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ON θ-HUREWICZ AND α-HUREWICZ TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we introduced α-Hurewicz and θ-Hurewicz properties in a topological space X and investigated their relationship with other selective covering properties. We have shown that for any extremally disconnected semi-regular spaces, the properties: Hurewicz, semi-Hurewicz, α-Hurewicz, θ-Hurewicz, almost-Hurewicz, nearly Hurewicz and mildly Hurewicz are equivalent. We have also proved that for an extremally disconnected space X, every finite power of X has the θ-Hurewicz property if and only if X has the selection principle $U_{fin}(\theta\text{-}\Omega, \theta\text{-}\Omega)$. The preservation under several types of mappings of α--Hurewicz and θ-Hurewicz properties are also discussed. Also, we have shown that if X is a mildly Hurewicz subspace of ω^ω, than X is bounded.

1. Introduction and Preliminaries

The classical Hurewicz property has a long history going back to the paper [12]. A topological space X has he Hurewicz property if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ such that for each $x \in X$, $x \in \bigcup \mathcal{B}_k$ for all but finitely many k. The Hurewicz property is weaker than the σ-compactness and stronger than the Lindelöf property.

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Recently, several weak variants of the Hurewicz property have been studied after applying the interior and closure operators in the definition of the Hurewicz property. Other ways have also been investigated where a sequence of open covers is replaced with generalized open sets. To study variants of Hurewicz spaces, the readers can see [2, 3, 13, 16–18, 29].

In [11, 15], the authors investigated the various classes of the Menger covering properties. Similarly, in this paper, we investigate the covering properties, namely α-Hurewicz, θ-Hurewicz, which are analogous to the classical Hurewicz property.

The following generalizations of open sets will be used for definitions of variations on the Hurewicz property:

A subset A of a topological space X is said to be:

- θ-open [31] if for every x in A, there exists an open set B in X such that $x \in B \subset \mathrm{Cl}(B) \subset A$.
- α-open [26] if $A \subset \mathrm{Int}\big(\mathrm{Cl}(\mathrm{Int}(A))\big)$, or equivalently, if there exists an open set B in X, such that $B \subset A \subset \mathrm{Int}\big(\mathrm{Cl}(B)\big)$. The complement of α-open set is α-closed. Moreover, a set A is α-closed in X if $\mathrm{Cl}\big(\mathrm{Int}(\mathrm{Cl}(A))\big) \subseteq A$.
- semi-open [20] if there exists an open set B in X such that $B \subset A \subset \mathrm{Cl}(B)$, or equivalently, if $A \subset \mathrm{Cl}\big(\mathrm{Int}(A)\big)$. The set $\mathrm{SO}(X)$ denotes the collection of all semi-open sets. The complement of a semi-open set is semi-closed, $\mathrm{sCl}(A)$ denotes the semi-closure of A, $\mathrm{sCl}(A)$ is the intersection of all semi--closed sets containing A. $A \subseteq X$ is semi-closed if and only if $\mathrm{sCl}(A) = A$.

Clearly, we have:

$$\text{clopen} \Rightarrow \theta\text{-open} \Rightarrow \text{open} \Rightarrow \alpha\text{-open} \Rightarrow \text{semi-open}.$$

par A space X is said to be α-compact [23] (θ-compact [19]) if every α-open (θ-open) cover of X has a finite subcover.

Recall that a space X is said to be semi-regular [5] if for each $x \in X$ and for each semi-closed set A such that $x \notin A$, there exist disjoint semi-open sets B and C of X such that $x \in B$ and $A \subset C$.

Lemma 1.1 ([5]). *For a space X, the following statements are equivalent:*

(i) *X is semi-regular;*

(ii) *For each $x \in X$ and $A \in \mathrm{SO}(X)$ such that $x \in A$, there exists a $B \in \mathrm{SO}(X)$ such that $x \in B \subset \mathrm{sCl}(B) \subset A$.*

Definition 1.2. A space X is called semi-Hurewicz [16] (resp. mildly Hurewicz [17], θ-Hurewicz, α-Hurewicz) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of semi-open (resp. clopen, θ-open, α-open) covers of X, there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$ such that for each $k \in \mathbb{N}$, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ and for each $x \in X$, $x \in \bigcup \mathcal{B}_k$ for all but finitely many k.

Evidently, we have the following implications:

semi-Hurewicz $\Rightarrow$ α-Hurewicz $\Rightarrow$ Hurewicz $\Rightarrow$ θ-Hurewicz $\Rightarrow$ mildly Hurewicz.

The above properties are also written in the form of a selection principle. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of the space X. Then, the space X satisfies the selection principle $U_{fin}(\mathcal{A}, \mathcal{B})$ if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ in $\mathcal{A}$ there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$ where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ such that $\{\cup\mathcal{B}_k : k = 1, 2, 3, ...\}$ is in $\mathcal{B}$ [28]. An infinite cover $\mathcal{C}$ of X is called γ-cover (resp. c-γ-cover, θ-γ-cover, α-γ-cover, s-γ-cover) if each element of $\mathcal{C}$ is open (resp. clopen, θ-open, α-open, semi-open) such that for each $x \in X$, the set $\{U \in \mathcal{C} : x \notin U\}$ is finite. Let Γ, c-Γ θ-Γ, α-Γ, s-Γ denote the collection of all γ, c-γ, θ-γ, α-γ, s-γ covers of X, respectively, and $\mathcal{O}$, $\mathcal{CO}$, θ-$\mathcal{O}$, α-$\mathcal{O}$, s-$\mathcal{O}$ denote the collection of all open, clopen, θ-open, α-open, semi-open covers of a space X, respectively. Then, the Hurewicz, mildly Hurewicz, θ-Hurewicz, α-Hurewicz, semi-Hurewicz property of X is equivalent to selection principles $U_{fin}(\mathcal{O}, \Gamma)$, $U_{fin}(\mathcal{CO},\ c\text{-}\Gamma)$, $U_{fin}(\theta\text{-}\mathcal{O},\ \theta\text{-}\Gamma)$, $U_{fin}(\alpha\text{-}\mathcal{O},\ \alpha\text{-}\Gamma)$, $U_{fin}(s\text{-}\mathcal{O},\ s\text{-}\Gamma)$, respectively. In this paper, we study the α-Hurewicz and the θ-Hurewicz properties in details.

