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ON BORSUK'S NON-RETRACT THEOREM

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ABSTRACT. Let X be a Hausdorff topological space and let A be both an F_σ and G_δ subset of X. Let also $f\colon A \to \mathbb{R}$ be a function for which the inverse image of every open subset $U \subset \mathbb{R}$ is F_σ in A. We will prove that there is a linear extension operator $\varphi^\star$ such that $\varphi^\star(f)$ has the same property on X. An analogous result is proved for the Baire-one function defined on an analogous subset of $\mathbb{R}$. We will also show that the extension map is (with a supremum norm) an isometry. In the second part of the paper, we deal with classical Borsuk's non-retract theorem. It says that a unit sphere in $\mathbb{R}^n$ is not a continuous retract of the unit closed ball. We will show that such a unit sphere is a piecewise continuous retract of the unit closed ball.

1. Introduction

Let X and Y be Hausdorff topological spaces, let A be an F_σ and G_δ subset of X and let $f\colon A \to \mathbb{R}$ be a function for which the inverse image of every open subset $U \subset \mathbb{R}$ is F_σ in A. The mapping $r\colon X \to A$ is an *algebraic retraction* if $r(x) = x$ for every $x \in A$. We say that a function $f\colon X \to \mathbb{R}$ is *piecewise continuous*, if there is an increasing sequence $(X_n) \subset X$ of nonempty closed sets such that $X = \bigcup_{n=0}^{\infty} X_n$ and the restriction $f_{\restriction X_n}$ is continuous for each $n \in \mathbb{N}$. The family of all piecewise continuous functions $X \to \mathbb{R}$ is denoted by $\mathcal{P}(X)$.

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Example 1. Let $E(t)$ be an *entier* of a real number t, $A = [0,1) \subset \mathbb{R}$ and let $g\colon \mathbb{R} \to A$ be a mapping given by the formula $g(x) = x - E(x)$. Since

$$[0,1) = \bigcup_{k=1}^{\infty} [0, 1 - \frac{1}{k}]$$

and

$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} \big(n + [0,1)\big) = \bigcup_{n=-\infty}^{\infty} [n, n+1),$$

g is a piecewise continuous retraction of $\mathbb{R}$ onto interval A.

Recall that $\mathcal{B}_1(X,Y)$ is the space of the pointwise limits of sequences of continuous functions defined on X. Moreover, $\mathcal{B}_1^\star(X,Y) \subsetneq \mathcal{B}_1(X,Y)$ is the space of the pointwise limits of those sequences of continuous functions (f_n) which are constant for every $n > n_\epsilon$ with $n_\epsilon \in \mathbb{N}$. In [5] it is shown that the inclusion $\mathcal{B}_1^\star \subset \mathcal{P}$ holds for metric spaces and the inclusion $\mathcal{P} \subset \mathcal{B}_1^\star$ holds for all complete metric spaces. We say that a function $f\colon X \to Y$ is

- *of the first Borel class*, if the f-inverse image of every open subset of Y is F_σ in X (see [2, p. 1]);
- *of the first level Borel class*, if the f-inverse image of every F_σ subset of Y is F_σ in X (see [2, p. 1]).

Note that in the last definition, the words "F_σ subset of Y" may be equivalently replaced by "closed subset of Y". The symbols $\mathfrak{fcBor}(X,Y)$ and $\mathfrak{flB}(X,Y)$ stand for the families of the first Borel and first level Borel classes of functions defined on X, respectively. The inclusion $\mathfrak{flB}(X,Y) \subset \mathfrak{fcBor}(X,Y)$ holds whenever any open set in Y is F_σ (for example, if Y is perfectly normal). A function witnessing the fact that $\mathfrak{flB}(\mathbb{R},\mathbb{R}) \neq \mathfrak{fcBor}(\mathbb{R},\mathbb{R})$ is the classical Riemann function (i.e., $f(x) = 0$ for x irrational and $f(x) = \frac{1}{q}$ if $x = \frac{p}{q}$, where p, q are mutually prime) which is of the first Borel class but not of the first level Borel class. If $f \in \mathfrak{fcBor}^b(A,Y)$ and $f \in \mathcal{B}_1^b(A,Y)$ are bounded elements of $\mathfrak{fcBor}(A,Y)$ and $\mathcal{B}_1(A,Y)$, respectively, the *supremum norm* $\|\cdot\|_A$ of f on A is defined as $\|f\|_A = \sup_{x\in A} |f(x)|$.

