I}\text{-br}(A)$. This proves that $\mathcal{I}\text{-}\operatorname{Cl}(A) \backslash \mathcal{I}\text{-}\operatorname{Int}(A) \subseteq \mathcal{I}\text{-br}(A)$. For the reverse inclusion, let $x \in \mathcal{I}\text{-br}(A)$. Then, $x \in \mathcal{I}\text{-}\operatorname{Cl}(A) \cap \mathcal{I}\text{-}\operatorname{Cl}(X \backslash A)$, which implies that $x \in \mathcal{I}\text{-}\operatorname{Cl}(A)$ but $x \notin \mathcal{I}\text{-}\operatorname{Int}(A)$. This completes the assertion.

(6) Obvious.

(7) First, suppose that A is $\mathcal{I}$-open. Then, $A = \mathcal{I}\text{-}\operatorname{Int}(A)$. By assertion (2), it follows that $A \cap \mathcal{I}\text{-br}(A) = \emptyset$. Conversely, suppose the condition holds. If $A = \emptyset$, then there is nothing to prove. Next, let $x \in A$. By the given condition, $x \notin \mathcal{I}\text{-br}(A)$, $\Rightarrow$ $x \in \mathcal{I}\text{-}\operatorname{Int}(A)$ by assertion (2).

(8) Follows from assertion (1).

(9) Follows from assertions (1) and (2).

$\square$

Theorem 3.3. *Let X, Y be two topological spaces. Then, a sequence*

$$(x_n,\ y_n) \xrightarrow{\mathcal{I}} (x,\ y)\ \text{in} X \times Y \quad \text{(w.r.t. the product topology)}$$

if and only if $x_n \xrightarrow{\mathcal{I}} x$ in X and $y_n \xrightarrow{\mathcal{I}} y$ in Y.

P r o o f. Follows from [58, Lemma 2.13]. □

In the following, "product space" refers to the space having product topology unless otherwise stated. Now, we have the following result.

Theorem 3.4. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces. Let $X = \Pi_{\alpha\in\Lambda} X_\alpha$ be the product space and $(x_n)_n$ a sequence in X. Then, $x_n \xrightarrow{\mathcal{I}} x$ in X if and only if $\pi_\alpha(x_n) \xrightarrow{\mathcal{I}} \pi_\alpha(x)$ for each projection map $\pi_\alpha : X \to X_\alpha$.*

Theorem 3.5. *Let $\{X_\alpha\}_{\alpha\in I}$ be a family of topological spaces.*

(1) *If $\prod_{\alpha\in I} X_\alpha$ is endowed with the product topology, then $\mathcal{I}\text{-}\mathrm{Cl}(\prod_{\alpha\in I} A_\alpha) = \prod_{\alpha\in I} \mathcal{I}\text{-}\mathrm{Cl}(A_\alpha)$, where $A_\alpha \subseteq X_\alpha$.*
(2) *If $\prod_{\alpha\in I} X_\alpha$ is endowed with the box topology, then $\mathcal{I}\text{-}\mathrm{Cl}(\prod_{\alpha\in I} A_\alpha) \subseteq \prod_{\alpha\in I} \mathcal{I}\text{-}\mathrm{Cl}(A_\alpha)$, where $A_\alpha \subseteq X_\alpha$.*

Corollary 3.6. *The set $\prod_{\alpha\in I} A_\alpha$, where $\emptyset \neq A_\alpha \subset X_\alpha$, is $\mathcal{I}$-closed in the product space $\prod_{\alpha\in I} X_\alpha$ if and only if A_α is $\mathcal{I}$-closed in X_α for every $\alpha \in I$.*

Theorem 3.7. *For a subset A of a topological space X, a point $x \in \mathcal{I}\text{-}\mathrm{br}(A)$ if and only if there is a sequence $(x_n,\ y_n)_n$ in $A \times (X\backslash A)$ with $(x_n,\ y_n) \xrightarrow{\mathcal{I}} (x,\ x)$ in X^2 (w.r.t. the product topology).*

P r o o f. Assume that $x \in \mathcal{I}\text{-}\mathrm{br}(A)$. Let $U \times V$ be an open neighborhood of $(x,\ x)$ in X^2. By definition, $x \in \mathcal{I}\text{-}\mathrm{Cl}(A) \cap \mathcal{I}\text{-}\mathrm{Cl}(X\backslash A)$. Therefore, there exist a sequence $(x_n)_n$ in A with $x_n \xrightarrow{\mathcal{I}} x$ in X and a sequence $(y_n)_n$ in $X\backslash A$ with $y_n \xrightarrow{\mathcal{I}} x$ in X. Consequently, $\{n : x_n \notin U\} \in \mathcal{I}$ and $\{n : y_n \notin V\} \in \mathcal{I}$. However, since $\{n : x_n \notin U\} \cup \{n : y_n \notin V\} = \{n : (x_n,\ y_n) \notin U \times V\}$, we have $\{n : (x_n,\ y_n) \notin U \times V\} \in \mathcal{I}$, hence $(x_n,\ y_n) \xrightarrow{\mathcal{I}} (x,\ x)$ in X^2. The converse part is obvious. □

Lemma 3.8. *Let $\mathcal{I} \subseteq \mathcal{J}$ be two ideals of $\mathbb{N}$. Then:*

(1) *If a sequence $(x_n)_n$ in a topological space X $\mathcal{I}$-converges to a point x of X, then $x_n \xrightarrow{\mathcal{J}} x$ in X;*
(2) *If A is a subset of X, then $\mathcal{I}\text{-}\mathrm{br}(A) \subseteq \mathcal{J}\text{-}\mathrm{br}(A)$.*

P r o o f.
(1) Let U be a neighborhood of x in X. Then, $\{n : x_n \notin U\} \in \mathcal{I}$ but $\mathcal{I} \subseteq \mathcal{J}$, therefore $\{n : x_n \notin U\} \in \mathcal{J}$. This implies that $x_n \xrightarrow{\mathcal{J}} x$ in X.

