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A STUDY ON DIMENSIONS OF CONTINUOUS MAPPINGS

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ABSTRACT. In Dimension Theory, there are "mapping theorems" that establish relationships between the dimensions of a domain and the range of a continuous mapping. Most of the theorems deal with mappings that satisfy additional conditions, such as being closed mappings. In the "environment" of such studies, the dimensions of continuous mappings have also been studied. In this paper, we introduce and investigate a new notion of dimension for continuous mappings between topological spaces, which is closer to the classical definition of the Lebesgue covering dimension of a space. We discuss various results concerning this dimension. Moreover, we present open questions and proposals of new dimensions of continuous mappings for further studies. They are based on known dimensions of continuous mappings between topological spaces.

1. Introduction and preliminaries

Notions of dimensions of topological spaces, like the covering dimension dim, the quasi covering dimension $\dim_q$, the small inductive dimension ind, the large inductive dimension Ind and the dimension Dind of A. Arhangel'skii, have attracted the interest of many studies, creating an important chapter in Topology, that is constantly developing (see [5, 6, 11, 14–16, 19, 25, 32–34, 43]).

Based on such investigations, the study of topological dimensions for continuous mappings between topological spaces was developed, which is an extension

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on related topics of Topological Dimension Theory. A review on these studies proves that the covering dimension of topological spaces was the motivation to start an interesting theory on topological dimensions for continuous mappings. The notion of the covering dimension is strictly connected with the following meanings.

Given a topological space X, a *cover* of X is a non-empty set of subsets of X, whose union is X. A cover $\mathcal{C}$ of X is said to be *open* if all elements of $\mathcal{C}$ are open subsets of X. Similarly, a cover $\mathcal{C}$ of X is said to be *closed* if all elements of $\mathcal{C}$ are closed subsets of X. A family $\mathcal{V}$ of subsets of X is said to be a *refinement* of a family $\mathcal{C}$ of subsets of X (briefly, $\mathcal{V} \succ \mathcal{C}$) if each element of $\mathcal{V}$ is contained in an element of $\mathcal{C}$. Also, a family $\mathcal{V} = \{V_1, \ldots, V_k\}$ of subsets of X is said to be a *shrinking* of a family $\mathcal{C} = \{U_1, \ldots, U_k\}$ of subsets of X if $V_i \subseteq U_i$ for each $i = 1, \ldots, k$. Additionally, we recall that a cover $\mathcal{V}$ of X is said to be a *star refinement* of another cover $\mathcal{U}$ of X if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $\mathrm{St}(V, \mathcal{V}) \subseteq U$, where $\mathrm{St}(V, \mathcal{V}) = \bigcup\{W \in \mathcal{V} : W \cap V \neq \emptyset\}$. It is well-known that every finite open cover of a normal space has a finite open star refinement (see [14]).

The *order* of a family $\mathcal{V}$ of subsets of X, denoted by $\mathrm{ord}\mathcal{V}$, is defined as follows:

(a) $\mathrm{ord}\mathcal{V} = -1$ if $\mathcal{V}$ consists of the empty set only (and therefore, $X = \emptyset$).

(b) $\mathrm{ord}\mathcal{V} = n$, where $n \in \mathbb{N}$, if the intersection of any $n + 2$ distinct elements of $\mathcal{V}$ is empty and there exist $n+1$ distinct elements of $\mathcal{V}$ whose intersection is not empty.

(c) $\mathrm{ord}\mathcal{V} = \infty$ if for every $n \in \mathbb{N}$, there exist n distinct elements of $\mathcal{V}$ whose intersection is not empty.

Definition 1.1. The *covering dimension*, dim, of a topological space X is defined as follows:

1. $\dim X = -1$ if $X = \emptyset$.
2. $\dim X \leqslant n$, where $n \in \mathbb{N}$, if for every finite open cover $\mathcal{C}$ of X there exists a finite open cover $\mathcal{V}$ of X such that $\mathcal{V} \succ \mathcal{C}$ and $\mathrm{ord}\mathcal{V} \leqslant n$.
3. $\dim X = n$, where $n \in \mathbb{N}$, if $\dim X \leqslant n$ and $\dim X \nleqslant n - 1$.
4. $\dim X = \infty$ if $\dim X \leqslant n$ does not hold for every $n \in \{-1, 0, 1, 2, \ldots\}$.

However, meanings of dimensions have also been studied for mappings between topological spaces, succeeding to insert a totally different approach to Topological Dimension Theory. Especially, given two topological spaces X and Y and a continuous mapping $f : X \to Y$, various dimensions of f and their properties have been investigated. Among them, we state the dimension $\dim f$ via inverse images of points, modifications of it, the covering dimension $\mathrm{cdim} f$ and the partition dimension pdim$f$. In this paper, we insert and study a new notion of dimension for continuous mappings which is closer to that of the covering dimension for topological spaces. Also, we present many open questions and proposals of new dimensions for continuous mappings giving the motivation for future investigations.

Especially, the paper is organized as follows. In Section 2, we insert a new notion of dimension for continuous mappings, denoted by $\dim_{(X,Y)} f$ whenever $f : X \to Y$ is a continuous mapping, and study properties of this dimension. In Section 3, we study other properties of this new dimension and make a special study of it in metric spaces. Finally, in Section 4, we present a framework of well-known notions of dimensions for continuous mappings and open questions that motivate further studies on related topics. The last is based on the dimension dim via inverse images of points, modifications of it, the covering dimension cdim and the partition dimension pdim.

2. A new covering dimension for mappings

In this section, we introduce a new type of dimension for continuous mappings and study its various properties. We state that all results of this section are always referred to continuous mappings and for more conditions for the given topological spaces, whenever it is necessary, we mention them in the corresponding results. Also, $f|A$ means the restriction of a mapping f to a set A. For well-known topological notions that are not presented in the paper, we refer to classical references such as [13].

Definition 2.1. Let $f : X \to Y$ be a continuous mapping from a topological space X to a topological space Y. A family $\mathcal{V}$ of subsets of X is said to be an *f-refinement* of a family $\mathcal{U}$ of subsets of Y, written $\mathcal{V} \succ_f \mathcal{U}$, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq f^{-1}(U) \Leftrightarrow f(V) \subseteq U$.

Definition 2.2. The *covering dimension* of a continuous mapping $f : X \to Y$, $\dim_{(X,Y)} f$, is defined as follows:

1. $\dim_{(X,Y)} f = -1$ if $X = \emptyset$.
2. $\dim_{(X,Y)} f \leqslant n$, where $n \in \mathbb{N}$, if for every finite open cover $\mathcal{U}$ of Y there exists a finite open cover $\mathcal{V}$ of X such that $\mathcal{V} \succ_f \mathcal{U}$ and $\mathrm{ord}\mathcal{V} \leqslant n$.
3. $\dim_{(X,Y)} f = n$, where $n \in \mathbb{N}$, if $\dim_{(X,Y)} f \leqslant n$ and $\dim_{(X,Y)} f \nleqslant n - 1$.
4. $\dim_{(X,Y)} f = \infty$ if the relation $\dim_{(X,Y)} f \leqslant n$ does not hold for every $n \in \{-1, 0, 1, 2, \ldots\}$.

If $X = Y$, then we write $\dim_X f$.

