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HILBERT C*-MODULE JORDAN HOMOMORPHISMS

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ABSTRACT. Let $\mathcal{M}$ be a Hilbert C*-module. A linear mapping $\psi : \mathcal{M} \to \mathcal{M}$ is said to be a Hilbert C*-module Jordan homomorphism on $\mathcal{M}$ if it satisfies the equation $\psi(\langle a, b\rangle a) = \langle \psi(a), \psi(b)\rangle \psi(a)$ for all $a, b \in \mathcal{M}$. In this paper, we show that if $\mathcal{M}$ is prime, then every Hilbert C*-module Jordan homomorphism ψ from $\mathcal{M}$ onto $\mathcal{M}$ is a Hilbert C*-module homomorphism or a Hilbert C*-module anti-homomorphism on $\mathcal{M}$. We also prove a similar result about generalized Hilbert C*-module Jordan homomorphism.

1. Introduction

An additive mapping h of a ring $\mathcal{R}$ into a 2-torsion free ring $\mathcal{R}'$ is called a homomorphism (anti-homomorphism) if $h(ab) = h(a)h(b)$ $\big(h(ab) = h(b)h(a)\big)$ for all $a, b \in \mathcal{R}$. If $h(ab + ba) = h(a)h(b) + h(b)h(a)$ for all $a, b \in \mathcal{R}$, h is called a Jordan homomorphism. Herstein [9] showed that every Jordan homomorphism of a ring $\mathcal{R}$ onto a prime ring $\mathcal{R}'$ of characteristic different from 2 and 3 is either a homomorphism or an anti-homomorphism. Later, Smilley [14] provided a brief proof of this result and also removed the requirement that the characteristic be different from 3. Similar statements as above, proved about Jordan triple homomorphism [4, 10].

Let $\mathcal{A}$ and $\mathcal{B}$ be complex algebras, $h : \mathcal{A} \to \mathcal{B}$ be a linear map, and n be an arbitrary and fixed positive integer. Then, h is called an n-homomorphism if for all $a_1, a_2, \dots, a_n \in \mathcal{A}$, $h(a_1a_2 \dots a_n) = h(a_1)h(a_2) \dots h(a_n)$. Moreover, h is called an n-Jordan homomorphism if for all $a \in \mathcal{A}$, $h(a^n) = h(a)^n$. Lee in [12] and Gselmann in [8] proved that every n-Jordan homomorphism between two commutative Banach algebras is an n-homomorphism. Later, this problem was

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solved in [3] based on the property of the Vandermonde matrix, which is different from the methods that are used in [8] and [12].

In the present paper, our goal is to study the concept of homomorphism and Jordan homomorphism on Hilbert C*-modules.

The notion of a Hilbert C*-module initiated as a generalization of a Hilbert space in which the inner product takes its values in a C*-algebra (see [11]). Let $\mathcal{A}$ be a C*-algebra. An inner product $\mathcal{A}$-module is a complex linear space $\mathcal{M}$ which is a left $\mathcal{A}$-module with compatible scalar multiplication $\lambda(ax) = (\lambda a)x = a(\lambda x)$ $(\lambda \in \mathbb{C}, x \in \mathcal{M}, a \in \mathcal{A})$, together with an $\mathcal{A}$-valued inner product $(x, y) \mapsto \langle x, y\rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ such that for each $x, y, z \in \mathcal{M}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$,

(i) $\langle x, x\rangle \geq 0$ and the equality holds if and only if $x = 0$,

(ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$,

(iii) $\langle ax, y\rangle = a\langle x, y\rangle$,

(iv) $\langle x, y\rangle^* = \langle y, x\rangle$.

It follows from the above conditions that $\langle x, x\rangle$ is a positive element in C*--algebra $\mathcal{A}$, the inner product is conjugate-linear in its second variable and $\langle x, ay\rangle = \langle x, y\rangle a^*$ for all $x, y \in \mathcal{M}$ and $a \in \mathcal{A}$. An inner product $\mathcal{A}$-module $\mathcal{M}$ which is complete with respect to the norm $\|x\|_{\mathcal{M}} = \|\langle x, x\rangle\|_{\mathcal{A}}^{\frac{1}{2}}$ is called a Hilbert $\mathcal{A}$-module or a Hilbert C*-module over the C*-algebra $\mathcal{A}$. For example, every C*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module under the $\mathcal{A}$-valued inner product $\langle a, b\rangle = ab^*$ $(a, b \in \mathcal{A})$. Every complex Hilbert space is a left Hilbert $\mathbb{C}$-module. The notion of a right Hilbert $\mathcal{A}$-module can be defined similarly.