(2) Suppose that $x \in \mathcal{I}$-br(A). By Theorem 3.7, there is a sequence $(x_n,\ y_n)_n$ in $A \times (X \backslash A)$ with $(x_n,\ y_n) \xrightarrow{\mathcal{I}} (x,\ x)$ in X^2. Then $(x_n,\ y_n) \xrightarrow{\mathcal{J}} (x,\ x)$ in X^2; hence $x \in \mathcal{J}$-br(A). □

Theorem 3.9. *Let $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \ldots, \mathcal{I}_k$ be ideals of $\mathbb{N}$ such that $\bigcup_{i=1}^{k} \mathcal{I}_i$ is an ideal on $\mathbb{N}$. For a subset A of a topological space X, a point $x \in (\bigcup_{i=1}^{k} \mathcal{I}_i)$-br$(A)$ if and only if $x \in \mathcal{I}_i$-br(A) for some $i \in \{1, 2, 3, \ldots, k\}$.*

Proof. Since $\mathcal{I}_i \subseteq \bigcup_{i=1}^{k} \mathcal{I}_i$, we have $\mathcal{I}_i$-br$(A) \subseteq (\bigcup_{i=1}^{k} \mathcal{I}_i)$-br$(A)$; hence $\bigcup_{i=1}^{k} \mathcal{I}_i$-br$(A) \subseteq (\bigcup_{i=1}^{k} \mathcal{I}_i)$-br$(A)$. Conversely, suppose that $x \in (\bigcup_{i=1}^{k} \mathcal{I}_i)$-br$(A)$. By Theorem 3.7, there exists a sequence

$$(x_n,\ y_n)_n \quad \text{in } A \times (X \backslash A) \quad \text{with} \quad (x_n,\ y_n) \xrightarrow{\bigcup_{i=1}^{k} \mathcal{I}_i} (x,\ x) \quad \text{in } X^2.$$

By Theorem 2.11 , we have

$$(x_n,\ y_n) \xrightarrow{\mathcal{I}_i} (x,\ x) \quad \text{in } X^2 \quad \text{for some} \quad 1 \leq i \leq k.$$

But then, $x \in \mathcal{I}_i$-br(A). It ends the proof. □

4. $\mathcal{I}$-hyperconnected spaces and $\mathcal{I}$-submaximal spaces

Definition 4.1. A subset A of a topological space X is said to be $\mathcal{I}$-dense in X if for every $x \in X$, there exists a sequence $(x_n)_n$ in A such that $x_n \xrightarrow{\mathcal{I}} x$ in X.

Theorem 4.2. *Let X be a topological space. A subset $A \subseteq X$ is $\mathcal{I}$-dense in X if and only if $\mathcal{I}$-$\mathrm{Cl}(A) = X$.*

Definition 4.3. A topological space X is said to be

(1) $\mathcal{I}$-hyperconnected if every non-empty open subset of X is $\mathcal{I}$-dense in X.

(2) $\mathcal{I}$-submaximal if every dense subset of X is $\mathcal{I}$-open in X.

Example.

(a) Let $X = \mathbb{R}$, the set of reals, have the co-countable topology. Then $\mathbb{P}$, the set of irrationals, is dense in $\mathbb{R}$, but it is not $\mathcal{I}$-dense in $\mathbb{R}$. Assume that r is a rational number and $(x_n)_n$ is a sequence of irrational numbers with $x_n \xrightarrow{\mathcal{I}} r$ in X. Let $U = \mathbb{R} \backslash \{x_n \colon n \in \mathbb{N}\}$. Then, U is open in $\mathbb{R}$. Since $\mathcal{I}$ is admissible, we have $\{n \colon x_n \notin U\} \notin \mathcal{I}$, which is a contradiction. Thus $r \notin \mathcal{I}$-$\mathrm{Cl}(\mathbb{P})$, i.e., $\mathcal{I}$-$\mathrm{Cl}(\mathbb{P}) = \mathbb{P}$. In fact, all subsets of X are $\mathcal{I}$-open as well as $\mathcal{I}$-closed. X with this topology is a hyperconnected space and an $\mathcal{I}$-submaximal space, but it is neither an $\mathcal{I}$--hyperconnected space nor a submaximal space.

(b) Consider $\mathbb{Z}$ the set of integers with the Khalimsky topology, i.e., the topology on $\mathbb{Z}$ generated by the family of sets of the form $\{2n-1,\ 2n,\ 2n+1\}$, where $n \in \mathbb{Z}$. More precisely, the topology is defined by:

- Every odd integer n forms a basic open set $\{n\}$.
- Every even integer n belongs to the basic open set $\{n-1,\ n,\ n+1\}$.