Proposition 2.3. *For every continuous mapping $f : X \to Y$, the following conditions are equivalent:*

1. $\dim_{(X,Y)} f \leqslant n$.
2. *For every finite open cover* $\{U_1, \ldots, U_m\}$ *of* Y *there exists a finite open cover* $\{V_1, \ldots, V_m\}$ *of* X *such that* $V_i \subseteq f^{-1}(U_i)$ *for every* $i = 1, \ldots, m$ *and* $\operatorname{ord}\{V_1, \ldots, V_m\} \leqslant n$.
3. *For every open cover* $\{U_1, \ldots, U_{n+2}\}$ *of* Y *we can find an open cover* $\{V_1, \ldots, V_{n+2}\}$ *of* X *such that* $V_i \subseteq f^{-1}(U_i)$ *for every* $i = 1, \ldots, n+2$ *and* $\bigcap_{i=1}^{n+2} V_i = \emptyset$.

Proof. The proof is similar to the corresponding result of the covering dimension of topological spaces (see, for example, [14, 34, 43]). □

Proposition 2.4. *Let $f : X \to Y$ be a continuous mapping from X to Y. Then,*

$$\dim_{(X,Y)} f \leqslant \min\{\dim X, \dim Y\}.$$

Proof. Let $\dim X = n$, where $n \in \mathbb{N}$, and $\{U_1, \ldots, U_m\}$ be a finite open cover of Y. Then, the family

$$\{f^{-1}(U_1), \ldots, f^{-1}(U_m)\}$$

is a finite open cover of X. Thus, there exists a finite open cover $\{W_1, \ldots, W_m\}$ of X such that $W_i \subseteq f^{-1}(U_i)$ for every $i = 1, \ldots, m$ and $\operatorname{ord}\{W_1, \ldots, W_m\} \leqslant n$.

Let $\dim Y = n$, where $n \in \mathbb{N}$, and $\{U_1, \ldots, U_m\}$ be a finite open cover of Y. Then, there exists a finite open cover $\{V_1, \ldots, V_m\}$ of Y such that $V_i \subseteq U_i$ for every $i = 1, \ldots, m$ and $\operatorname{ord}\{V_1, \ldots, V_m\} \leqslant n$. The family

$$\{f^{-1}(V_1), \ldots, f^{-1}(V_m)\}$$

is a finite open cover of X such that $f^{-1}(V_i) \subseteq f^{-1}(U_i)$ for every $i = 1, \ldots, m$. Since the inverse image preserves intersections, $\operatorname{ord}\{f^{-1}(V_1), \ldots, f^{-1}(V_m)\} \leqslant n$. □

Proposition 2.5. *Let $f : X \to Y$ be a homeomorphism. Then,*

$$\dim_{(X,Y)} f = \dim X = \dim Y.$$

Proof. According to Proposition 2.4, it suffices to prove that

$$\dim X \leqslant \dim_{(X,Y)} f.$$

Let $\dim_{(X,Y)} f = n$, where $n \in \mathbb{N}$, and $\{U_1, \ldots, U_m\}$ be a finite open cover of X. Since f is open and surjective, the family $\{f(U_1), \ldots, f(U_m)\}$ is a finite open cover of Y. Therefore, there exists a finite open cover $\{V_1, \ldots, V_m\}$ of X such that $V_i \subseteq f^{-1}\big(f(U_i)\big) = U_i$ for every $i = 1, \ldots, m$ (since f is injective) and $\operatorname{ord}\{V_1, \ldots, V_m\} \leqslant n$. □

A mapping $f : X \to X$ determines a topology

$$\tau(f) = \{U \subseteq X : f^{-1}(U) \subseteq U\}.$$

A topological space (X, τ) is *functional Alexandroff* [2, 30] if $\tau = \tau(f)$ for some mapping f. Obviously, the mapping $f : \big(X, \tau(f)\big) \to \big(X, \tau(f)\big)$ is continuous.

Proposition 2.6. *Let f be a mapping from the topological space $\big(X, \tau(f)\big)$ to itself. Then,*

$$\dim_X f = \dim X.$$

P r o o f. According to Proposition 2.4, it suffices to prove that

$$\dim X \leqslant \dim_X f.$$

Let $\dim_X f = n$, where $n \in \mathbb{N}$, and $\{U_1, \ldots, U_m\}$ be a finite open cover of $\big(X, \tau(f)\big)$. Then, there exists a finite open cover $\{V_1, \ldots, V_m\}$ of $\big(X, \tau(f)\big)$ such that $V_i \subseteq f^{-1}(U_i) \subseteq U_i$ for every $i = 1, \ldots, m$ and $\mathrm{ord}\{V_1, \ldots, V_m\} \leqslant n$. □

Fact 2.7. *For every constant mapping $f : X \to Y$ we have* $\dim_{(X,Y)} f = 0$.

Example. We study the following spaces and continuous mappings:

(1) Let $X = \{x_1, x_2, x_3\}$. We consider the topology on X generated by the base of minimal open neighbourhoods:

$$\mathbf{U}_{x_1} = \{x_1\},\ \mathbf{U}_{x_2} = \{x_1, x_2\},\ \mathbf{U}_{x_3} = \{x_1, x_3\}.$$

Also let $Y = \{y_1, y_2, y_3\}$ be a copy of X. We consider the continuous mapping $f : X \to Y$ defined as follows: $f(x_1) = f(x_2) = y_1,\ f(x_3) = y_2$. Then, $\dim X = \dim Y = 1$ and $\dim_{(X,Y)} f = 0$.

(2) Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}$ be the real plane and line, respectively, with the usual (Euclidean) topology. We also consider the projection mapping $f : X \to Y$ defined as $f(x, y) = x$. We prove that $\dim_{(X,Y)} f = 1$. Let $\{U_1, \ldots, U_m\}$ be a finite open cover of Y. Without loss of generality, we suppose that $\mathrm{ord}\{U_1, \ldots, U_m\} = 1$. We consider the finite open cover $\{V_1, \ldots, V_m\}$ of X, where

$$V_i = U_i \times \mathbb{R},\ i = 1, \ldots, m.$$

Then, $V_i \subseteq f^{-1}(U_i)$ for every $i = 1, \ldots, m$ and $\mathrm{ord}\{V_1, \ldots, V_m\} = 1$.