Throughout the paper, a space X and (X, τ) mean a topological space and $|X|$ denotes the cardinality of X. For a subset A of a space X, $\mathrm{Int}(A)$ and $\overline{A}$ or $\mathrm{Cl}(A)$ denote the interior and closure of A, respectively. Further, ω and ω_1 denote the first infinite cardinal and uncountable cardinal, respectively.

2. θ-Hurewicz Spaces and α-Hurewicz Spaces

First, recall that the family of all θ-open (resp. α-open) sets of a space (X, τ) are form topologies on X denoted by τ_θ [21] (resp. τ_α [26]). Further, $\tau_\theta \subseteq \tau \subseteq \tau_\alpha$. The role of θ-open and α-open sets have been invastigated in many papers (see, [19, 23])

Clearly, a space (X, τ) is θ-Hurewicz (resp. α-Hurewicz) if and only if the space (X, τ_θ) (resp. (X, τ_α)) is Hurewicz.

Theorem 2.1. *Every countable space X has the α-Hurewicz property.*

Proof. Let $X = \{x_1, x_2, \ldots, x_n, \ldots, \}$ be a countable space. Let $(\mathcal{A}_k)_{k\in\mathbb{N}}$ be a sequence of α-open covers of X. For each $k \in \mathbb{N}$, consider

$$\mathcal{B}_k = \{A_{k,1}, A_{k,2}, \ldots, A_{k,k}\},$$

where for each $i \in \{1, 2, \ldots, k\}$, $A_{k,i}$ is α-open such that $x_i \in A_{k,i}$. Then, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ and for each $x \in X$, $x \in \bigcup\mathcal{B}_k$ for all but finitely many k. $\square$

Similarly, we can prove that every countable space has the θ-Hurewicz property.

Example.

(1) Every α-compact space is α-Hurewicz. However, the converse is not true. The real line $\mathbb{R}$ with the cocountable topology is α-Hurewicz being semi--Hurewicz [16], but it is not α-compact since every α-compact space is compact.

(2) Every θ-compact space is θ-Hurewicz. However, the converse is not true. Let X be a countably infinite discrete space. Then, the space X has the θ-Hurewicz property, but the θ-open cover $\{\{x\} : x \in X\}$ has no finite subcover.

(3) The real line $\mathbb{R}$ is a Hurewicz space, but it is not semi-Hurewicz [16].

(4) The Sorgenfrey line S does not have the α-Hurewicz property because it does not have the Hurewicz property.

Example. Let A be a nonempty finite subset of an uncountable set X. Then, $\tau = \{\phi, A, X\}$ is a topology on X. Clearly, the space (X, τ) is Hurewicz. Moreover, the sets of the forms $A \cup \{p\}$, for $p \in X \setminus A$, are α-open in (X, τ). For each $k \in \mathbb{N}$, put $\mathcal{A}_k = \{A \cup \{p\} : p \in X \setminus A\}$. Then, the sequence $(\mathcal{A}_k : k \in \mathbb{N})$ witnesses that (X, τ) is not an α-Hurewicz space because the cover $\mathcal{A}_k$ does not have a countable subcover.

Example. Let p be a fixed point of an uncountable set X. Then, $\tau_p = \{O \subseteq X : p \in O\}$ together with an empty set is an uncountable particular point topology on X. In [30], it is shown that the space X is not Lindelöf, so X cannot be Hurewicz since every Hurewicz space is Lindelöf. Note that the space X is the only closed set containing p. Then, $\mathrm{Cl}(A) = X$ for each $A \neq \emptyset$, $A \in \tau_p$. Hence, ϕ and X are the only θ-open sets. Therefore, X is θ-Hurewicz space.

A space X is said to be nearly Hurewicz [2] (resp. almost Hurewicz [29]) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X, there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each $k \in \mathbb{N}$, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ such that for each $x \in X$, $x \in \cup\{\mathrm{Int}(\mathrm{Cl}(B)) : B \in \mathcal{B}_k\}$ (resp. $x \in \cup\{\mathrm{Cl}(B) : B \in \mathcal{B}_k\}$) for all but finitely many k.

Evidently, from the definitions, the following implications follow:

$$\text{Hurewicz} \Rightarrow \text{nearly Hurewicz} \Rightarrow \text{almost Hurewicz}.$$

The following theorem describes a relation between almost Hurewicz and θ-Hurewicz space.

Theorem 2.2. *Every almost Hurewicz space is θ-Hurewicz.*

P r o o f. Let X be an almost Hurewicz space and $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X. Then, for each $k \in \mathbb{N}$ and each $x \in X$, there is an open set $B_{x,k}$ such that $x \in B_{x,k} \subset \mathrm{Cl}(B_{x,k}) \subset A_k$ for some $A_k \in \mathcal{A}_k$. For each k, put $\mathcal{B}_k = \{B_{x,k} : x \in X\}$. Then, each $\mathcal{B}_k$ is an open cover of X. Since X is an almost Hurewicz, for each $k \in \mathbb{N}$, there is a finite subset $\mathcal{B}'_k$ of $\mathcal{B}_k$ such that $x \in X$, $x \in \cup\{\mathrm{Cl}(B') : B' \in \mathcal{B}'_k\}$ for all but finitely many k. Since for each $B' \in \mathcal{B}'_k$, there is a $A'_{k,B'} \in \mathcal{A}_k$ such that $\mathrm{Cl}(B') \subset A'_{k,B'}$. Let

$$\mathcal{A}'_k = \{A_{k,B'} \in \mathcal{A}_k : B' \in \mathcal{B}'_k\}.$$

Then, the sequence $(\mathcal{A}'_k : k \in \mathbb{N})$ witnesses that X is θ-Hurewicz. $\square$

Next, we determine a class of spaces in which the above variants of the Hurewicz property are equivalent. Recall that a space X is called extremally disconnected if the closure of an open set is open.