In the first part of the paper, we deal with a general problem in the theory of real functions, which is inspired by the Tietze extension theorem:

> (P): *Let A be a nonempty subset of a topological space X and let $f_0 \in \mathbb{R}^A$ be a function with a certain property (W). Can f_0 be extended to a function $f \in \mathbb{R}^X$ with the same property (W)?*

Classical Tietze-Urysohn's theorem says that every continuous map defined on a closed subset of a normal topological space can be extended to a function continuous on all that space. In 1982, Jayne and Rogers [2] proved the following

Theorem 1. *Let X and Y be metric spaces, Y complete. Let $A \subset X$ and $f \in \mathcal{P}(A, Y)$. Then, there exists $A_1 \subset X$ which is the intersection of the sets G_δ and F_σ and the function $\overline{f} \in \mathcal{P}(A_1, Y)$ extending f.*

Finally, in 2008, Karlova [4] showed that if X is perfectly paracompact and A is its complete metrizable subspace, then every continuous function $f\colon A \to Y$ has an extension $\overline{f} \in \mathfrak{fc}\mathfrak{Bor}(X)$.

In this paper, we prove an analogous results for the classes of first Borel and Baire one functions. In (key) Lemma 3, we construct an algebraic retraction φ whose properties allow us to formulate and prove these results.

2. Properties and useful lemmas

The first Lemma is a well-known fact and follows, for example, from [6, Theorem 2, Section VI of § 31].

Lemma 1. *Let X be a topological space. If*

$$f, g \in \mathfrak{fc}\mathfrak{Bor}(X, \mathbb{R}) \quad \textit{then} \quad f + g \in \mathfrak{fc}\mathfrak{Bor}(X, \mathbb{R}).$$

Since $|f|$ is a composition of f and a continuous mapping $x \mapsto |x|$, we have

Lemma 2. *Let X be a topological space. If*

$$f \in \mathfrak{fc}\mathfrak{Bor}(X, \mathbb{R}) \quad \textit{then} \quad |f| \in \mathfrak{fc}\mathfrak{Bor}(X, \mathbb{R}).$$

From Lemma 1 and Lemma 2 it follows

Corollary 1. *Let X be a topological space. Then $\mathfrak{fc}\mathfrak{Bor}(X, \mathbb{R})$ is a linear lattice.*

The following lemma is crucial for the main result.

Lemma 3. *Let X be the Hausdorff topological space and let A be both an F_σ and G_δ subset of X. Let also $g\colon X \setminus A \to A \subset X$ be a continuous map. Then, the algebraic retraction $\varphi\colon X \to A \subset X$ given by the formula*

$$\varphi(x) = \begin{cases} x & : \ x \in A, \\ g(x) & : \ x \in X \setminus A \end{cases} \tag{1}$$

belongs to the class $\mathcal{P}(X, X) \cap \mathfrak{fl}\mathfrak{B}(X, X)$. Furthermore, if X is perfectly normal, then $\varphi \in \mathfrak{fc}\mathfrak{Bor}(X, X)$. Finally, if $X = \mathbb{R}$, then $\varphi \in \mathcal{B}_1(X, X)$.

P r o o f. We first show that $\varphi \in \mathcal{P}(X, X)$. Let $i_A \colon A \to A$ be an identity on A. Since A is both an F_σ and G_δ subset of X, there are increasing sequences (F_n) and (G_n) of closed sets such that

$$\bigcup_n F_n = A \qquad \text{and} \qquad \bigcup_n G_n = X \setminus A.$$

Obviously

$$F_j \cap G_k = \emptyset \qquad \text{for every} \qquad j, k \in \mathbb{N}.$$

Since g is continuous, the restrictions $i_{A \restriction F_n}$ and $g_{\restriction G_n}$ are continuous for every $n \in \mathbb{N}$. Set

$$H_n = F_n \cup G_n \qquad \text{for every} \qquad n \in \mathbb{N}.$$

Obviously

$$\bigcup_n H_n = X.$$

Furthermore, the continuity of restrictions $i_{A \restriction F_n}$ and $g_{\restriction G_n}$ implies continuity of φ on every H_n. That completes this part of the proof.