Observe that a subset $D \subseteq \mathbb{Z}$ is dense in the Khalimsky topology if and only if it contains all odd integers and it intersects every even-integer cluster $\{n-1,\ n,\ n+1\}$. It is now easy to see that every dense subset of this space is open, and thereby it is $\mathcal{I}$-open. Therefore, this space is $\mathcal{I}$-submaximal. However, it is not an $\mathcal{I}$-hyperconnected space.

(c) Consider the set $\mathbb{R}$ of reals with the topology $\tau = \{\emptyset,\ \mathbb{P},\ \mathbb{R}\}$ where $\mathbb{P}$ denotes the set of irrationals. Let r be a rational number and $(x_n)_n$ be any sequence of irrationals. Then, $\{n\colon x_n \notin \mathbb{R}\} = \emptyset \in \mathcal{I}$. This means that $\mathcal{I}\text{-}\operatorname{Cl}(\mathbb{P}) = \mathbb{R}$. Hence, $(\mathbb{R},\ \tau)$ is an $\mathcal{I}$-hyperconnected space. However, it is not an $\mathcal{I}$-submaximal space because $\mathbb{P} \cup \{r\}$, where r is a rational, is dense in this space, which is not $\mathcal{I}$-open.

Theorem 4.4. *Let A be a subset of a topological space X. If for every $x \in X$ there exists a sequence $(x_n)_n$ in A with $x_n \xrightarrow{\mathcal{I}} x$ in X, then $A \cap U \neq \emptyset$ for every non-empty $\mathcal{I}$-open subset U of X.*

Proof. Let x be an element of X. By the given condition, it follows that $x \in \mathcal{I}\text{-}\operatorname{Cl}(A)$; $\mathcal{I}\text{-}\operatorname{Cl}(A) = X$. Let U be a non-empty $\mathcal{I}$-open set in X. Suppose for the sake of contradiction that $A \cap U = \emptyset$. Then, $A \subseteq X \backslash U$. But then, as U is $\mathcal{I}$-open, $X \backslash U$ is $\mathcal{I}$-closed. Therefore, $\mathcal{I}\text{-}\operatorname{Cl}(A) \subseteq X \backslash U$, which is a contradiction. Thus, $A \cap U \neq \emptyset$. $\square$

Arens space demonstrates that the converse of Theorem 4.4 is not true.

Definition 4.5. Let X be a topological space. Denote by $\mathcal{G}$ the class of dense sets in X and by $\mathcal{I}$-$\mathcal{G}$ the class of $\mathcal{I}$-dense sets in X.

(1) The smallest cardinal number of the form $|A|$ with $A \in \mathcal{G}$ is called density of the space X; this cardinal number is denoted by $d(X)$.

(2) The smallest cardinal number of the form $|A|$ with $A \in \mathcal{I}\text{-}\mathcal{G}$ is called $\mathcal{I}$-density of the space X; this cardinal number is denoted by $\mathcal{I}$-$d(X)$.

Definition 4.6. Let $X,\ Y$ be topological spaces and $f\colon X \to Y$ a mapping.

(1) f is called preserving $\mathcal{I}$-convergence if for each sequence $(x_n)_n$ in X with $x_n \xrightarrow{\mathcal{I}} x$ in X, the sequence $\big(f(x_n)\big)_n$ $\mathcal{I}$-converges to $f(x)$ in Y.

(2) f is called $\mathcal{I}$-continuous if to every $\mathcal{I}$-open subset U of Y, the set $f^{-1}(U)$ is $\mathcal{I}$-open in X.

Theorem 4.7. *For every topological space X, the following statements are valid:*

(1) *every $\mathcal{I}$-dense subset of X is dense in X;*

(2) *$d(X) \leq \mathcal{I}$-$d(X)$;*

(3) *if $\mathcal{I}$ is a maximal ideal of $\mathbb{N}$ and there is a surjective $\mathcal{I}$-continuous function f from X to a topological space Y, then $\mathcal{I}$-$d(Y) \leq \mathcal{I}$-$d(X)$.*

Proof.
(1) Let A be an $\mathcal{I}$-dense subset of X. Then for every $x \in X$, there is a sequence $(x_n)_n$ in A with $x_n \xrightarrow{\mathcal{I}} x$ in X. By Theorem 4.4, $A \cap U \neq \emptyset$ for every non-empty $\mathcal{I}$-open set U in X. In particular, $A \cap U \neq \emptyset$ for every non-empty open set U in X. So, $\mathrm{Cl}(A) = X$; A is dense in X.
(2) Let $d(X) = p$ and $\mathcal{I}$-$d(X) = q$. Let A be an $\mathcal{I}$-dense subset of X with $|A| = q$. Since every $\mathcal{I}$-dense set is dense, A is dense in X. Consequently, $|A| \geq d(X)$; hence $d(X) \leq \mathcal{I}$-$d(X)$.
(3) To prove this, it suffices to show that for every $\mathcal{I}$-dense set A in X, $f(A)$ is $\mathcal{I}$-dense in Y. For, let A be an $\mathcal{I}$-dense subset of X. Then, $\mathcal{I}\text{-}\mathrm{Cl}(A) = X$. By Theorem 4.11, $f\big(\mathcal{I}\text{-}\mathrm{Cl}(A)\big) \subseteq \mathcal{I}\text{-}\mathrm{Cl}\big(f(A)\big)$; that is, $f(X) \subseteq \mathcal{I}\text{-}\mathrm{Cl}\big(f(A)\big)$. Consequently, $f(A)$ is $\mathcal{I}$-dense in Y. □