Proposition 2.8. *Let $f : X \to Y$ be a continuous mapping from X to Y. Then,*

$$\dim_{(X,Y)} f \leqslant \dim_{(X,f(X))} f.$$

P r o o f. Let $\dim_{(X,f(X))} f = n$, where $n \in \mathbb{N}$, and $\{U_1, \ldots, U_m\}$ be a finite open cover of Y. The family $\{U_1 \cap f(X), \ldots, U_m \cap f(X)\}$ is a finite open cover of $f(X)$. Therefore, there exists a finite open cover $\{V_1, \ldots, V_m\}$ of X such that $f(V_i) \subseteq U_i \cap f(X) \subseteq U_i$ for every $i = 1, \ldots, m$, and $\mathrm{ord}\{V_1, \ldots, V_m\} \leqslant n$. □

Corollary 2.9. *Let $f : X \to Y$ be a continuous mapping from X to Y. Then,*

$$\dim_{(X,Y)} f \leqslant \min\{\dim X, \dim f(X), \dim Y\}.$$

Proof. It follows by Propositions 2.4 and 2.8. □

Proposition 2.10. *Let $f : X \to Y$ be a continuous mapping from X to Y. If $f(X)$ is closed in Y, then*

$$\dim_{(X,f(X))} f = \dim_{(X,Y)} f.$$

Proof. According to Proposition 2.8, it suffices to prove that

$$\dim_{(X,f(X))} f \leqslant \dim_{(X,Y)} f.$$

Let $\dim_{(X,Y)} f = n$, where $n \in \mathbb{N}$, and $\{U_1, \ldots, U_m\}$ be a finite open cover of $f(X)$. We consider open sets $W_1, \ldots, W_m$ in Y such that $U_i = W_i \cap f(X)$ for every $i = 1, \ldots, m$. The family

$$\{W_1, \ldots, W_m, Y \setminus f(X)\}$$

is a finite open cover of Y. Therefore, there exists a finite open cover $\{V_1, \ldots, V_m\}$ of X such that $f(V_i) \subseteq W_i$ for every $i = 1, \ldots, m$ and $\mathrm{ord}\{V_1, \ldots, V_m\} \leqslant n$. Since $f(V_i) \subseteq W_i \cap f(X) = U_i$ for every $i = 1, \ldots, m$, we conclude that $\dim_{(X,f(X))} f \leqslant n$. □

Proposition 2.11. *For every triad (n, m, k), where $k < \min\{n, m\}$, of natural numbers, there exist topological spaces X, Y and a continuous mapping $f : X \to Y$ such that $\dim_{(X,Y)} f = k$, $\dim X = n$, $\dim Y = m$ and $\dim f(X) = k + 1$.*

Proof. Let $X_1 = \{x_0, \ldots, x_k, x_{k+1}, x_{k+2}\}$. We consider the topology τ_{X_1} on X_1 generated by the base of minimal open neighbourhoods:

$$\mathbf{U}_{x_0} = \{x_0\}, \quad \mathbf{U}_{x_1} = \{x_1\}, \quad \mathbf{U}_{x_i} = \{x_0, x_i\},\ i = 2, \ldots, k+2.$$

Also let $Y_1 = \{y_0, \ldots, y_k, y_{k+1}, y_{k+2}\}$. We consider the topology τ_{Y_1} on Y_1 generated by the basis of minimal open neighborhoods:

$$\mathbf{U}_{y_0} = \{y_0\}, \quad \mathbf{U}_{y_i} = \{y_0, y_i\},\ i = 1, \ldots, k+2.$$

We consider two spaces (X_2, τ_{X_2}) and (Y_2, τ_{Y_2}) such that $\dim X_2 = n$ and $\dim Y_2 = m$. Let $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. We consider a fixed point $y_0 \in Y_2$ and the mapping $f : X \to Y$ defined as follows:

$$f(x) = \begin{cases} y_0 & \text{if} \quad x \in X_2, \\ y_i & \text{if} \quad x = x_i. \end{cases}$$

Since $f^{-1}(U) \in \{\emptyset, X_2\}$ for every $U \in \tau_{Y_2}$ and $f^{-1}(U) \in \tau_{X_1}$ for every $U \in \tau_{Y_1}$, by the definition of the topological sum of two spaces, the mapping f is continuous. Moreover, we have

$$\begin{aligned} \dim X &= \max\{\dim X_1, \dim X_2\} &= n, \\ \dim Y &= \max\{\dim Y_1, \dim Y_2\} &= m \end{aligned}$$

and since $f(X) = \{y_0\} \oplus Y_1$, we have $\dim f(X) = \dim Y_1 = k+1$. Finally, it is easy to see that $\dim_{(X,Y)} f = k$. □

Proposition 2.12. *Let $f : X \to Y$ be a continuous mapping from X to Y and $A \subseteq X$. If $f|A : A \to Y$ is the restriction of f to A, then*

$$\dim_{(A,Y)} f|A \leqslant \dim_{(X,Y)} f.$$

Proof. Let $\dim_{(X,Y)} f = n$, where $n \in \mathbb{N}$, and $\{U_1, \ldots, U_m\}$ be a finite open cover of Y. There exists a finite open cover $\{V_1, \ldots, V_m\}$ of X such that $f(V_i) \subseteq U_i$ for every $i = 1, \ldots, m$, and $\operatorname{ord}\{V_1, \ldots, V_m\} \leqslant n$. The family

$$\{V_1 \cap A, \ldots, V_m \cap A\}$$

is a finite open cover of A such that

$$f|A(V_i \cap A) = f(V_i \cap A) \subseteq f(V_i) \subseteq U_i, \quad i = 1, \ldots, m$$

and $\operatorname{ord}\{V_1 \cap A, \ldots, V_m \cap A\} \leqslant n$. Hence, $\dim_{(A,Y)} f|A \leqslant n$. □

Proposition 2.13. *Let $f : X \to Y$ and $g : Y \to Z$ be two continuous mappings. Then,*

$$\dim_{(X,Z)}(g \circ f) \leqslant \dim_{(X,Y)} f.$$

Proof. Let $\dim_{(X,Y)} f = n$, where $n \in \mathbb{N}$, and $\{U_1, \ldots, U_m\}$ be a finite open cover of Z. We consider the open cover

$$\{g^{-1}(U_1), \ldots, g^{-1}(U_m)\} \quad \text{of } Y.$$

There exists a finite open cover $\{V_1, \ldots, V_m\}$ of X such that $f(V_i) \subseteq g^{-1}(U_i)$ for every $i = 1, \ldots, m$ and $\operatorname{ord}\{V_1, \ldots, V_m\} \leqslant n$. We observe that for every $i = 1, \ldots, m$ we have $(g \circ f)(V_i) = g\big(f(V_i)\big) \subseteq g\big(g^{-1}(U_i)\big) \subseteq U_i$. □

Proposition 2.14. *For every continuous mapping $f : X \to Y$ from a normal space X to a space Y, the following conditions are equivalent:*

1. $\dim_{(X,Y)} f \leqslant n$.
2. *For every finite open cover $\mathcal{U}$ of Y there exists a finite closed cover $\mathcal{F}$ of X such that $\mathcal{F} \succ_f \mathcal{U}$ and $\operatorname{ord}\mathcal{F} \leqslant n$.*
3. *For every finite open cover $\{U_1, \ldots, U_m\}$ of Y there exists a finite closed cover $\{F_1, \ldots, F_m\}$ of X such that $F_i \subseteq f^{-1}(U_i)$ for every $i = 1, \ldots, m$ and $\operatorname{ord}\{F_1, \ldots, F_m\} \leqslant n$.*

4. *For every open cover* $\{U_1, \dots, U_{n+2}\}$ *of* Y *we can have a closed cover* $\{F_1, \dots, F_{n+2}\}$ *of* X *such that* $F_i \subseteq f^{-1}(U_i)$ *for every* $i = 1, \dots, n+2$ *and* $\bigcap_{i=1}^{n+2} F_i = \emptyset$.

P r o o f. It follows immediately from Proposition 2.3. □

We recall that a subset L of a topological space X is called a *partition* between two disjoint subsets A and B of X if there exist open subsets U and V of X satisfying the conditions $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$ and $X \setminus L = U \cup V$.