A Hilbert C*-module $\mathcal{M}$ is said to be *prime* if for elements a, b of $\mathcal{M}$, $\langle a, \mathcal{M}\rangle b = 0$ implies that $a = 0$ or $b = 0$. Equivalently, $\mathcal{M}$ is called prime if for elements a, b of $\mathcal{M}$, validity the equation $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$, implies that $a = 0$ or $b = 0$. $\mathcal{M}$ is said to be *semiprime* if $\langle a, \mathcal{M}\rangle a = 0$ implies that $a = 0$. Trivially, any prime Hilbert C*-module $\mathcal{M}$ is semiprime.

A linear mapping $\psi : \mathcal{M} \to \mathcal{M}$ is called a

(i) *Hilbert C*-module homomorphism* on $\mathcal{M}$ if for all $a, b, c \in \mathcal{M}$,

$$\psi(\langle a, b\rangle c) = \langle \psi(a), \psi(b)\rangle \psi(c), \tag{1}$$

(ii) *Hilbert C*-module anti-homomorphism* on $\mathcal{M}$ if for all $a, b, c \in \mathcal{M}$,

$$\psi(\langle a, b\rangle c) = \langle \psi(c), \psi(b)\rangle \psi(a), \tag{2}$$

(iii) *Hilbert C*-module Jordan homomorphism* on $\mathcal{M}$ if for all $a, b, c \in \mathcal{M}$,

$$\psi(\langle a, b\rangle a) = \langle \psi(a), \psi(b)\rangle \psi(a). \tag{3}$$

Note that every Hilbert C*-module homomorphism or anti-homomorphism is a Hilbert C*-module Jordan homomorphism. However, the opposite is generally not true.

Example. Let $M_2(\mathbb{C})$ be a C*-algebra of 2×2 complex matrices. The mapping $\psi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$\psi(A) = \tfrac{\sqrt{2}}{2}\begin{bmatrix} a_{11}+a_{21} & a_{12}+a_{22} \\ a_{21}-a_{11} & a_{22}-a_{12} \end{bmatrix},$$

for all $A = [a_{ij}] \in M_2(\mathbb{C})$, is a Hilbert C*-module homomorphism on $M_2(\mathbb{C})$.

Example. The set

$$\mathcal{M} = \left\{ X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}; X_1, X_1 \in M_2(\mathbb{C}) \right\},$$

under usual matrix operations, is a C*-algebra. The mapping $\psi : \mathcal{M} \to \mathcal{M}$ defined by

$$\psi(X) = \begin{bmatrix} X_1^* & 0 \\ 0 & X_2 \end{bmatrix},$$

for all $X \in \mathcal{M}$, is a Hilbert C*-module Jordan homomorphism. Since

$$\begin{aligned}
\langle \psi(X), \psi(Y)\rangle \psi(X) &= \psi(X)\psi(Y)^*\psi(X) \\
&= \begin{bmatrix} X_1^* & 0 \\ 0 & X_2 \end{bmatrix}\begin{bmatrix} Y_1 & 0 \\ 0 & Y_2^* \end{bmatrix}\begin{bmatrix} X_1^* & 0 \\ 0 & X_2 \end{bmatrix} \\
&= \begin{bmatrix} X_1^* Y_1 X_1^* & 0 \\ 0 & X_2 Y_2^* X_2 \end{bmatrix} \\
&= \psi\left(\begin{bmatrix} X_1 Y_1^* X_1 & 0 \\ 0 & X_2 Y_2^* X_2 \end{bmatrix}\right) \\
&= \psi(XY^*X) \\
&= \psi(\langle X, Y\rangle X),
\end{aligned}$$

for all $X, Y \in \mathcal{M}$. However, it is neither a Hilbert C*-module homomorphism nor a Hilbert C*-module antihomomorphism. Since