Theorem 2.3. *For an extremally disconnected semi-regular space X, the following statements are equivalent:*

(1) *X is semi-Hurewicz;*

(2) *X is α-Hurewicz;*

(3) *X is Hurewicz;*

(4) *X is nearly Hurewicz;*

(5) *X is almost Hurewicz;*

(6) *X is θ-Hurewicz;*

(7) *X is mildly Hurewicz.*

P r o o f. Already proven (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7). For (7) $\Rightarrow$ (1), let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of semi-open covers of X. Then, by Lemma 1.1, for each $x \in X$, we have a $B_{k,x} \in \mathrm{SO}(X)$ such that $x \in B_{k,x} \subset \mathrm{sCl}(B_{k,x}) \subset A$ for some $A \in \mathcal{A}_k$. For $k \in \mathbb{N}$, put $\mathcal{B}_k = \{B_{k,x} : x \in X\}$. Then, $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of semi-open covers of X. As X is extremally disconnected, by [14, Proposition 4.1], we have $B \subset \mathrm{Int}\big(\mathrm{Cl}(B)\big)$. For each $B \in \mathrm{SO}(X)$, $\mathrm{Cl}\big(\mathrm{Int}(\mathrm{Cl}(B))\big)$ is clopen in X. Put $\mathcal{C}_k = \{\mathrm{Cl}\big(\mathrm{Int}(\mathrm{Cl}(B))\big) : B \in \mathcal{B}_k\}$. Then, $(\mathcal{C}_k : k \in \mathbb{N})$ is a sequence of clopen covers of X. As X is mildly Hurewicz, there exists a sequence $(\mathcal{C}'_k : k \in \mathbb{N})$, where for each k, $\mathcal{C}'_k$ is a finite subset of $\mathcal{C}_k$ such that for $x \in X$, $x \in \cup\mathcal{C}'_k$ for all but finitely many k. Observe that for each subset A of X, $\mathrm{Int}(\mathrm{Cl}(A)) \subset \mathrm{sCl}(A)$ and from the extremal disconnectedness of X, $\mathrm{sCl}(A) = \mathrm{Cl}(A)$ for each $A \in \mathrm{SO}(X)$. From the above construction, for each $C' \in \mathcal{C}'_k$ we have a $A_{C'} \in \mathcal{A}_k$ such that $C' \subset A_{C'}$. Then, for $k \in \mathbb{N}$, let

$\mathcal{A}'_k = \{A_{C'} : C' \in \mathcal{C}'_k\}$. Hence the sequence $(\mathcal{A}'_k : k \in \mathbb{N})$ witnesses that X is semi-Hurewicz. $\square$

In the following examples, we show that the extremal disconnectedness and semi-regularity are necessary conditions in Theorem 2.3.

Example. Consider a real line $\mathbb{R}$ with the usual topology. Then, $\mathbb{R}$ is a semi-regular mildly Hurewicz space, but it is not an extremally disconnected space. On the other hand, $\mathbb{R}$ is not semi-Hurewicz [16].

Example. Let A be a nonempty finite subset of an uncountable set X. Then, $\tau = \{\phi, A, X\}$ is a topology on X. Then, X is an extremally disconnected space. Moreover, X is a mildly Hurewicz space, but does not hold the semi-Hurewicz property (see, Example 2).

In an extremally disconnected space, zero-dimensionality and semi-regularity are equivalent ([27], Theorem 6.4). We have the following corollary:

Corollary 2.4. *For an extremally disconnected, zero-dimensional space X, the following statements are equivalent:*

(1) *X is semi-Hurewicz;*

(2) *X is α-Hurewicz;*

(3) *X is Hurewicz;*

(4) *X is nearly Hurewicz;*

(5) *X is almost Hurewicz;*

(6) *X is θ-Hurewicz;*

(7) *X is mildly Hurewicz.*

A space X is called S-paracompact [1] if for every open cover of X there exists a locally finite semi-open refinement. A S-paracompact Hausdorff space X is semi-regular [1]. Hence, the properties mentioned in Theorem 2.3 are also equivalent for an extremally disconnected S-paracompact Hausdorff spaces.

It is known that the Stone-Čech compactification of a discrete space is extremally disconnected compact Hausdorff space. Thus, the class of Stone-Čech compactification of discrete spaces is contained in the class of extremally disconnected S-paracompact Hausdorff spaces, and it turns out to be a subclass of extremally disconnected semi-regular spaces.

Theorem 2.5. *For a space X, the following statements are equivalent:*

(1) *X has the θ-Hurewicz property;*

(2) *X satisfies $U_{fin}(\theta\text{-}\Omega, \theta\text{-}\mathcal{O})$.*

Proof.

1⇒ 2.: It follows from the fact that each θ-γ-cover of X is a θ-open cover of X.

2⇒ 1.: Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X.