Proposition 2.15. *Let* $f : X \to Y$ *be a continuous mapping from a normal space* X *to a space* Y. *If* $\dim_{(X,Y)} f \leqslant n$, *then for every family*

$$\{(A_1, B_1), \dots, (A_{n+1}, B_{n+1})\}$$

of $n+1$ *pairs of disjoint closed subsets of* Y *there exist closed subsets* $L_1, \dots, L_{n+1}$ *of* X *such that* L_i *is a partition between* $f^{-1}(A_i)$ *and* $f^{-1}(B_i)$ *in the space* X *and* $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

P r o o f. Let $\dim_{(X,Y)} f \leqslant n$ and let $\{(A_1, B_1), \dots, (A_{n+1}, B_{n+1})\}$ be a family of $n+1$ pairs of disjoint closed subsets of Y. Consider the open cover

$$\left\{ Y \setminus A_1, \dots, Y \setminus A_{n+1}, Y \setminus \bigcup_{i=1}^{n+1} B_i \right\} \quad \text{of } Y.$$

By Proposition 2.3, there exists an open cover $\{V_1, \dots, V_{n+2}\}$ of X such that $f(V_i) \subseteq Y \setminus A_i$ for $i = 1, \dots, n+1$, $f(V_{n+2}) \subseteq Y \setminus \bigcup_{i=1}^{n+1} B_i$ and $\bigcap_{i=1}^{n+2} V_i = \emptyset$. Since X is normal, there exists a closed cover $\{F_1, \dots, F_{n+2}\}$ of X such that $F_i \subseteq V_i$ for each $i = 1, \dots, n+2$. Now, we consider the closed subsets $A'_i = (F_{n+2} \setminus V_i) \cup f^{-1}(A_i)$ and $B'_i = F_i \cup f^{-1}(B_i)$ of X for $i = 1, \dots, n+1$. Then,

$$\bigcup_{i=1}^{n+1} F_i \subseteq \bigcup_{i=1}^{n+1} B'_i, \quad F_{n+2} \subseteq \bigcup_{i=1}^{n+1} A'_i \quad \text{and} \quad A'_i \cap B'_i = \emptyset,\ i = 1, \dots, n+1.$$

By Urysohn's lemma, for each index $i = 1, \dots, n+1$ there exists a continuous function $f_i : X \to [0,1]$ such that $f_i(A'_i) \subseteq \{0\}$ and $f_i(B'_i) \subseteq \{1\}$. The subset $L_i = f_i^{-1}(\{\frac{1}{2}\})$ of X is a partition between A'_i and B'_i in the space X. Since $f^{-1}(A_i) \subseteq A'_i$ and $f^{-1}(B_i) \subseteq B'_i$, L_i is also a partition between $f^{-1}(A_i)$ and $f^{-1}(B_i)$ in the space X. Moreover,

$$\bigcap_{i=1}^{n+1} L_i \subseteq \bigcap_{i=1}^{n+1} \big(X \setminus (A'_i \cup B'_i)\big) = X \setminus \bigcup_{i=1}^{n+1} (A'_i \cup B'_i) \subseteq X \setminus \bigcup_{i=1}^{n+2} F_i = \emptyset,$$

proving the proposition. □

Proposition 2.16. *Let $f : X \to Y$ be a continuous mapping from a space X to a normal space Y and assume that for every family*

$$\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$$

of $n+1$ pairs of disjoint closed subsets of Y there exist closed subsets $L_1, \ldots, L_{n+1}$ of X such that L_i is a partition between $f^{-1}(A_i)$ and $f^{-1}(B_i)$ in the space X and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. Then, $\dim_{(X,Y)} f \leqslant n$.

Proof. Let $\{U_1, \ldots, U_{n+2}\}$ be an open cover of Y. Since Y is normal, there exists a closed cover $\{B_1, \ldots, B_{n+2}\}$ of Y such that $B_i \subseteq U_i$ for each $i = 1, \ldots, n+2$. Consider the family

$$\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$$

of $n+1$ pairs of disjoint closed subsets of Y, where $A_i = Y \setminus U_i$ for $i = 1, \ldots, n+1$. By assumption, there exist closed subsets $L_1, \ldots, L_{n+1}$ of X such that L_i is a partition between $f^{-1}(A_i)$ and $f^{-1}(B_i)$ in the space X and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. Therefore, for each $i = 1, \ldots, n+1$ there exist open subsets W_i and V_i of X satisfying the conditions $f^{-1}(A_i) \subseteq W_i$, $f^{-1}(B_i) \subseteq V_i$, $W_i \cap V_i = \emptyset$ and $X \setminus L_i = W_i \cup V_i$. Let $V_{n+2} = f^{-1}(U_{n+2}) \cap \bigcup_{i=1}^{n+1} W_i$. Then, $\{V_1, \ldots, V_{n+2}\}$ is an open cover of X such that $f(V_i) \subseteq U_i$ for each $i = 1, \ldots, n+2$ and $\bigcap_{i=1}^{n+2} V_i = \emptyset$. Thus, by Proposition 2.3, $\dim_{(X,Y)} f \leqslant n$. □

Corollary 2.17. *Let $f : X \to Y$ be a continuous mapping from a normal space X to a normal space Y and n a non-negative integer. The following statements are equivalent:*

1. $\dim_{(X,Y)} f \leqslant n$.
2. *For every family $\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$ of $n+1$ pairs of disjoint closed subsets of Y there exist closed subsets $L_1, \ldots, L_{n+1}$ of X such that L_i is a partition between $f^{-1}(A_i)$ and $f^{-1}(B_i)$ in the space X and $\bigcap_{i=1}^{n+1} L_i = \emptyset$.*
3. *For every family $\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$ of $n+1$ pairs of disjoint closed subsets of Y there exist disjoint open subsets U_i, V_i of X such that $f^{-1}(A_i) \subseteq U_i$, $f^{-1}(B_i) \subseteq V_i$ and $\bigcup_{i=1}^{n+1} (U_i \cup V_i) = X$.*
4. *For every family $\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$ of $n+1$ pairs of disjoint closed subsets of Y there exist disjoint closed subsets F_i, K_i of X such that $f^{-1}(A_i) \subseteq F_i$, $f^{-1}(B_i) \subseteq K_i$ and $\bigcup_{i=1}^{n+1} (F_i \cup K_i) = X$.*

Proof. It follows from Propositions 2.15, 2.16 and [5, Lemma 3.2]. □

3. Additional properties of $\dim_{(X,Y)} f$

In this section, we continue the study of the new dimension of continuous mappings given in Section 2, investigating sum and product properties of it. Also, we present a special study of it in metric spaces. Similarly, all results of this section are referred to continuous mappings and for more conditions of the given topological spaces, whenever it is necessary, we mention them in the corresponding results.