Theorem 4.8. *Let $\mathcal{I} \subseteq \mathcal{J}$ be two ideals of $\mathbb{N}$. If A is $\mathcal{I}$-dense in a topological space X, then it is also $\mathcal{J}$-dense in X.*

Proof. Follows from Theorem 2.8. □

Theorem 4.9. *The set $\prod_{\alpha \in I} A_\alpha$, where $\emptyset \neq A_\alpha \subset X$, is $\mathcal{I}$-dense in the product space $\prod_{\alpha \in I} X_\alpha$ if and only if A_α is $\mathcal{I}$-dense in X_α for every $\alpha \in I$.*

Proof. A simple consequence of Theorem 3.4. □

Lemma 4.10 ([56], Theorem 4.2)**.** *Let X, Y be topological spaces and $f\colon X \to Y$ a function.*

(1) *If f is continuous, then f preserves $\mathcal{I}$-convergence.*

(2) *If f preserves $\mathcal{I}$-convergence, then f is $\mathcal{I}$-continuous.*

Theorem 4.11. *Let $\mathcal{I}$ be a maximal ideal of $\mathbb{N}$. For a function f of a topological space X to a topological space Y, the following assertions are equivalent:*

(1) *f is $\mathcal{I}$-continuous;*

(2) *the inverse images of all $\mathcal{I}$-closed subsets of Y are $\mathcal{I}$-closed in X;*

(3) *for every $A \subseteq X$ we have $f\big(\mathcal{I}\text{-}\mathrm{Cl}(A)\big) \subseteq \mathcal{I}\text{-}\mathrm{Cl}\big(f(A)\big)$;*

(4) *for every $B \subseteq Y$ we have $\mathcal{I}\text{-}\mathrm{Cl}\big(f^{-1}(B)\big) \subseteq f^{-1}\big(\mathcal{I}\text{-}\mathrm{Cl}(B)\big)$.*

P r o o f. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are obvious.

Now, we will prove (1) $\Rightarrow$ (3). Let $y \in f\big(\mathcal{I}\text{-}\operatorname{Cl}(A)\big)$. Then, $y = f(x)$ for some $x \in \mathcal{I}\text{-}\operatorname{Cl}(A)$. Let $(x_n)_n$ be a sequence in A such that $x_n \xrightarrow{\mathcal{I}} x$ in X. Let U be an open neighborhood of y in Y. Then, $f^{-1}(U)$ is $\mathcal{I}$-open in X containing x. By Lemma 2.5, $\{n\colon x_n \in f^{-1}(U)\} \notin \mathcal{I}$. This implies that $\{n\colon f(x_n) \in U\} \notin \mathcal{I}$. But then, $\mathcal{I}$ is maximal, $\{n\colon f(x_n) \notin U\} \in \mathcal{I}$. Consequently, $y \in \mathcal{I}\text{-}\operatorname{Cl}\big(f(A)\big)$. This proves that $f\big(\mathcal{I}\text{-}\operatorname{Cl}(A)\big) \subseteq \mathcal{I}\text{-}\operatorname{Cl}\big(f(A)\big)$.

To prove (3) $\Rightarrow$ (4), applying (3) to $A = f^{-1}(B)$, we have

$$f\Big(\mathcal{I}\text{–}\operatorname{Cl}\big(f^{-1}(B)\big)\Big) \subseteq \mathcal{I}\text{–}\operatorname{Cl}\big(ff^{-1}(B)\big) \subseteq \mathcal{I}\text{–}\operatorname{Cl}(B).$$

This results in $\mathcal{I}\text{-}\operatorname{Cl}\big(f^{-1}(B)\big) \subseteq f^{-1}\big(\mathcal{I}\text{-}\operatorname{Cl}(B)\big)$.

To complete the proof, it suffices to prove (4) $\Rightarrow$ (2). For, let F be an $\mathcal{I}$--closed subset of Y. Then, $\mathcal{I}\text{-}\operatorname{Cl}\big(f^{-1}(F)\big) \subseteq f^{-1}\big(\mathcal{I}\text{-}\operatorname{Cl}(F)\big) = f^{-1}(F)$. Therefore, $\mathcal{I}\text{-}\operatorname{Cl}\big(f^{-1}(F)\big) = f^{-1}(F)$, that is, $f^{-1}(F)$ is $\mathcal{I}$-closed in X. Hence, f is $\mathcal{I}-$continuous. □

Theorem 4.12. *Let $\mathcal{I}$ be a maximal ideal of $\mathbb{N}$ and f be a surjective $\mathcal{I}$-continuous function from a topological space X to a topological space Y. If A is an $\mathcal{I}$-dense subset of X, then $f(A)$ is $\mathcal{I}$-dense in Y.*

P r o o f. Let A be an $\mathcal{I}$-dense subset of X. By Theorem 4.11, $f\big(\mathcal{I}\text{-}\operatorname{Cl}(A)\big) \subseteq \mathcal{I}\text{-}\operatorname{Cl}\big(f(A)\big)$. Since A is $\mathcal{I}$-dense, $f(X) \subseteq \mathcal{I}\text{-}\operatorname{Cl}\big(f(A)\big)$. But f is onto, so

$$\mathcal{I}\text{–}\operatorname{Cl}\big(f(A)\big) = Y.$$

It ends the proof. □

Theorem 4.13. *Let X be a Hausdorff space. If $(x_n)_n$ is an $\mathcal{I}$-convergent sequence in X, then it has one $\mathcal{I}$-limit point.*