Let $\mathbb{N} = Y_1 \cup Y_2 \cup \cdots \cup Y_m \cup \cdots$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets. For each k, let $\mathcal{B}_k$ contain all sets of the form

$$A_{k_1} \cup A_{k_2} \cup \cdots \cup A_{k_n}, \quad k_1 \leq \cdots \leq k_n,$$

$$k_i \in Y_k, \quad A_{k_i} \in \mathcal{A}_k, \quad i \leq n, \quad n \in \mathbb{N}.$$

Then, for each k, $\mathcal{B}_k$ is a θ-ω-cover of X. Applying $U_{fin}(\theta\text{-}\Omega, \theta\text{-}\mathcal{O})$ on the sequence $(\mathcal{B}_k : k \in \mathbb{N})$, there is a sequence $(\mathcal{C}_k : k \in \mathbb{N})$, where for each k, $\mathcal{C}_k$ is a finite subset of $\mathcal{B}_k$ such that $x \in X$ $x \in \cup\mathcal{C}_k$ for all but finitely many k. Assume that $\mathcal{C}_k = \{C_k^1, \ldots, C_k^{m_k}\}$, then, by the above construction,

$$C_k^i = A_k^{k_{i_1}} \cup, \ldots, \cup A_k^{k_{i_n}}, \quad C_k^i \in \mathcal{C}_k.$$

Thus for each k, we have a finite subset $\mathcal{A}'_k$ of $\mathcal{A}_k$ such that $\cup\mathcal{C}_k \subseteq \cup\mathcal{A}'_k$. Hence, X has the θ-Hurewicz property. □

On the similar lines, we can prove that a space X has the α-Hurewicz property if and only if X satisfies the selection principle $U_{fin}(\alpha\text{-}\Omega, \alpha\text{-}\mathcal{O})$.

Theorem 2.6. *If each finite power of the space X is θ-Hurewicz, then X satisfies $U_{fin}(\theta\text{-}\Omega, \theta\text{-}\Omega)$.*

Proof. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open θ-ω-covers of X. For each $l \in \mathbb{N}$, we put $\mathcal{B}_k = \{A^l : A \in \mathcal{A}_k\}$. For each $l \in \mathbb{N}$, applying the θ-Hurewicz property to the sequence $(\mathcal{B}_k : k \in \mathbb{N})$ of θ-open covers of X^l, for each $k \in \mathbb{N}$ we have finite subfamilies $\mathcal{C}_k$ of $\mathcal{B}_k$ such that $x \in X^l$, $x \in \cup C_k$ for all but finitely many k. For $k \in \mathbb{N}$, let $\mathcal{A}'_k = \{A \in \mathcal{A}_k : A^l \in \mathcal{C}_k\}$. Then, the sequence $(\mathcal{A}'_k : k \in \mathbb{N})$ witnesses that X satisfies $U_{fin}(\theta\text{-}\Omega, \theta\text{-}\Omega)$. □

In a similar way, we can prove that if each finite power of the space X is α-Hurewicz, then X satisfies $U_{fin}(\alpha\text{-}\Omega, \alpha\text{-}\Omega)$.

Lemma 2.7 ([11]). *Let X be an extremally disconnected space. Then, for each θ-ω-cover $\mathcal{A}$ of X^k, $k \in \mathbb{N}$, there exists a θ-ω-cover $\mathcal{B}$ of X such that the θ-open cover $\{B^k : B \in \mathcal{B}\}$ of X^k refines $\mathcal{A}$.*

Theorem 2.8. *Let X be an extremally disconnected space. If X has the property $U_{fin}(\theta\text{-}\Omega, \theta\text{-}\Omega)$, then for each $n \in \mathbb{N}$, X^n also has this property.*

Proof. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-ω-covers of X^n. Then, by Lemma 2.7, there exists a θ-ω-cover $\mathcal{B}_k$ of X such that $\{B^n : B \in \mathcal{B}_k\}$ refines $\mathcal{A}_k$. Apply the condition $U_{\text{fin}}(\theta\text{-}\Omega, \theta\text{-}\Omega)$ of X on the sequence $(\mathcal{B}_k : k \in \mathbb{N})$, then, for each $k \in \mathbb{N}$, there exists a finite subset $\mathcal{C}_k$ of $\mathcal{B}_k$ such that $\{\cup C_k : k \in \mathbb{N}\}$ forms a θ-ω-cover of X. Since $\{B^n : B \in \mathcal{B}_k\}$ refines $\mathcal{A}_k$, for each $C \in \mathcal{C}_k$, we have $A_C \in \mathcal{A}_k$ such that $C^n \subset A_C$. For $k \in \mathbb{N}$, let

$$\mathcal{A}'_k = \{A_C \in \mathcal{A}_k : C \in \mathcal{C}_k\}.$$

Thus, the sequence $(\mathcal{A}'_k : k \in \mathbb{N})$ witnesses that X^n has the property $U_{\text{fin}}(\theta\text{-}\Omega, \theta\text{-}\Omega)$. □

Thus, from Theorem 2.5, Theorem 2.6 and Theorem 2.8, we obtained the following corollary.

Corollary 2.9. *Let X be an extremally disconnected space. Then, every finite power of X is θ-Hurewicz if and only if X satisfies $U_{fin}(\theta\text{-}\Omega, \theta\text{-}\Omega)$.*

3. Preservation in Subspaces and Mappings

In this section, we analyse the properties of α-Hurewicz and θ-Hurewicz spaces. We investigate the behaviour of these properties under subspaces and various type of mappings. In the following example, we show that α-Hurewicz is not a hereditary property.

Example. Let x_0 be a fixed point of an uncountable set X. Then, the family

$$\tau = \{A \subset X : x_0 \notin A\} \cup \{A \subset X : X \setminus A \quad \text{is finite}\}$$

of subsets of X forms a topology on X. It is easy to prove that the space (X, τ) is α-Hurewicz. Consider the subspace $Y = X \setminus \{x_0\}$ of (X, τ). Then, one point set $\{x\}, x \in Y$ is α-open in Y. Then, the α-open cover $\mathcal{A} = \{\{x\} : x \in Y\}$ of Y has no countable subcover. Hence, the subspace Y of the space (X, τ) is not α-Hurewicz.

Note that Y is also an open (α-open) subset of (X, τ). Hence, an open (α-open) subspace of a α-Hurewicz space need not be α-Hurewicz.

Remark 1. Let X be the space considered in Example 3. It is also easy to prove that X is a θ-Hurewicz space. On the other hand, the open subspace

$$Y = X \setminus \{x_0\} \quad \text{of } X \text{ is not } \theta\text{-Hurewicz.}$$

It means that the θ-Hurewicz property is also not hereditary.