Let $X = X_1 \oplus X_2$ be the sum of two spaces X_1 and X_2. If $f_1 : X_1 \to Y$ is a continuous mapping from X_1 to Y and $f_2 : X_2 \to Y$ is a continuous mapping from X_2 to Y, then we denote by $f_1 \oplus f_2 : X \to Y$ the continuous mapping which is defined as follows:

$$(f_1 \oplus f_2)(x) = \begin{cases} f_1(x) & \text{if } x \in X_1, \\ f_2(x) & \text{if } x \in X_2. \end{cases}$$

Proposition 3.1. *If $f_1 : X_1 \to Y$ is a continuous mapping from X_1 to Y and $f_2 : X_2 \to Y$ is a continuous mapping from X_2 to Y such that $X_1 \cap X_2 = \emptyset$, $\dim_{(X_1,Y)} f_1 \leqslant n$ and $\dim_{(X_2,Y)} f_2 \leqslant n$, then $\dim_{(X_1 \oplus X_2,Y)}(f_1 \oplus f_2) \leqslant n$.*

Proof. Let $\mathcal{U} = \{U_1, \dots, U_k\}$ be a finite open cover of Y. As $\dim_{(X_1,Y)} f_1 \leqslant n$, there exists a finite open cover $\mathcal{V}_1$ of X_1 such that $\mathcal{V}_1 \succ_{f_1} \mathcal{U}$ and $\operatorname{ord}\mathcal{V}_1 \leqslant n$. Similarly, since $\dim_{(X_2,Y)} f_2 \leqslant n$, there exists a finite open cover $\mathcal{V}_2$ of X_2 such that $\mathcal{V}_2 \succ_{f_2} \mathcal{U}$ and $\operatorname{ord}\mathcal{V}_2 \leqslant n$. Then, the set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is the required finite open cover of $X_1 \oplus X_2$ with $\mathcal{V} \succ_f \mathcal{U}$ and $\operatorname{ord}\mathcal{V} \leqslant n$. Thus, $\dim_{(X_1 \oplus X_2,Y)}(f_1 \oplus f_2) \leqslant n$. $\square$

The inductively-open mappings were introduced by A. Arhangel'skiĭ [3]. A mapping f of a space X onto a space Y is called *inductively-open* if there is a subspace Z of X such that $f(Z) = Y$ and the restriction $f|Z$ of f to Z is an open mapping of Z onto Y.

Definition 3.2. Let $f : X \to Y$ be a mapping from a set X to a set Y and $A \subseteq X$. The mapping f is called *A-injective* if for any $x_1, x_2 \in X$, $x_1 \neq x_2$ and $x_1 \in A$ imply $f(x_1) \neq f(x_2)$.

Proposition 3.3. *Let $f : X \to Y$ be a continuous inductively-open mapping from a space X onto a space Y. If X can be represented as a union of two closed subspaces A and B such that f is A-injective, $\dim_{(A,Y)} f|A \leqslant n$ and $\dim_{(B,Y)} f|B \leqslant n$, then $\dim_{(X,Y)} f \leqslant n$.*

Proof. Since f is inductively-open, there exists a subspace Z of X such that $f(Z) = Y$ and the restriction $f|Z$ of f to Z is an open mapping of Z onto Y. Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a finite open cover of Y. Since $\dim_{(A,Y)} f|A \leqslant n$, there exists a finite open cover $\{V_1^A, \dots, V_m^A\}$ of A such that $f(V_i^A) \subseteq U_i$

for every $i = 1, \ldots, m$ and $\mathrm{ord}\{V_1^A, \ldots, V_m^A\} \leqslant n$. For each $i = 1, \ldots, m$ let V_i be an open subset of X such that $V_i^A = A \cap V_i$. We consider the finite open cover $\mathcal{W} = \{W_1, \ldots, W_m\}$ of X, where

$$W_i = \big(f^{-1}(U_i) \setminus A\big) \cup \big(f^{-1}(U_i) \cap V_i\big), \quad i = 1, \ldots, m.$$

It is clear that $f(W_i) \subseteq U_i$, for each $i = 1, \ldots, m$. We prove that

$$\mathrm{ord}\{A \cap W_1, \ldots, A \cap W_m\} \leqslant n. \tag{1}$$

Indeed, for every $i = 1, \ldots, m$ we have

$$A \cap W_i = A \cap \Big(\big(f^{-1}(U_i) \setminus A\big) \cup \big(f^{-1}(U_i) \cap V_i\big)\Big) = f^{-1}(U_i) \cap V_i^A \subseteq V_i^A.$$

Now, we consider the finite open cover

$$\{f(W_1 \cap Z), \ldots, f(W_m \cap Z)\}$$

of Y. Since $\dim_{(B,Y)} f|B \leqslant n$, there exists a finite open cover $\{V_1^B, \ldots, V_m^B\}$ of B such that $f(V_i^B) \subseteq f(W_i \cap Z) \Leftrightarrow V_i^B \subseteq f^{-1}\big(f(W_i \cap Z)\big)$ for every $i = 1, \ldots, m$ and $\mathrm{ord}\{V_1^B, \ldots, V_m^B\} \leqslant n$. For each $i = 1, \ldots, m$ let V_i' be an open subset of X such that $V_i^B = B \cap V_i'$. We consider the finite open cover $\mathcal{H} = \{H_1, \ldots, H_m\}$ of X, where

$$H_i = (W_i \setminus B) \cup \Big(f^{-1}\big(f(W_i \cap Z)\big) \cap V_i'\Big), \quad i = 1, \ldots, m.$$

For every $i = 1, \ldots, m$ we have

$$f(H_i) \subseteq f(W_i) \subseteq U_i.$$

We prove that

$$\mathrm{ord}\{B \cap H_1, \ldots, B \cap H_m\} \leqslant n. \tag{2}$$

Indeed, for every $i = 1, \ldots, m$ we have

$$B \cap H_i = B \cap \bigg((W_i \setminus B) \cup \Big(f^{-1}\big(f(W_i \cap Z)\big) \cap V_i'\Big)\bigg) = f^{-1}\big(f(W_i \cap Z)\big) \cap V_i^B \subseteq V_i^B.$$

Finally, since $X = A \cup B$ and $A \cap H_i \subseteq A \cap W_i$ for every $i = 1, \ldots, m$ (here, we need the condition that the mapping f is A-injective), by relations (1) and (2), we conclude that $\mathrm{ord}\mathcal{H} \leqslant n$. □

Proposition 3.4. *Let $f : X \to Y$ be a continuous closed mapping from a normal space X to a space Y. If X can be represented as the union of a sequence $X_1, X_2, \ldots$ of closed subspaces such that f is injective on X_i and $\dim_{(X_i,Y)} f|X_i \leqslant n$ for every $i = 1, 2, \ldots,$ then $\dim_{(X,Y)} f \leqslant n$.*

Proof. By assumption, the mapping $f|X_i : X_i \to f(X_i)$ is also a closed mapping for each i. Further, since $f|X_i$ is injective, $f|X_i : X_i \to f(X_i)$ is homeomorphism. Hence, by Propositions 2.5 and 2.10, we have

$$\dim X_i = \dim_{(X_i, f(X_i))} f|X_i = \dim_{(X_i,Y)} f|X_i \leqslant n$$

for each $i = 1, 2, \dots$ By the countable sum theorem for dim in normal spaces (see for example [5]), we have $\dim X \leqslant n$. Hence, it follows from Proposition 2.4 that

$$\dim_{(X,Y)} f \leqslant \dim X \leqslant n. \qquad \square$$

Proposition 3.5. *Let $f : X \to Y$ be a continuous mapping from a normal space X to a space Y. If X can be represented as the union of a sequence $X_1, X_2, \dots$ of closed subspaces such that $\dim X_i \leqslant n$ for every $i = 1, 2, \dots$, then $\dim_{(X, f(X))} f \leqslant n$.*