To show that $\varphi \in \mathfrak{fl}\mathfrak{B}(X, X)$, we assume F to be a closed subset of X. We have

$$\varphi^{-1}(F) = (A \cap F) \cup g^{-1}(F).$$

Since A is F_σ and F is closed, $A \cap F$ is F_σ. Further, $g^{-1}(F)$ is a relatively closed subset of an F_σ-set $X \setminus A$, hence, it is again F_σ. So, $\varphi^{-1}(F)$ is F_σ.

If X is perfectly normal then any open set is F_σ. Hence, it follows from the already proved fact that the inverse image of any open set is F_σ. Thus

$$\varphi \in \mathfrak{fc}\mathfrak{Bor}(X, X).$$

To see that $\varphi \in \mathcal{B}_1(\mathbb{R}, \mathbb{R})$, it suffices to use the well-known fact that

$$\mathfrak{fc}\mathfrak{Bor}(\mathbb{R}, \mathbb{R}) = \mathcal{B}_1(\mathbb{R}, \mathbb{R})$$

(cf., the Lebesgue-Hausdorff theorem, [6, §31, Section IX]). □

3. The main extension results

The first main theorem considers extending functions from the first Borel class.

Theorem 2. *Let X be a perfectly normal space and let A be its both an F_σ and G_δ subset. Let also $f \in \mathfrak{fc}\mathfrak{Bor}(A, \mathbb{R})$. Then there exists a non-negative, preserving unity and linear extension operator $\varphi^\star \colon \mathfrak{fc}\mathfrak{Bor}(A, \mathbb{R}) \to \mathfrak{fc}\mathfrak{Bor}(X, \mathbb{R})$ such that its restriction to $\mathfrak{fc}\mathfrak{Bor}^b(A, \mathbb{R})$ is an isometry.*

P r o o f. Let $g\colon X\setminus A \to A \subset X$ be a continuous map. Set $\varphi^\star(f) = f\circ\varphi$, where φ is an algebraic retraction given by (1). Positivity of $\varphi^\star$ is due to the fact that the class $\mathfrak{fcBor}(X)$ equipped with the order by the coordinates is a linear lattice (see Corollary 1). We shall show that

$$\varphi^\star(f) \in \mathfrak{fcBor}(X) \quad \text{for every} \quad f \in \mathfrak{fcBor}(A).$$

Let U be an open subset of $\mathbb{R}$. There is

$$\big(\varphi^\star(f)\big)^{-1}(U) = (f\circ\varphi)^{-1}(U) = \varphi^{-1}\left(f^{-1}(U)\right).$$

Since $f \in \mathfrak{fcBor}(A)$, $f^{-1}(U)$ is F_σ in A. Hence $f^{-1}(U) = \bigcup_n K_n$ with every K_n closed in A. Thus

$$f^{-1}(U) = \bigcup_n K_n = \bigcup_n (A\cap G_n) = A \cap \bigcup_n G_n$$

with every G_n closed in X.
Therefore

$$\begin{aligned}\big(\varphi^\star(f)\big)^{-1}(U) &= \varphi^{-1}\Bigg(A\cap\bigcup_n G_n\Bigg) = \varphi^{-1}(A)\cap\varphi^{-1}\Bigg(\bigcup_n G_n\Bigg)\\ &= X\cap\bigcup_n \varphi^{-1}(G_n) = \bigcup_n \varphi^{-1}(G_n).\end{aligned}$$