However, the α-Hurewicz and θ-Hurewicz properties are preserved under clopen subsets as shown below in Proposition 3.1.

Proposition 3.1. *A clopen subspace of the α-Hurewicz (θ-Hurewicz) space is α-Hurewicz (θ-Hurewicz).*

P r o o f. Let Y be a clopen subspace of the α-Hurewicz space X. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of α-open covers of Y. Then, $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of α-open covers of X, where $\mathcal{B}_k = \mathcal{A}_k \cup \{X \setminus Y\}$ for each k. Since X is α-Hurewicz, there is a sequence $(\mathcal{B}'_k : k \in \mathbb{N})$, where $\mathcal{B}'_k$ is a finite subset of $\mathcal{B}_k$ such that $x \in X$, $x \in \bigcup \mathcal{B}'_k$ for all but finitely many k. We observe that for each $y \in Y$,

$$y \in \bigcup \mathcal{B}'_k \setminus \{X \setminus Y\}.$$

That means that Y is an α-Hurewicz space. We can prove a similar result for θ--Hurewicz space. □

Proposition 3.2. *Let Y be a subspace of the space X. If Y is θ-Hurewicz, then for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of covers of Y by θ-open sets of X, there is a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ such that for each $y \in Y$, $y \in \cup\mathcal{B}_k$ for all except finitely many k.*

P r o o f. Let Y be θ-Hurewicz subspace of the space X. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of covers of Y by θ-open sets of X. Put

$$\mathcal{B}_k = \{Y \cap A : A \in \mathcal{A}_k\}.$$

Then, $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of θ-open covers of Y and Y is θ-Hurewicz, there exists a finite subset $\mathcal{C}_k$ of $\mathcal{B}_k$ such that $y \in Y$, $y \in \cup\mathcal{C}_k$ for all but finitely many k. Let

$$\mathcal{A}'_k = \{A \in \mathcal{A}_k : A \cap Y \in \mathcal{C}_k\}.$$

Then, the sequence $(\mathcal{A}'_k : k \in \mathbb{N})$ witnesses our requirement. □

In the following example, we show that the converse of the above theorem does not hold.

Example. Let

$$U = \{u_\alpha : \alpha < \omega_1\}, \quad V = \{v_i : i \in \omega\} \quad \text{and} \quad W = \{\langle u_\alpha, v_i\rangle : \alpha < \omega_1, i \in \omega\}.$$

Let

$$X = W \cup U \cup \{x'\}, \quad x \notin W \cup U.$$

Topologize X as follows: for $u_\alpha \in U$, $\alpha < \omega_1$ the basic neighborhood takes of the form

$$A_{u_\alpha}(i) = \{u_\alpha\} \cup \{\langle u_\alpha, v_j\rangle : j \geq i, \quad i \in \omega\},$$

the basic neighborhood of x' takes of the form

$$A_{x'}(\alpha) = \{x'\} \cup \bigcup \{\langle u_\beta, v_i\rangle : \beta > \alpha, \quad i \in \omega\}, \quad \alpha < \omega_1$$

and each point of W is isolated.

Consider the subspace $Y = \{u_\alpha : \alpha < \omega_1\} \cup \{x'\}$ of the space X. Observe that the singleton set $\{y\}$, $y \in Y$, is θ-open in Y. Thus, the family $\{\{y\} : y \in Y\}$ is an uncountable θ-open cover of Y which has no countable subcover. Hence, Y is not θ-Hurewicz.

Next, we show that Y for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of covers of Y by θ-open sets of X, there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ such that for each $y \in Y$, $y \in \cup\mathcal{B}_k$ for all but finitely many k.

Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a family of θ-open sets of X such that for each $k \in \mathbb{N}$, $Y \subseteq \cup\mathcal{A}_k$. Then for each $k \in \mathbb{N}$, there is an open set B_k and $A_k \in \mathcal{A}_k$ such that $x' \in B_k \subset \overline{B_k} \subset A_k$. From the construction of topology on X for each k, there exists a $\beta_k < \omega_1$ such that $A_{x'}(\beta_k) \subseteq B_k$, $\overline{A_{x'}(\beta_k)} \subseteq \overline{B_k}$. Thus, for each k, $\{u_\alpha : \alpha > \beta_k\} \cup \{x'\} \subseteq \overline{B_k} \subset A_k$ and $Y'_k = \bigcup_{\alpha \leq \beta_k} u_\alpha$ is countable. Thus, $\left(\bigcup_{k\in\mathbb{N}} Y'_k\right) \cap Y$ is countable. As similiar to Theorem 2.1, we can find a subset $\mathcal{A}'_k$ of $\mathcal{A}_k$ such that for each $y \in \left(\bigcup_{k\in\mathbb{N}} Y'_k\right) \cap Y$, $y \in \cup\mathcal{A}'_k$ for all but finitely many k. For each $k \in \mathbb{N}$, let $\mathcal{A}''_k = \mathcal{A}'_k \cup \{A_k\}$. Then, $\mathcal{A}''_k$ is a finite subset of $\mathcal{A}_k$ and for each $y \in Y$, $y \in \cup\mathcal{A}''_k$ for all but finitely many k.

The mapping $f : X \to Y$ from a space X to a space Y is said to be:

1. α-continuous [25] (α-irresolute [24]) if the preimage of each open (α-open) set of Y is α-open in X.
2. α-open (strongly α-open) if the image of each α-open set of X is α-open (open) in Y.
3. θ-continuous ([9], [10]) (resp. strongly θ-continuous [22]) if for each $x \in X$ and each open set B of Y containing $f(x)$ there exists an open set A of X containing x such that $f\big(\mathrm{Cl}(A)\big) \subset \mathrm{Cl}(B)$ (resp. $f\big(\mathrm{Cl}(A)\big) \subset B$).