P r o o f. The proof follows from [32, Theorem 1]. □

Theorem 4.14. *Let X, Y be two topological spaces and let $f, g\colon X \to Y$ be two mappings that preserve $\mathcal{I}$-convergence. If Y is Hausdorff and $\{x \in X\colon f(x) = g(x)\}$ is $\mathcal{I}$-dense in X, then $f(x) = g(x)$ for all $x \in X$.*

P r o o f. Let x be an element of X and $(x_n)_n$ be a sequence in X with $x_n \xrightarrow{\mathcal{I}} x$ and $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$. Since f and g are preserving $\mathcal{I}$-convergence, $f(x_n) \xrightarrow{\mathcal{I}} f(x)$ and $g(x_n) \xrightarrow{\mathcal{I}} g(x)$. By Theorem 4.13, we have $f(x) = g(x)$. Thus, $f(x) = g(x)$ for all $x \in X$. □

4.1. Properties of an $\mathcal{I}$-hyperconnected space

Theorem 4.15. *Let $\mathcal{I}$, $\mathcal{J}$ be two ideals of $\mathbb{N}$. For a topological space X, the following assertions hold:*

(1) *If X is $\mathcal{I}$-hyperconnected and $\mathcal{I} \subseteq \mathcal{J}$, then it is also $\mathcal{J}$-hyperconnected;*

(2) *if X is $(\mathcal{I} \cap \mathcal{J})$-hyperconnected, then it is $\mathcal{I}$-hyperconnected as well as $\mathcal{J}$-hyperconnected;*

(3) *if X is $\mathcal{I}$-hyperconnected or $\mathcal{J}$-hyperconnected, then it is $(\mathcal{I} \cup \mathcal{J})$-hyperconnected provided $\mathcal{I} \cup \mathcal{J}$ is an ideal on $\mathbb{N}$.*

Proof.
(1) Let U be a non-empty open subset of X. By Theorem 2.8, $\mathcal{I}\text{-}\operatorname{Cl}(U) \subseteq \mathcal{J}\text{-}\operatorname{Cl}(U)$. Since U is $\mathcal{I}$-dense in X, we have $\mathcal{I}\text{-}\operatorname{Cl}(U) = X$ and this implies that $\mathcal{J}\text{-}\operatorname{Cl}(U) = X$, i.e., X is $\mathcal{J}$-hyperconnected.

Similarly, (2) and (3) can be proven. □

Theorem 4.16. *$\mathcal{I}$-hyperconnected spaces are preserved by surjective continuous mappings provided $\mathcal{I}$ is a maximal ideal of $\mathbb{N}$.*

Proof. Let $f\colon X \to Y$ be a surjective continuous function and X be an $\mathcal{I}$--hyperconnected space. Let U be a non-empty open subset of Y. Since f is continuous, $f^{-1}(U)$ is open in X. But X is $\mathcal{I}$-hyperconnected, so $\mathcal{I}\text{-}\operatorname{Cl}\big(f^{-1}(U)\big) = X$. According to Theorem 4.11,

$$\mathcal{I}\text{-}\operatorname{Cl}\big(f^{-1}(U)\big) \subseteq f^{-1}\big(\mathcal{I}\text{-}\operatorname{Cl}(U)\big);$$

therefore $f^{-1}\big(\mathcal{I}\text{-}\operatorname{Cl}(U)\big) = X$. Thus, $f(X) = \mathcal{I}\text{-}\operatorname{Cl}(U)$. As f is surjective, Y is $\mathcal{I}$-hyperconnected. □

Corollary 4.17. *$\mathcal{I}$-hyperconnected spaces are preserved by quotient mappings provided $\mathcal{I}$ is a maximal ideal of $\mathbb{N}$.*

Theorem 4.18. *If $(X,\ \tau)$ is an $\mathcal{I}$-hyperconnected space, then for every $(U,\ V) \in (\tau \backslash \{\emptyset\}) \times (\tau_{\mathcal{I}} \backslash \{\emptyset\})$, $U \cap V \neq \emptyset$.*

Proof. Suppose that $(X,\ \tau)$ is $\mathcal{I}$-hyperconnected. Let U be a non-empty open subset of X and V be a non-empty $\mathcal{I}$-open subset of X. Since X is $\mathcal{I}$-hyperconnected, $\mathcal{I}\text{-}\operatorname{Cl}(U) = X$. By Theorem 4.4, we have $U \cap V \neq \emptyset$. □

Definition 4.19. A subset A of a topological space X is said to be

(1) $\mathcal{I}$-semi-open if there exists an open set $U \subseteq X$ such that $U \subseteq A \subseteq \mathcal{I}\text{-}\operatorname{Cl}(U)$.

(2) semi-$\mathcal{I}$-open if there exists an $\mathcal{I}$-open set $V \subseteq X$ such that $V \subseteq A \subseteq \operatorname{Cl}(V)$.