According to Lemma 3, every set $\varphi^{-1}(G_n)$ is F_σ in X. Hence, the set

$$\big(\varphi^\star(f)\big)^{-1}(U) = \bigcup_n \varphi^{-1}(G_n)$$

is F_σ in X as well. Thus $\varphi^\star(f) \in \mathfrak{fcBor}(X)$. $\varphi^\star$ is of course an extension of f. Its linearity and preserving of the unity are obvious. Furthermore, if f is a bounded element of $\mathfrak{fcBor}(A)$, we have

$$\|\varphi^\star(f)\|_X = \sup_{x\in X}\big|f\big(\varphi(x)\big)\big| = \sup_{a\in A}|f(a)| = \|f\|_A. \qquad \square$$

Note that in [8, Theorem 3] we have shown that if X is Hausdorff and A is its both an F_σ and G_δ subset, then $f \in \mathcal{P}(A)$ has a linear extension $\varphi^\star(f) = f\circ\varphi$. In 2005, Kalenda and Spurný proved [3, Example 10] that there is a bounded function of the first class $f\colon \mathbb{Q}\cap[0,1] \to \mathbb{R}$, which cannot be extended to a function of the first class on $[0,1]$. From these considerations and by Theorem 1, we obtain the following

Remark 1. Let A be a nonempty subset of a metric space X. Let also $f \in \mathcal{P}(A,\mathbb{R})$. If $\overline{f} \in \mathcal{P}(A_1,\mathbb{R})$ extends f, then in Theorem 1 we cannot require A_1 to be simultanoeusly F_σ and G_δ.

P r o o f. Let $A = [0,1] \cap \mathbb{Q}$ and $X = \mathbb{R}$. A is of course an F_σ subset of X. Moreover, $\mathcal{P}(A) = \mathcal{B}_1(A) = \mathbb{R}^A$. Let $f \colon A \to \mathbb{R}$. Suppose that there is an extension $\overline{f} \in \mathcal{P}(A_1)$ such that $A_1 \supset A$ is both an F_σ and G_δ subset of $\mathbb{R}$ and $\overline{f}_{\restriction A} = f$. Then, by [8, Theorem 3], the mapping $\varphi^\star(\overline{f}) = \overline{f} \circ \varphi$ is a piecewise continuous (and thus Baire one class) extension of $\overline{f}$. Those arguments would imply that every function $f \in \mathcal{B}_1(A)$ has an extension $\overline{f} \in \mathcal{B}_1(\mathbb{R})$, a contradiction. □

The second main theorem deals with extensions of Baire one class functions.

Theorem 3. *Let A be both an F_σ and G_δ subset of $\mathbb{R}$ and let $f \in \mathcal{B}_1(A,\mathbb{R})$. There exists a non-negative, preserving unity and linear extension operator $\varphi^\star \colon \mathcal{B}_1(A,\mathbb{R}) \to \mathcal{B}_1(\mathbb{R},\mathbb{R})$ such that its restriction to $\mathcal{B}_1^b(A)$ is an isometry.*

P r o o f. Let $g \colon \mathbb{R} \setminus A \to A \subset \mathbb{R}$ be a continuous map. Set $\varphi^\star(f) = f \circ \varphi$, where φ is an algebraic retraction (1). Since $f \in \mathcal{B}_1(A,\mathbb{R})$, there is a sequence (f_n) of continuous mappings on A such that $f(a) = \lim_{n\to\infty} f_n(a)$ for every $a \in A$. Since (by Lemma 3) $\varphi \in \mathcal{B}_1(A,\mathbb{R})$, we have $\varphi = \lim_{n\to\infty} \phi_n$ with every ϕ_n continuous on A. Obviously, every composition $f_n \circ \phi_n$ is continuous on A. Let $x \in X$. Since (again by Lemma 3) $\varphi \in \mathcal{P}(A,\mathbb{R})$ and $\mathcal{P}(A,\mathbb{R}) = \mathcal{B}_1^\star(A,\mathbb{R})$ (see Introduction), there is $n_x \in \mathbb{N}$ such that $\phi_n(x) = \varphi(x)$ for every $n > n_x$. Therefore

$$(f_n \circ \phi_n)(x) = f_n\bigl(\phi_n(x)\bigr) = f_n\bigl(\varphi(x)\bigr) \to f\bigl(\varphi(x)\bigr)$$

and

$$f_n \circ \phi_n \xrightarrow{n\to\infty} f \circ \varphi = \varphi^\star \in \mathcal{B}_1(\mathbb{R},\mathbb{R}).$$