Theorem 3.3. *An α-continuous image of an α-Hurewicz space is Hurewicz.*

Proof. Let X be an α-Hurewicz space and $f : X \to Y$ be an α-continuous map from X onto a space Y. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of Y. Since f is α-continuous, then for each $k \in \mathbb{N}$, $\{f^{-1}(A_k) : A_k \in \mathcal{A}_k\}$ is an α-open cover of X. Since X is α-Hurewicz, there is a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ such that $x \in X$, $x \in \bigcup\{f^{-1}(B) : B \in \mathcal{B}_k\}$ for all but finitely many k. Consider $A_{B_k} = f(B_k)$, $k \in \mathbb{N}$. Then, the sequence $\big(f(B) : B \in \mathcal{B}_k \ \&\ k \in \mathbb{N}\big)$ witness that Y is Hurewicz. □

Similarly, we can prove the following theorem.

Theorem 3.4. *An α-irresolute image of an α-Hurewicz space is α-Hurewicz.*

Since each continuous map is α-continuous, we have the following corollary:

Corollary 3.5. *A continuous image of an α-Hurewicz space is Hurewicz.*

Theorem 3.6. *A strongly θ-continuous image of a θ-Hurewicz space X is Hurewicz.*

Proof. Let X be a θ-Hurewicz space and $f : X \to Y$ be a strongly θ-continuous map from X onto a space Y. Consider the sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of Y. Then, for each k and for each $x \in X$, $f(x) \in A_k$ for some $A_k \in \mathcal{A}_k$. From the strongly θ-continuity of f, there is an open set $B_{x,k}$ containing x such that $f\big(\mathrm{Cl}(B_{x,k})\big) \subset A_k$. This means that $f^{-1}(A_k)$ is θ-open. Then for each $k \in \mathbb{N}$, $\{f^{-1}(A) : A \in \mathcal{A}_k\}$ is a θ-open cover of X. As X is θ-Hurewicz, there is a sequence $(\mathcal{C}_k : k \in \mathbb{N})$, where for each k, $\mathcal{C}_k$ is a finite subset of $\mathcal{A}_k$ such that $x \in X$, $x \in \bigcup\{f^{-1}(C) : C \in \mathcal{C}_k\}$ for all but finitely many k. Then, we have

$$Y = f(X) = f\left(\bigcup_{C \in \mathcal{C}_k} f^{-1}(C)\right) = \bigcup \mathcal{C}_k.$$

Hence, Y is Hurewicz. □

Theorem 3.7. *A θ-continuous image of a θ-Hurewicz space X is θ-Hurewicz.*

Proof. Let $f : X \to Y$ be a θ-continuous map from a θ-Hurewicz space X onto a space Y. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of Y. Then, for each $k \in \mathbb{N}$, $\{f^{-1}(A) : A \in \mathcal{A}_k\}$ is a θ-open cover of X because f is θ-continuous. Since X is θ-Hurewicz, there is a sequence $(\mathcal{B}_k : k \in \mathbb{N})$ where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ such that $x \in X$, $x \in \bigcup_{B \in \mathcal{B}_k} f^{-1}(B)$ for all but finitely many k. For each k and for each $B_k \in \mathcal{B}_k$, we may choose $A_k \in \mathcal{A}_k$ such that $B_k = f^{-1}(A_k)$. Then, we have

$$Y = f(X) = f\left(\bigcup_{B \in \mathcal{B}_k} f^{-1}(B)\right) = \bigcup \mathcal{B}_k.$$

Hence, Y is θ-Hurewicz. □

Since continuity implies θ-continuity, we have the following corollary:

Corollary 3.8. *A continuous image of a θ-Hurewicz space is θ-Hurewicz.*

Theorem 3.9. *For a space (X, τ), the following statements are equivalent:*

(1) *(X, τ) is α-Hurewicz;*

(2) *(X, τ) admits a strongly α-open bijection onto a Hurewicz space (Y, τ').*

Proof.

(1)⇒(2): Let (X, τ) be an α-Hurewicz space, then (X, τ_α) is Hurewicz. The identity map $I_X : (X, \tau) \to (X, \tau_\alpha)$ is a strongly α-open bijection.

(2)⇒(1): Assume that $f : (X, \tau) \to (Y, \tau')$ is a strongly α-open bijection from a space (X, τ) onto a Hurewicz space (Y, τ'). Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of α-open covers of (X, τ). Then, for each $k \in \mathbb{N}$, $\{f(A_k) : A_k \in \mathcal{A}_k\}$ is an open cover of Y. Since Y is Hurewicz space, there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ such that for each $y \in Y$, $y \in \bigcup\{f(B) : B \in \mathcal{B}_k\}$ for all but finitely many k. Hence, for each $x \in X$, $x \in \bigcup \mathcal{B}_k$ for all but finitely many k. □

4. Characterizations of Variants of Hurewicz Spaces

Let ω^ω be a set of all functions $f : \omega \to \omega$. The set ω^ω is equipped with the product topology. Define a relation $\leq^*$ on ω^ω as follows:

$$f \leq^* g \quad \text{if} \quad f(n) \leq g(n)$$

for all but finitely many n. Then, the relation $\leq^*$ on ω^ω is reflexive and transitive. Let H be a subset of ω^ω. We say H is a bounded if H has an upper bound with respect to $\leq^*$, otherwise H is unbounded. We say H is dominating if it is cofinal in $(\omega^\omega, \leq^*)$. Let $\mathfrak{b}$ be the smallest cardinality of an unbounded subset of ω^ω with respect to $\leq^*$. The cardinal $\mathfrak{b}$ is known as the bounding number. It is not difficult to prove that $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$ and it is known that $\omega_1 < \mathfrak{b} = \mathfrak{c}$, $\omega_1 < \mathfrak{d} = \mathfrak{c}$ and $\omega_1 \leq \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$ are all consistent with the axioms of ZFC; for more details, see [6].