Theorem 4.20. *Let X be an $\mathcal{I}$-hyperconnected space. Then, the following statements are valid:*

(1) $A \bigcap B \neq \emptyset$, *for every non-empty open set* $A \subseteq X$ *and for every non-empty* $\mathcal{I}$*-open set* $B \subseteq X$*;*

(2) $A \bigcap B \neq \emptyset$, *for every non-empty open set* $A \subseteq X$ *and for every non-empty* $\mathcal{I}$*-semi-open set* $B \subseteq X$*;*

(3) $A \bigcap B \neq \emptyset$, *for every non-empty open set* $A \subseteq X$ *and for every non-empty semi-*$\mathcal{I}$*-open set* $B \subseteq X$*;*

(4) $A \bigcap B \neq \emptyset$, *for every non-empty semi-open set* $A \subseteq X$ *and for every non-empty* $\mathcal{I}$*-open set* $B \subseteq X$*;*

(5) $A \bigcap B \neq \emptyset$, *for every non-empty semi-open set* $A \subseteq X$ *and for every non-empty* $\mathcal{I}$*-semi-open set* $B \subseteq X$*;*

(6) $A \bigcap B \neq \emptyset$, *for every non-empty semi-open set* $A \subseteq X$ *and for every non-empty semi-*$\mathcal{I}$*-open set* $B \subseteq X$*.*

Proof.
(1) It follows from Theorem 4.18.
(2) Let A be a non-empty open subset of X and B be a non-empty $\mathcal{I}$-semi-open subset of X. Let U be an open set in X such that $U \subseteq B \subseteq \mathcal{I}\text{-}\mathrm{Cl}(U)$. By the given condition, $A \cap U \neq \emptyset$; this proves that $A \cap B \neq \emptyset$.
(3) Suppose that A is a non-empty open subset of X and B is a non-empty semi-$\mathcal{I}$-open subset of X. Let U be an $\mathcal{I}$-open subset of X such that $U \subseteq B \subseteq \mathrm{Cl}(U)$. Then, $A \cap U \neq \emptyset$, and thus, $A \cap B \neq \emptyset$.
(4) Let A be a non-empty semi-open subset of X and B be a non-empty $\mathcal{I}$-semi-open subset of X. Then, there exists an open set U in X with $U \subseteq A \subseteq \mathrm{Cl}(U)$. Consequently, $U \cap B \neq \emptyset$; hence $A \cap B \neq \emptyset$.
(5) Let A be a non-empty semi-open subset of X and B be a non-empty $\mathcal{I}$-semi-open subset of X. Let V be an open set in X with $V \subseteq B \subseteq \mathcal{I}\text{-}\mathrm{Cl}(V)$. Then $A \cap V \neq \emptyset$; so $A \cap B \neq \emptyset$.
(6) Let A be a non-empty semi-open subset of X and B be a non-empty semi-$\mathcal{I}$-open subset of X. Let V be an $\mathcal{I}$-open set in X with $V \subseteq B \subseteq \mathrm{Cl}(V)$. Then, $A \cap V \neq \emptyset$; so $A \cap B \neq \emptyset$. □

Theorem 4.21. *Let* $\mathcal{I}$ *be a maximal ideal of* $\mathbb{N}$ *and* X *a first countable topological space.* X *is hyperconnected if and only if it is* $\mathcal{I}$*-hyperconnected.*

Proof. A simple consequence in [10, Theorem 7]. □

Theorem 4.22. *Let* $\mathcal{I}$ *be a maximal ideal of* $\mathbb{N}$*. A product space* $X = \prod_{\alpha \in I} X_\alpha$ *is* $\mathcal{I}$*-hyperconnected if and only if each factor* X_α *is an* $\mathcal{I}$*-hyperconnected space.*

Proof. The direct part follows from Theorem 4.16, whereas the converse part follows from Theorem 3.5. □

Theorem 4.23. *Let $f\colon X \to Y$ be a function and $h\colon X \to X \times Y$ the graph of f defined by $h(x) = \big(x,\ f(x)\big)$ for every $x \in X$.*

(1) *If h preserves $\mathcal{I}$-convergence, then f preserves $\mathcal{I}$-convergence.*

(2) *If h is $\mathcal{I}$-continuous, then f is $\mathcal{I}$-continuous.*

Proof.
(1) Let $(x_n)_n$ be a sequence in X with $x_n \xrightarrow{\mathcal{I}} x$ in X. Since h preserves $\mathcal{I}$--convergence, $h(x_n) \xrightarrow{\mathcal{I}} h(x)$ in $X \times Y$. Then by the continuity of projection map $p_Y\colon X \times Y \to Y$, we have $f(x_n) \xrightarrow{\mathcal{I}} f(x)$ in Y. This shows that f preserves $\mathcal{I}$-convergence.
(2) Let F be an $\mathcal{I}$-closed subset of X. Then $X \times F = \mathcal{I}\text{-}\operatorname{Cl}(X \times F)$, so $X \times F$ is $\mathcal{I}$-closed in $X \times Y$. Since h is $\mathcal{I}$-continuous, $h^{-1}(X \times F)$ is $\mathcal{I}$-closed in X. However, $h^{-1}(X \times F) = f^{-1}(F)$, it follows that f is $\mathcal{I}$-continuous. □

4.2. Properties of an $\mathcal{I}$-submaximal space

We start this section with some equivalents of an $\mathcal{I}$-submaximal space which follow quickly from the definition.

Theorem 4.24. *For a topological space X, the following are equivalent:*

(1) *X is an $\mathcal{I}$-submaximal;*

(2) $\operatorname{Cl}(A)\backslash A$ *is $\mathcal{I}$-closed for every subset A of X;*

(3) *every subset A of X, for which* $\operatorname{Int}(A)$ *is empty, is $\mathcal{I}$-closed.*

Proof.