P r o o f. It follows from the countable sum theorem for dim and Proposition 2.4. □

Furthermore, there are some product theorems for the covering dimension. In fact, in the following cases, the inequality $\dim(X \times Y) \leqslant \dim X + \dim Y$ holds (see, for example, [14, 43]):

1. X is compact Hausdorff and Y is paracompact Hausdorff,
2. X is metrizable and Y is perfectly normal Hausdorff.

Proposition 3.6. Let X and Y be Hausdorff spaces and $\mathrm{pr}_X : X \times Y \to X$ the projection onto X. If $\dim(X \times Y) \leqslant \dim X + \dim Y$ holds, then we have that

$$\dim_{(X \times Y, X)} \mathrm{pr}_X \leqslant \dim X.$$

P r o o f. By Proposition 2.4, we have

$$\dim_{(X \times Y, X)} \mathrm{pr}_X \leqslant \min\{\dim(X \times Y), \dim X\}.$$

Thus, by the assumption, it follows that

$$\dim_{(X \times Y, X)} \mathrm{pr}_X \leqslant \min\{\dim(X) + \dim(Y), \dim X\} = \dim X. \qquad \square$$

Remark 1. Let $X \times Y$ be a product space. A subset of the form $U \times V$ in $X \times Y$ is called a *rectangle*. A cover $\mathcal{U}$ of $X \times Y$ is called *rectangular* if each member of $\mathcal{U}$ is a rectangle. A product space $X \times Y$ is said to be *rectangular* if every finite cozero-set cover of $X \times Y$ has a σ-locally finite rectangular cozero refinement. Pasynkov [38] proved the following product theorem for dim: *Let X and Y be Tychonoff spaces. If the product $X \times Y$ is rectangular, then* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

Finally, in the case study of $\dim_{(X,Y)} f$ for metric spaces, we consider two metric spaces (X, d_X) and (Y, d_Y) and we denote the uniform distance in the set $B(X, Y)$ of all bounded continuous mappings of X into Y by d, that is,

$$d(f, g) = \sup\big\{d_Y\big(f(x), g(x)\big) : x \in X\big\}.$$

Proposition 3.7. *Let X be a metric space, Y a compact metric space and $\mathcal{F} \subseteq B(X, Y)$ such that $\dim_{(X,Y)} f \leqslant n$ for every $f \in \mathcal{F}$. If $g \in \mathrm{Cl}(\mathcal{F})$, then $\dim_{(X,Y)} g \leqslant n$.*

P r o o f. Let $g \in \mathrm{Cl}(\mathcal{F})$ and $\mathcal{U}$ be a finite open cover of Y. Since Y is normal, the finite open cover $\mathcal{U}$ of Y has a finite open star refinement, that is, there exists a finite open cover $\mathcal{W}$ of Y such that for every $W \in \mathcal{W}$ there is $U \in \mathcal{U}$ with $\mathrm{St}(W, \mathcal{W}) \subseteq U$. We choose a Lebesgue number $\varepsilon > 0$ for $\mathcal{W}$. Let $B(g, \varepsilon)$ be the open ball of radius ε centered at g, that is,

$$B(g, \varepsilon) = \{f \in B(X, Y) : d(f, g) < \varepsilon\}.$$

Since $g \in \mathrm{Cl}(\mathcal{F})$, there exists $f \in \mathcal{F}$ such that $f \in B(g, \varepsilon)$. Since $\dim_{(X,Y)} f \leqslant n$, there exists a finite open cover $\mathcal{V}$ of X such that $\mathcal{V} \succ_f \mathcal{W}$ and $\mathrm{ord}\mathcal{V} \leqslant n$. It suffices to prove that $\mathcal{V} \succ_g \mathcal{U}$.

Let $V \in \mathcal{V}$. We prove that $g(V) \subseteq U$ for some $U \in \mathcal{U}$. Since $\mathcal{V} \succ_f \mathcal{W}$, there exists $W \in \mathcal{W}$ such that $f(V) \subseteq W \subseteq \mathrm{St}(W, \mathcal{W}) \subseteq U$, for some $U \in \mathcal{U}$. Let $x \in V$. Then, $f(x) \in W$. Moreover, the open ball $B(f(x), \varepsilon)$ is contained in some element W_0 of $\mathcal{W}$. This element W_0 intersects W at $f(x)$. Therefore,

$$B\big(f(x), \varepsilon\big) \subseteq W_0 \subseteq \mathrm{St}\big(W, \mathcal{W}\big) \subseteq U.$$

Finally, since $f \in B(g, \varepsilon)$, we have

$$d_Y\big(f(x), g(x)\big) \leqslant \sup\big\{d_Y\big(f(x), g(x)\big) : x \in X\big\} < \varepsilon.$$

Hence, $g(x) \in B\big(f(x), \varepsilon\big)$ and thus, $g(x) \in U$. Since x is an arbitrary element of V, we conclude that $g(V) \subseteq U$. □

Corollary 3.8. *Let X be a metric space, Y a compact metric space and*

$$\mathcal{F}_n = \{f \in C(X, Y) : \dim_{(X,Y)} f \leqslant n\}, \ n = 0, 1, 2 \ldots,$$

where $C(X, Y)$ denotes the set of all continuous mappings from X into Y. Then, $\mathcal{F}_n$ is closed in $C(X, Y)$.

Corollary 3.9. *Let $(f_k)_{k=1}^{\infty}$ be a sequence of points in $B(X, Y)$, where X is a metric space and Y is a compact metric space, which converges (with respect to the uniform distance) to f. If $\dim_{(X,Y)} f_k \leqslant n$ for every $k = 1, 2, \ldots$, then $\dim_{(X,Y)} f \leqslant n$.*

4. Open questions

Given a continuous mapping $f : X \to Y$ between topological spaces X and Y, we focus on the meaning of dim via the inverses of points. It is well-known that the dimension of a continuous mapping $f : X \to Y$ is usually defined as the supremum of the dimensions of point-inverses $f^{-1}(y)$ over all $y \in Y$. For many years, B. A. Pasynkov (see, for example, [37, 39, 41]) has developed the dimension theory of mappings in this fiberwise treatment. We state that according to the corresponding references, in this section, all mappings are continuous as well.

Definition 4.1 ([1,5,43]). Let $f : X \to Y$ be a continuous mapping. The dimension of f is defined as follows:

$$\dim f = \sup\bigl\{\dim\bigl(f^{-1}(y)\bigr) : y \in f(X)\bigr\}.$$

Among many important results, we present the following result of Pasynkov, on the basis of which many other researches have been developed. For that, a mapping f is n-dimensional means that $\dim f \leqslant n$. The *diagonal product* of two mappings $f : X \to Y$ and $g : X \to Z$ is the mapping $f \times g : X \to Y \times Z$ defined by $(f \times g)(x) = \bigl(f(x), g(x)\bigr)$. Moreover, we shall use the notations $\mathbb{I} = [0, 1]$ and $\mathbb{I}^n$ for the n-dimensional cube equipped with the Euclidean metric.