Theorem 4.1. *Let $X \subset \omega^\omega$. If X is a mildly Hurewicz space, then X is bounded.*

Proof. Let us assume that X be an unbounded subset of ω^ω. For $f_x \in X$ and $n \in \omega$, let

$$A_n^{f_x} = \{h \in X : h(i) \in \{f_x(1), f_x(2), \ldots, f_x(n)\}, 1 \leq i \leq n\}.$$

Then, $A_n^{f_x}$ is a basic open set of X containing f_x. Moreover, if $g \notin A_n^{f_x}$, then there exists an $i \in \omega$ such that $1 \leq i \leq n$ and $g(i) \notin \{f_x(1), f_x(2), \ldots, f_x(n)\}$. Then, we have an open set $B_n^g = \big\{h \in X : h(i) \in \omega \setminus \{f_x(1), f_x(2), \ldots, f_x(n)\}\big\}$ of X containing g such that $B_n^g \cap A_n^{f_x} = \emptyset$. This implies that $g \notin \mathrm{Cl}_X(A_n^{f_x})$. Hence, $\mathrm{Cl}_X(A_n^{f_x}) \subseteq A_n^{f_x}$ which implies that $A_n^{f_x}$ is closed. For each $n \in \omega$, put $\mathcal{A}_n = \{A_n^{f_x} : f_x \in X\}$. Then, $\mathcal{A}_n$ is a clopen cover of X and $(\mathcal{A}_n : n \in \omega)$ is a sequence of clopen covers of X. For $n \in \omega$ and for any finite subset $\mathcal{B}_n$ of $\mathcal{A}_n$, let $n_{f_x} = \max\{f_x(1), f_x(2), \ldots, f_x(n) : f_x \in A_n^{f_x}\}$. Define a function $f : \omega \to \omega$ as follows:

$$f(n) = \max\big\{n_{f_x} : A_n^{f_x} \in \mathcal{B}_n\big\} + 1.$$

From the assumption of unboundedness of X,there exists $f' \in X$ such that $f' \not\leq^* f$, that is, $f(n) < f'(n)$ for infinitely many n. Hence, for infinitely many n, $f' \notin A_n^{f_x}$, $A_n^{f_x} \in \mathcal{B}_n$. Thus, $f' \notin \bigcup \mathcal{B}_n$ for infinitely many n. This means that X does not have the mildly Hurewicz property. This completes the proof. □

Corollary 4.2. *The dominating subset D of ω^ω is not mildly Hurewicz.*

By Theorem 4.1, the following corollaries follows directly:

Corollary 4.3. *Let X be a θ-Hurewicz subspace of ω^ω, then X is bounded.*

Corollary 4.4. *Let X be a nearly Hurewicz subspace of ω^ω, then X is bounded.*

Corollary 4.5. *Let X be an almost Hurewicz subspace of ω^ω, then X is bounded.*

Theorem 4.6. *Let X be a θ-Hurewicz space. Then, every θ-continuous image of X in ω^ω is bounded.*

Proof. Let $F : X \to \omega^\omega$ be a θ-continuous map from the θ-Hurewicz space X to ω^ω. Then, $F(X)$ is a θ-Hurewicz space. Hence, $F(X)$ is bounded. □

Corollary 4.7. *Every continuous image of a θ-Hurewicz space X in ω^ω is bounded.*

Corollary 4.8. *Every continuous image of a Hurewicz space X in ω^ω is bounded.*

Corollary 4.9. *Every continuous image of a nearly Hurewicz space X in ω^ω is bounded.*

Corollary 4.10. *Every continuous image of an almost Hurewicz space X in ω^ω is bounded.*

Definition 4.11. A space X is said to be θ-Lindelöf if each θ-open cover of X has a countable subcover.

Theorem 4.12. *Let X be a θ-Lindelöf space. If the cardinality of X is less than $\mathfrak{b}$, then X is θ-Hurewicz.*

Proof. Let X be a θ-Lindelöf space with $|X| < \mathfrak{b}$. If X is not a θ-Hurewicz space, then there exists a sequence $(\mathcal{A}_n : n \in \omega)$ of θ-open covers of X such that for each n and for each finite subset $\mathcal{B}_n$ of $\mathcal{A}_n$, there exists a $x \in X$ such that $x \notin \cup\mathcal{B}_n$ for infinitely many n. Since X is θ-Lindelöf, assume that for each

$$n,\ \ \mathcal{A}_n = \{A_n^j : j \in \omega\}.$$

For each $x \in X$, define $f_x : \omega \to \omega$ as :

$$f_x(n) = \min\{j : x \in A_n^j\}.$$

Let $D = \{f_x : x \in X\}$. Then, D is an unbounded set. If D is bounded, then there exists a $f \in \omega^\omega$ such that $f_x \leq^* f$ for all $f_x \in D$. For $n \in \omega$, put $\mathcal{B}_n = \{A_n^j : j \leq f(n)\}$. Then for each $x \in X$, $x \in \cup\mathcal{B}_n$ for all but finitely many n. This leads to a contradiction with the fact that there is a $x \in X$ such that $x \notin \cup\mathcal{B}_n$ for infinitely many n. Thus, D is an unbounded set. Hence, $\mathfrak{b} \leq |D|$.

Since $|X| < \mathfrak{b}$, it is mapped surjectively to D. This leads to a contradiction. Hence, X must be a θ-Hurewicz space. □

Theorem 4.13. *Let X be a topological space with $|X| < \mathfrak{b}$. The following statements are equivalent.*

(1) *X is mildly Hurewicz.*

(2) *X is mildly Lindelöf.*

Proof.

(1) $\Longrightarrow$ (2): Trivial.