(1) ⇒ (2): Suppose X is $\mathcal{I}$-submaximal. We prove $\operatorname{Cl}(A)\backslash A$ is $\mathcal{I}$-closed for every $A \subseteq X$.
Let A be any subset of X. Define $B = \operatorname{Cl}(A)\backslash A$. Observe that $X\backslash B$ is a dense subset of X. By the assumption of $\mathcal{I}$-submaximality, $X\backslash B$ is $\mathcal{I}$-open. This proves that $\operatorname{Cl}(A)\backslash A$ is $\mathcal{I}$-closed in X.

(2) ⇒ (1): Suppose for every subset $A \subseteq X$, $\operatorname{Cl}(A)\backslash A$ is $\mathcal{I}$-closed. We want to show that X is $\mathcal{I}$-submaximal. For this, let D be a dense subset of X, i.e., $\operatorname{Cl}(D) = X$. By the given condition, $\operatorname{Cl}(D)\backslash D$ is $\mathcal{I}$-closed. This means that $X\backslash D$ is $\mathcal{I}$-closed. Therefore, D is $\mathcal{I}$-open, which shows that X is $\mathcal{I}$-submaximal.

(1) ⇒ (3): Suppose X is $\mathcal{I}$-submaximal. Let $A \subseteq X$ be such that $\operatorname{Int}(A) = \emptyset$. Then, $X\backslash A$ is dense in X and hence $\mathcal{I}$-open. This means that A is $\mathcal{I}$-closed.

(3) ⇒ (1): Suppose every subset A with $\operatorname{Int}(A) = \emptyset$ is $\mathcal{I}$-closed. Let D be a dense subset of X. Then, $X\backslash D$ has empty interior. By assumption, $X\backslash D$ is $\mathcal{I}$-closed, this means that D is $\mathcal{I}$-open. Hence, X is $\mathcal{I}$-submaximal.

□

Theorem 4.25. *Let X, Y be topological spaces and $f\colon X \to Y$ a homeomorphism. Then:*

(1) $f\big(\mathcal{I}\text{-}\operatorname{Cl}(A)\big) = \mathcal{I}\text{-}\operatorname{Cl}\big(f(A)\big)$ *for each* $A \subseteq X$*;*

(2) $\mathcal{I}\text{-}\operatorname{Cl}\big(f^{-1}(B)\big) = f^{-1}\big(\mathcal{I}\text{-}\operatorname{Cl}(B)\big)$ *for each* $B \subseteq Y$.

Proof.
(1) Let y be an element of $\mathcal{I}\text{-}\operatorname{Cl}\big(f(A)\big)$ and $(y_n)_n$ a sequence in $f(A)$ with $y_n \xrightarrow{\mathcal{I}} y$ in Y. Let $(x_n)_n$ be a sequence in A with $y_n = f(x_n)$. Since f is surjective, there exists an element $x \in X$ such that $f(x) = Y$.

Claim. $x_n \xrightarrow{\mathcal{I}} x$ *in* X.

Suppose, to derive a contradiction, that $(x_n)_n$ is not $\mathcal{I}$-convergent to the point x. Let U be an open neighborhood of x in X such that $\{n\colon x_n \notin U\} \notin \mathcal{I}$. Then, $\{n\colon y_n \notin f(U)\} \notin \mathcal{I}$. Since f is open, $f(U)$ is an open neighborhood of y; hence the sequence $(y_n)_n$ is not $\mathcal{I}$-convergent to the point Y, which is a contradiction. Therefore, $x_n \xrightarrow{\mathcal{I}} x$, which gives $y \in f\big(\mathcal{I}\text{-}\operatorname{Cl}(A)\big)$. Thus,

$$\mathcal{I}\text{-}\operatorname{Cl}\big(f(A)\big) \subseteq f\big(\mathcal{I}\text{-}\operatorname{Cl}(A)\big).$$

By Theorem 4.11, the assertion follows.

Similarly, (2) can be proven. □

Corollary 4.26. *$\mathcal{I}$-openness is a topological property.*

Theorem 4.27. *$\mathcal{I}$-submaximality is a topological property.*

Proof. Let $f\colon X \to Y$ be a homeomorphism and X an $\mathcal{I}$-submaximal space. Let A be a dense subset of Y. Let U be a non-empty open subset of X. Since f is open, $f(U)$ is open in Y; hence $A \cap f(U) \neq \emptyset$. But then, $f^{-1}(A) \cap f^{-1}f(U) \neq \emptyset$, which gives $f^{-1}(A) \cap U \neq \emptyset$. This means that $f^{-1}(A)$ is dense in X but X is $\mathcal{I}$-submaximal, so $f^{-1}(A)$ is $\mathcal{I}$-open in X. Since f is surjective, $A = ff^{-1}(A)$; A is $\mathcal{I}$-open in Y. Hence, Y is $\mathcal{I}$-submaximal. □

Theorem 4.28. *Every submaximal space is $\mathcal{I}$-submaximal.*

Theorem 4.29. *Let $\mathcal{I} \subseteq \mathcal{J}$ be two ideals of $\mathbb{N}$. If X is a $\mathcal{J}$-submaximal space, then it is also $\mathcal{I}$-submaximal.*

Proof. Let A be an $\mathcal{I}$-dense subset of X. Then, A is $\mathcal{J}$-dense in X. But then, by $\mathcal{J}$-submaximality of X, A is $\mathcal{J}$-open in X. This means that $A = \mathcal{J}\text{-}\operatorname{Int}(A) \subseteq \mathcal{I}\text{-}\operatorname{Int}(A) \subseteq A$. Thus, $A = \mathcal{I}\text{-}\operatorname{Int}(A)$, i.e., A is $\mathcal{I}$-open in X, showing that X is $\mathcal{I}$-submaximal. □

Theorem 4.30. *Suppose that $A \subseteq S$ are two subsets of a space X. If S is a subspace of X, then the following are valid:*

(1) $\mathcal{I}\text{-}\mathrm{Cl}_S(A) \subseteq \mathcal{I}\text{-}\mathrm{Cl}_X(A)$;

(2) $\mathcal{I}\text{-}\mathrm{Cl}_S(A) = S \cap \mathcal{I}\text{-}\mathrm{Cl}_X(A)$.