Linearity and preserving of the unity are obvious for $\varphi^\star$. Furthermore, if f is a bounded element of $\mathcal{B}_1(A,\mathbb{R})$, we have

$$\|\varphi^\star(f)\|_X = \sup_{x\in X}\bigl|f\bigl(\varphi(x)\bigr)\bigr| = \sup_{a\in A}|f(a)| = \|f\|_A. \qquad \square$$

4. Borsuk's non-retract theorem

The classical Borsuk's non-retract theorem [7, Theorem 8] asserts that

Theorem 4 (Borsuk). *A unit sphere in $\mathbb{R}^n$ is not a continuous retract of the unit closed ball.*

It turns out that if the word "continuous retract" is replaced by "piecewise continuous retract", the Borsuk's claim is no longer true.

Theorem 5. *Let $(X, \|\cdot\|)$ be a normed space (or X is simply a closed unit ball in a normed space Y) and let $U = X \setminus \{0\}$. Let also A be the unit sphere of X and fix $t \in A$. The mapping $\Phi\colon X \to A$ given by the formula*

$$\Phi(x) = \begin{cases} \frac{x}{\|x\|} & : \quad x \in U, \\ t & : \quad x = 0 \end{cases} \tag{2}$$

is piecewise continuous. Hence, the unit sphere is a piecewise contiunous retract of the unit closed ball, as well as of Y.

Proof. As elements of the increasing sequence (X_n), it suffices to take

$$X_n = \left\{ x \in X : \|x\| \geqslant \frac{1}{n} \right\}. \qquad \square$$

Remark 2. Let X be a closed unit ball in a normed space and let $U = X \setminus \{0\}$. Let also A be the unit sphere of X. Let us consider any retraction $\varphi\colon X \to A$. To put an image $\varphi(x)$ of any element $x \in U$ into A we must have $\|\varphi(x)\| = 1$, so $\varphi(x)$ must be a normalization of x.

On the other hand, an element $0 \in X \setminus U$ must be transformed to A without normalization. So, the image of $0 \in X \setminus U$ must be a point $t \in A$. From these considerations we get that general form of every retraction $\Phi\colon X \to A$ is given by (2).

Our last theorem describes a retract of a normed space X onto the open unit ball. It is an extension of Theorem 5.

Theorem 6. *Let $(X, \|\cdot\|)$ be at least 1-dimensional normed space and let K^0 be the open unit ball in X. Then the function $\Phi\colon X \to K^0$ given by*

$$\Phi(0) = 0 \quad \textit{and} \quad \Phi(x) = \left(1 - \frac{E(\|x\|)}{\|x\|}\right) \cdot x \quad \textit{for every } x \neq 0,$$

is a piecewise continuous retraction from X onto K^0.

Proof. Note first that

$$\Phi(x) = \begin{cases} x & \text{if} \quad \|x\| < 1, \\ 0 & \text{if} \quad \|x\| = 1. \end{cases}$$

Since

$$\Phi(x) = (\|x\| - E(\|x\|)) \cdot \frac{x}{\|x\|} \quad \text{and} \quad \|\Phi(x)\| = \|x\| - E(\|x\|) < 1,$$

Φ is a surjection from X onto K^0.

Set

$$\begin{aligned} B_n &= \{x \in X : n \leqslant \|x\| < n+1\} \\ &= \bigcup_{j=1}^{\infty} \left\{ x \in X : n \leqslant \|x\| \leqslant n+1-\frac{1}{j} \right\} \\ &= \bigcup_{j=1}^{\infty} B_{n,j}. \end{aligned}$$

Obviously

$$X = \bigcup_{n=0}^{\infty} \bigcup_{j=1}^{\infty} B_{n,j},$$

every set $B_{n,j}$ is closed and every restriction $\Phi_{\restriction B_{n,j}}$ is continuous. □

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