(2) $\Longrightarrow$ (1): Similar to the proof of Theorem 4.11. □

Corollary 4.14. *Let X be a subset of real line $\mathbb{R}$. If X is not θ-Hurewicz, then $|X| \geq \mathfrak{b}$.*

Corollary 4.15. *Let X be a subset of real line $\mathbb{R}$. If X is not mildly Hurewicz, then $|X| \geq \mathfrak{b}$.*

Remark 2.
In [31], Velicko defined the θ-closure operator which is denoted by $\mathrm{Cl}_\theta(A)$. For $A \subset X$,

$$\mathrm{Cl}_\theta(A) = \{x \in X : \text{for each neighbourhood } U \text{ of } x, \mathrm{Cl}(U) \cap A \neq \phi\}$$

and

$$\mathrm{Cl}(A) \subseteq \mathrm{Cl}_\theta(A).$$

Many papers have been published on θ-closure operator (see [4, 7, 8, 19]). Using the θ-closure operator, it is interesting to investigate the following class of spaces. A space X is called θ-almost Hurewicz if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ for each k, such that for each

$$x \in X, \quad x \in \bigcup\big\{\mathrm{Cl}_\theta\big(\mathrm{Cl}(B)\big) : B \in \mathcal{B}_k\big\}$$

for all but finitely many k. Observe that every almost Hurewicz space is almost θ-Hurewicz.

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REFERENCES

[1] AL-ZOUBI, K. Y.: *S-paracompact spaces*, Acta Math. Hungar. **110** (2006), 165–174.

[2] AQSA—KHAN, MOIZ UD DIN: *On nearly Hurewicz spaces*, Open Math. **17** (2019), 1310–1318.
`https://doi.org/10.1515/math-2019-0095`

[3] BONANZINGA, M.—CAMMAROTO, F.—KOČINAC, LJ. D. R.: *Star-Hurewicz and related properties*, Appl. Gen. Topol. **5** (2004), 79–89.

[4] DICKMAN, R. F.—PORTER, J. R.: *θ-closed subsets of Hausdorff spaces*, Pac. J. Appl. Math. **59** (1975), 407–415.

[5] DORSETT, C.: *Semi-regular spaces*, Soochow J. Math. **8** (1982), 45–53.

[6] ENGELKING, R.: *General Topology*. (Revised and Cmpleted Edition). In: Sigma Series in Pure Mathematics, Vol.6, Heldermann, Berlin, 1989.

[7] ESPELIE, M. S.—JOSEPH, J. E.: *Some properties of θ-closure*, Canadian J. Math. **33**, (1981), no. 1, 142–149.

[8] EVANS, D.: *Role of θ-Closure in the Study of H-Closed Spaces*. Master's Thesis, Morgan State University. 2021.

[9] FOMIN, S. V.: *Extensions of topological spaces*, Doklady Akad. Nauk SSSR, **32** (1941), 114–116.

[10] FOMIN, S.: *Extensions of topological spaces*, Ann. Math. **44** (1943), 471–480.

[11] KUMAR, G.—TYAGI, B. K.: *On variants of θ-Menger spaces*, Filomat **37** (2023), no. 15, 5075–5085.

[12] HUREWICZ, W.: *Über die Verallgemeinerung des Borelschen Theorems*, Math. Z. **24** (1925), 401–425.

[13] HUREWICZ, W.: *Über Folgen stetiger Functionen*, Fund. Math. **9** (1927), 193–204,

[14] JANKOVIĆ, D. S.: *A note on mappings of extremally disconnected spaces*, Acta Math. Hungar. **46** (1985), no 1–2, 83–92.

[15] KOČINAC, L. D. R.: *Generalized open sets and selection properties*, Filomat **33** (2019), no. 5, 1485–1493.

[16] KOČINAC, L. D. R.—SABAH, A. —KHAN, M. D.—DJAMILA, SEBA: *Semi-Hurewicz spaces*, Hacettepe J. Math. Statist. **46** (2017), no. 11, 53–66.

[17] KOČINAC, L. D. R.: *On mildly Hurewicz spaces*, Internat. Math. Forum **11**, (2016), no. 12, 573–582.

[18] KOČINAC, L. D. R.: *The Pixley-Roy topology and selection principles*, Quest. Answers Gen. Topology **19** (2001), no. 2, 219–225.

[19] KOHLI, J. K.—DAS, A. K.: , *A class of spaces containing all generalized absolutely closed (almost compact) spaces*, Appl. Gen. Topol. **7** (2006), 233–244.

[20] LEVINE, N.: *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[21] LONG, P. E.—HERRINGTON, L. L.: *The T_θ-topology and faintly continuous functions*, Kyungpook Math. J. **22** (1982), no. 1, 7–14.

[22] LONG, P. E.—HERRINGTON, L. L.: *Strongly θ-continuous functions*, J. Korean Math. Soc. **18** (1981), 21–28.

[23] MAHESHWARI, S. N.—THAKUR, S. S.: *On α-compact spaces*, Bull. Inst. Math. Acad. Sinica, **13** (1985), 341–347.

[24] MAHESHWARI, S. N.—THAKUR, S. S.: *On α-irresolute mappings*, Tamkang J. Math. **11** (1980), 209–214.

[25] MASHHOUR, A. S.—ABD EL-MONSEF, M. E.—EL-DEEP, S. N.: *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.

[26] NJÅSTAD, O.: *On some classes of nearly open sets*, Pacific J. Math.**15** (1965), no. 3, 961–970.

[27] PORTER, J. R.—WOODS, G.: *Extensions and Absolutes of Hausdorff Spaces*. Springer--Verlag Berlin. 2012.

[28] SCHEEPERS, M.: *Combinatorics of open covers I: Ramsey theory*, Topology Appl. **69** (1996), 31–62.

[29] SONG, Y.-K—LI, R.: *The almost Hurewicz spaces*, Quest. Ans. Gen. Topology. **31** (2013), 131–136.

[30] STEEN, L. A.—SEEBACH, J. A.: *Counterexamples in Topology*. Springer-Verlag, United States, 1970.

[31] VE1ICKO, N. V.: *H-closed topological spaces*, (Russian) Mat. Sb. (N.S.) **70** (1966), 98–112; Amer. Math. Soc. Transl. **78** (1968), 102–118).
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