Proof.
(1) Take $x \in \mathcal{I}\text{-}\mathrm{Cl}_S(A)$, and let $(x_n)_n$ be a sequence in A with $x_n \xrightarrow{\mathcal{I}} x$ in S. Let U be a neighborhood of x in X. Then, $U \cap S$ is a neighborhood of x in S and so, $\{n\colon x_n \notin U \cap S\} \in \mathcal{I}$. But $\{n\colon x_n \notin U\} \subseteq \{n\colon x_n \notin U \cap S\}$, therefore $\{n\colon x_n \notin U\} \in \mathcal{I}$. Consequently,

$$x_n \xrightarrow{\mathcal{I}} x \ \text{ in } X, \quad \text{i.e.,} \quad x \in \mathcal{I}\text{-}\underset{X}{\mathrm{Cl}}(A).$$

Thus, $\mathcal{I}\text{-}\mathrm{Cl}_S(A) \subseteq \mathcal{I}\text{-}\mathrm{Cl}_X(A)$.

(2) Clearly, $S \cap \mathcal{I}\text{-}\mathrm{Cl}_X(A) \subseteq \mathcal{I}\text{-}\mathrm{Cl}_S(A)$. Now, take $x \in \mathcal{I}\text{-}\mathrm{Cl}_S(A)$, and let $(x_n)_n$ be a sequence in A with $x_n \xrightarrow{\mathcal{I}} x$ in S. As it is shown previously, $x_n \xrightarrow{\mathcal{I}} x$ in X. So, $x \in \mathcal{I}\text{-}\mathrm{Cl}_X(A)$. Thus, $S \cap \mathcal{I}\text{-}\mathrm{Cl}_X(A) \subseteq \mathcal{I}\text{-}\mathrm{Cl}_S(A)$ and hence, $\mathcal{I}\text{-}\mathrm{Cl}_S(A) = S \cap \mathcal{I}\text{-}\mathrm{Cl}_X(A)$. □

Theorem 4.31. *Every subspace of an $\mathcal{I}$-submaximal space is $\mathcal{I}$-submaximal.*

Remark 1. The product of two $\mathcal{I}$-submaximal spaces need not be $\mathcal{I}$-submaximal. For example,

(1) Let $X = \{a, b\}$ have the Sierpinski topology $\tau = \{\emptyset, \{a\}, X\}$. Then, X is an $\mathcal{I}$-submaximal space, but X^2 is not $\mathcal{I}$-submaximal because $A = \{(a, a), (b, b)\}$ is dense in X^2 which is not $\mathcal{I}$-open.

(2) Let $X = \{0\} \bigcup \{\frac{1}{n} \colon n = 1, 2, \ldots\}$ have the usual real subspace topology. Then, X is $\mathcal{I}$-submaximal, but the product space X^2 is not $\mathcal{I}$-submaximal.

Theorem 4.32. *If a product space $X \times Y$ is an $\mathcal{I}$-submaximal, then X and Y are $\mathcal{I}$-submaximal spaces.*

Proof. Suppose $X \times Y$ is $\mathcal{I}$-submaximal. Consider the projection maps,

$$\pi_X \colon X \times Y \to X \quad \text{and} \quad \pi_Y \colon X \times Y \to Y.$$

Let $D \subseteq X$ be a dense subset of X. Then, $D \times Y$ is dense in $X \times Y$ and hence, $\mathcal{I}$-open. Let $(x_n)_n$ be a sequence in X such that $x_n \xrightarrow{\mathcal{I}} x \in D$ in X. Then, for any $y \in Y$, the sequence (x_n, y) $\mathcal{I}$-converges to (x, y) in $X \times Y$. Since $(x, y) \in D \times Y$, by Lemma 2.5, $|\{n\colon (x_n, y) \in D \times Y\}| = \omega$. This gives

$$|\{n\colon x_n \in D\}| = \omega.$$

Again, by Lemma 2.5, D is $\mathcal{I}$-open. Hence, X is $\mathcal{I}$-submaximal. Similarly, Y is $\mathcal{I}$-submaximal. □

5. Conclusion

In this paper, we have explored some basic properties of $\mathcal{I}$-closure and $\mathcal{I}$-boundary points of a set in a topological space. An important example has been presented to illustrate the behavior of $\mathcal{I}$-closure and $\mathcal{I}$-boundary points. We have also investigated the notions of $\mathcal{I}$-hyperconnected spaces and $\mathcal{I}$-submaximal spaces. We have extended existing results and provided a broader understanding of how ideals influence topological properties. Future work may include exploring further properties of these concepts and studying the relationships between $\mathcal{I}$-hyperconnectedness and other ideal-based topological constructs.

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