Proposition 4.2 ([37]). *Let f be an n-dimensional surjection between compact metric spaces X and Y with n and $\dim Y$ finite. Then, there exists a mapping $g : X \to \mathbb{I}^n$ such that $\dim(f \times g) = 0$. Moreover, the set of all such g is dense and G_δ in $C(X, \mathbb{I}^n)$ with respect to the uniform convergence topology.*

Proposition 4.3 ([51]). *Let X and f be as in Proposition 4.2. For each $0 \leqslant k \leqslant n - 1$ there is a σ-compact subset A_k of X such that $\dim A_k \leqslant k$ and $\dim\bigl(f|(X \setminus A_k)\bigr) \leqslant n - k - 1$.*

A lot of papers and researchers have developed a very interesting theory for this dimension. A review on this topic can be found in [4, 8, 9, 12, 17, 18, 27–29, 36, 40–42, 45–50, 52, 53]. Combining the new dimension of Section 2 with this dimension, we have the following result.

Proposition 4.4. *Let $f : X \to Y$ be a continuous mapping from a normal space X to a T_1-space Y. If the image $f(X)$ is countable, then*

$$\dim_{(X, f(X))} f \leqslant \dim f.$$

Proof. It follows by Proposition 3.5. □

Pasynkov inserted the dimension $\dim f$ using the notion of the covering dimension dim. However, there are many other topological dimensions that can be used in order to define new types of dimensions for continuous mappings. This fact leads to the following questions.

Question 1. Given a continuous mapping $f : X \to Y$, can we define new dimensions following the arguments of Pasynkov's studies? Can we study the properties of these dimensions?

Based on the above meaning of $\dim f$ types of it have been also studied. The first step was the following result.

Proposition 4.5 ([19]). *If $f : X \to Y$ is a continuous closed mapping of a metric separable space X onto a metric separable space Y, then*

$$\dim X \leqslant \dim Y + \dim f.$$

The above result has attracted the interest of many researches considering more general spaces X and Y (see, for example, [31,32]), related results in [26,35] and geometrical aspects of it in [22, 54]. In [44], M. Reichaw presented another approach to Proposition 4.5, modifying the dimension dim of mappings as follows.

Definition 4.6 ([44]). Let X and Y be topological spaces and $f : X \to Y$ be a continuous mapping. The dimension $\dim_0$ of f is defined as follows:

$$\dim_0 f = \sup \left\{ \dim f^{-1}(A) : \dim A \leqslant 0,\ A \subset Y \text{ and } A \text{ closed} \right\}.$$

Proposition 4.7 ([44]). *For any closed mapping $f : X \to Y$, $\dim f = \dim_0 f$.*

Question 2. Do there exist more properties/theorems of dimension $\dim_0 f$?

As a different point of view of Proposition 4.5, in [44], considering a class P larger than the class of closed and continuous mappings (stating, for example, the local homeomorphisms with compact point inverses belong to P), a new notion of dimension of mappings is given as follows.

Definition 4.8 ([44]). Let $f : X \to Y$ be a continuous mapping. The dimension $\dim_k$ of f, for $k \geqslant 0$, is defined as follows:

$$\dim_k f = \sup\{\dim f^{-1}(A) : \dim A \leqslant k,\ A \subset Y \text{ and } A \text{ closed}\}.$$

Also, $\dim_{-1} f = \dim X$.

Question 3. Do there exist more properties/theorems of dimension $\dim_k f$, $k \geqslant 1$?

Question 4. Given a continuous mapping $f : X \to Y$, can we define new dimensions following the same arguments as $\dim_0 f$ and $\dim_k f$? Can we study properties of these dimensions?

Additionally, in [23, 24], J. Krzempek extended the classical covering dimension of topological spaces to dimension of closed continuous mappings, and obtained the so-called covering dimension of mappings. Following the arguments for these papers [23, 24], the order $\mathrm{ord}\mathcal{A}$ of a collection $\mathcal{A} = (A_s)_{s \in S}$ of a fixed set X is the largest integer n such that $A_{s_1} \cap A_{s_2} \cap \cdots \cap A_{s_n} \neq \emptyset$ for some distinct indices $s_1, s_2, \ldots, s_n \in S$. If for every n there exists such a non-empty intersection, we write $\mathrm{ord}\mathcal{A} = \infty$. The author also defines $\mathrm{ord} f$ as the order of the collection $f(X)$ indexed by elements of X.

Definition 4.9 ([23, 24]). To every closed mapping f from a normal space X, the *covering dimension of* f, denoted by $\mathrm{cdim}f$, is the least integer n such that every finite open cover of X has a finite closed refinement $\mathcal{V}$ with $\mathrm{ord}f(\mathcal{V}) \leqslant n+1$. If such n's do not exist, we write $\mathrm{cdim}f = \infty$, and if X is empty, we write $\mathrm{cdim}f = -1$.

In the above definition, the set $f(\mathcal{V})$ denotes the image of the family $\mathcal{V}$, that is $f(\mathcal{V}) = \{f(V) : V \in \mathcal{V}\}$. Also, since images of distinct elements of $\mathcal{V}$ may be equal, then such an image is counted two or more times among the elements of the collection $f(\mathcal{V})$. Some results of this dimension are given as follows and are always referred to closed continuous mappings.

Proposition 4.10 ([23]). *The following are satisfied:*

1. *If X is a normal space, then* $\mathrm{dim}X = \mathrm{cdim}id_X$, *where id_X denotes the identity mapping on X.*
2. $\mathrm{dim}X \leqslant \mathrm{cdim}f$ *for mappings f from normal spaces X.*
3. $\mathrm{dim}f(X) \leqslant \mathrm{cdim}f$ *for mappings f from normal spaces X.*

Proposition 4.11 ([23]). *(The subspace theorem for cdim) If f is a closed mapping from a normal space X and M is a closed subset of X, then*

$$\mathrm{cdim}f|M \leqslant \mathrm{cdim}f.$$

Proposition 4.12 ([23]). *(The countable sum theorem for cdim) Suppose that f is a closed mapping from a normal space X. If X is the union of countably many closed subsets M_i, $i = 1, 2, \ldots$, such that* $\mathrm{cdim}f|M_i \leqslant n$ *for each $i = 1, 2, \ldots$, and every point-inverse $f^{-1}(y)$ is contained in some M_i, then* $\mathrm{cdim}f \leqslant n$.

Question 5. Do there exist more properties (like product theorems) of $\mathrm{cdim}f$?

In [23, 24], we can also find another meaning of dimension of mappings, called the partition dimension. We remind that by a *partition* of a space X we mean a two element cover (U, V) of X and by a *partition system* on X we mean a collection $(\mathcal{U}, \mathcal{V}) = (U_s, V_s)_{s \in S}$ of partitions (U_s, V_s) of X. Especially, a partition system $(\mathcal{U}, \mathcal{V})$ is *closed* (respectively, *open*) if all sets U_s and V_s are closed (respectively, open). A partition system $(W_s, T_s)_{s \in S}$ on X is a *shrinking* of $(\mathcal{U}, \mathcal{V})$ if every partition (W_s, T_s) is a shrinking of (U_s, V_s).

Definition 4.13 ([23, 24]). To every closed mapping f from a non-empty normal space X we assign the *partition dimension of* f, denoted by $\mathrm{pdim}f$, which is the least non-negative integer n such that every open partition system $(U_i, V_i)_{i=0}^n$ on X has a closed shrinking $(F_i, G_i)_{i=0}^n$ with $\bigcap_{i=0}^n f(F_i \cap G_i) = \emptyset$. If such n's do not exist, we write $\mathrm{pdim}f = \infty$ and if X is empty, we write $\mathrm{pdim}f = -1$.

Some results of this dimension are given as follows and are always referred to closed continuous mappings.

Proposition 4.14 ([23]). *The following are satisfied:*

1. *If X is a normal space, then* $\dim X = \mathrm{pdim}\, id_X$*, where id_X denotes the identity mapping on X.*
2. $\dim X \leqslant \mathrm{pdim} f$ *for mappings f from normal spaces X.*
3. *If f is a closed-and-open mapping from a normal space X, then*

$$\dim f(X) \leqslant \mathrm{pdim} f.$$

Proposition 4.15 ([23]). *(The subspace theorem for pdim) If f is a closed mapping from a normal space X and M is a closed subset of X, then*

$$\mathrm{pdim} f|M \leqslant \mathrm{pdim} f.$$

Proposition 4.16 ([23]). *(The countable sum theorem for pdim). Suppose that f is a closed mapping from a normal space X. If X is the union of countably many closed subsets M_i, $i = 1, 2, \ldots$, such that* $\mathrm{pdim} f|M_i \leqslant n$ *for each $i = 1, 2, \ldots$, and every point-inverse $f^{-1}(y)$ is contained in some M_i, then* $\mathrm{pdim} f \leqslant n$.

Question 6. Do there exist more properties (like product theorems) of $\mathrm{pdim} f$?

Combining the new dimension of Section 2 with the above dimensions $\mathrm{cdim} f$ and $\mathrm{pdim} f$, we get the following results.

Proposition 4.17. *If X is a normal space, then*

$$\dim_X id_X = \mathrm{cdim}\, id_X = \mathrm{pdim}\, id_X,$$

where id_X is the identity mapping on X.

Proof. It follows by Propositions 2.5, 4.10 and 4.14. □

Proposition 4.18. *If f is a continuous closed mapping from a normal space X to itself, then:*

1. $\dim_X f \leqslant \mathrm{cdim} f$.
2. $\dim_X f \leqslant \mathrm{pdim} f$.

Proof. It follows by Propositions 2.4, 4.10 and 4.14. □

Also, we provide proposals of new notions of dimensions for continuous mappings. As in [16], a *quasi cover* of X is a non-empty set of subsets of X, whose union is dense in X. A quasi cover $\mathcal{C}$ of X is said to be *open* if all elements of $\mathcal{C}$ are open in the space X. Two quasi covers are called *similar* if their unions are the same dense set. Based on the quasi covers, the quasi covering dimension, $\dim_q$, for topological spaces was studied in [16], proving among other results that it is greater than or equal to the covering dimension. Inspired by this dimension, we present the following notions.

Definition 4.19. Let $f : X \to Y$ be a continuous mapping from a topological space X to a topological space Y. A family $\mathcal{V}$ of subsets of X is called f*-similar* of a family $\mathcal{U}$ of subsets of Y, written $\mathcal{V} \sim_f \mathcal{U}$, if the unions of $\mathcal{V}$ and $f^{-1}(\mathcal{U})$ are the same dense subset of X, where

$$f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\},$$

that is, the families $\mathcal{V}$ and $f^{-1}(\mathcal{U})$ are similar.

Definition 4.20. The *quasi covering dimension* of a continuous mapping $f : X \to Y$, denoted by $\dim_{q(X,Y)} f$, is defined as follows:

1. $\dim_{q(X,Y)} f = -1$ if $X = \emptyset$.
2. $\dim_{q(X,Y)} f \leqslant n$, where $n \in \mathbb{N}$, if for every finite open quasi cover $\mathcal{U}$ of Y there exists a finite open quasi cover $\mathcal{V}$ of X such that $\mathcal{V} \sim_f \mathcal{U}$, $\mathcal{V} \succ_f \mathcal{U}$ and $\mathrm{ord}\mathcal{V} \leqslant n$.
3. $\dim_{q(X,Y)} f = n$, where $n \in \mathbb{N}$, if $\dim_{q(X,Y)} f \leqslant n$ and $\dim_{q(X,Y)} f \nleqslant n - 1$.
4. $\dim_{q(X,Y)} f = \infty$, if the relation $\dim_{q(X,Y)} f \leqslant n$ does not hold for every $n \in \{-1, 0, 1, 2, \ldots\}$.

If $X = Y$, then we write $\dim_{q(X)} f$.

Question 7. Does there exist a corresponding theory (similar to $\dim_{(X,Y)} f$ of Section 2) for $\dim_{q(X,Y)} f$? Can we have subspace, union, product theorems for this dimension?

Another dimension for topological spaces that has been studied is the so-called Dind of A. Arhangel'skii, [6, 7, 11, 15, 25]. Inspired by this dimension, we present the following notion.

Definition 4.21. The dimension of a continuous mapping $f : X \to Y$, denoted by $\mathrm{Dind}_{(X,Y)} f$, is defined as follows:

1. $\mathrm{Dind}_{(X,Y)} f = -1$, if $X = \emptyset$.
2. $\mathrm{Dind}_{(X,Y)} f \leqslant n$, where $n \in \mathbb{N}$, if for any finite open cover $\mathcal{U}$ of Y, there exists a finite family $\mathcal{V}$ of pairwise disjoint open subsets of X such that $V \succ_f \mathcal{U}$ and $\mathrm{Dind}_{(A,Y)} f|A \leqslant n - 1$, where $A = X \setminus \bigcup\{V : V \in \mathcal{V}\}$.
3. $\mathrm{Dind}_{(X,Y)} f = n$, where $n \in \mathbb{N}$, if $\mathrm{Dind}_{(X,Y)} f \leqslant n$ and $\mathrm{Dind}_{(X,Y)} f \nleqslant n - 1$.
4. $\mathrm{Dind}_{(X,Y)} f = \infty$, if the relation $\mathrm{Dind}_{(X,Y)} f \leqslant n$ does not hold for every $n \in \{-1, 0, 1, 2, \ldots\}$.

If $X = Y$, then we write $\mathrm{Dind}_X f$.

Question 8. Does there exist a corresponding theory (similar to $\dim_{(X,Y)} f$ of Section 2) for $\mathrm{Dind}_{(X,Y)} f$? Can we have subspace, union, product theorems for this dimension?

Finally, we make a special note on the so-called universality problem. This problem focuses on finding universal elements in classes of topological spaces, frames and continuous mappings (see [10,20,21]. The study of universal elements is considered to be an important branch of Topology since these elements are considered to be the "representative" of their corresponding classes. Especially, this "representative" gives information for all members of the class. Thus, if we consider classes of continuous mappings which are determined by dimensions given in this study, we get the following question.

Question 9. Can we have a study of the existence of universal elements in classes of continuous mappings which are determined by dimensions given in this paper?

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