$$\psi(\langle X, Y\rangle Z) = \psi(XY^*Z) = \psi\left(\begin{bmatrix} Z_1^* Y_1 X_1^* & 0 \\ 0 & X_2 Y_2^* Z_2 \end{bmatrix}\right),$$

while

$$\langle \psi(X), \psi(Y)\rangle \psi(Z) = \psi(X)\psi(Y)^*\psi(Z) = \begin{bmatrix} X_1^* Y_1 Z_1^* & 0 \\ 0 & X_2 Y_2^* Z_2 \end{bmatrix},$$

and

$$\langle \psi(Z), \psi(Y)\rangle \psi(X) = \psi(Z)\psi(Y)^*\psi(X) = \begin{bmatrix} Z_1^* Y_1 X_1^* & 0 \\ 0 & Z_2 Y_2^* X_2 \end{bmatrix},$$

for all $X, Y, Z \in \mathcal{M}$.

In this paper, we prove that if $\mathcal{M}$ is a prime Hilbert C*-module, then every Hilbert C*-module Jordan homomorphism from $\mathcal{M}$ onto $\mathcal{M}$ is a Hilbert C*-module homomorphism or anti-homomorphism on $\mathcal{M}$. We also prove a similar result about generalized Hilbert C*-module Jordan homomorphism.

Information on homomorphisms on algebras can be found in [1, 4, 6, 13, 15]. Also, for more information on Hilbert C*-modules and linear operators on them, see [2, 5, 7].

2. Hilbert C*-module Jordan homomorphism

In this section, we prove that on a prime Hilbert C*-module $\mathcal{M}$, every Hilbert C*-module Jordan homomorphism from $\mathcal{M}$ onto $\mathcal{M}$ is a Hilbert C*-module homomorphism or a Hilbert C*-module anti-homomorphism. Before proving the result, we need some lemmas.

Lemma 2.1. *Let $\mathcal{M}$ be a semiprime Hilbert C^*-module and $a, b \in \mathcal{M}$. If* $\langle a, x\rangle b + \langle b, x\rangle a = 0$, *for all* $x \in \mathcal{M}$, *then* $\langle a, x\rangle b = \langle b, x\rangle a = 0$ *for all* $x \in \mathcal{M}$. *If*

$$\langle a, x\rangle b = 0 \text{ for all } x \in \mathcal{M}, \text{ then } \langle b, x\rangle a = 0 \text{ for all } x \in \mathcal{M}.$$

P r o o f. Let $a, b \in \mathcal{M}$. Suppose that $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. Then, we have

$$\begin{aligned}
\big\langle \langle a, x\rangle b, y\big\rangle \langle a, x\rangle b &= -\big\langle \langle b, x\rangle a, y\big\rangle \langle a, x\rangle b = -\langle b, x\rangle \langle a, y\rangle \langle a, x\rangle b \\
&= -\big\langle b, \langle y, a\rangle x\big\rangle \langle a, x\rangle b = -\big\langle \langle b, \langle y, a\rangle x\rangle a, x\big\rangle b \\
&= \big\langle \langle a, \langle y, a\rangle x\rangle b, x\big\rangle b = \big\langle \langle a, x\rangle \langle a, y\rangle b, x\big\rangle b \\
&= \langle a, x\rangle \langle a, y\rangle \langle b, x\rangle b = \langle a, x\rangle \big\langle \langle a, y\rangle b, x\big\rangle b \\
&= -\langle a, x\rangle \big\langle \langle b, y\rangle a, x\big\rangle b = -\langle a, x\rangle \langle b, y\rangle \langle a, x\rangle b \\
&= -\big\langle \langle a, x\rangle b, y\big\rangle \langle a, x\rangle b,
\end{aligned}$$

for all $y \in \mathcal{M}$, which implies that $\big\langle \langle a, x\rangle b, y\big\rangle \langle a, x\rangle b = 0$ for all $y \in \mathcal{M}$. Since $\mathcal{M}$ is semiprime, we get $\langle a, x\rangle b = 0$, and so, $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$.

Now, suppose that $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$. Then for all $y \in \mathcal{M}$, we have

$$\big\langle \langle b, x\rangle a, y\big\rangle \langle b, x\rangle a = \langle b, x\rangle \langle a, y\rangle \langle b, x\rangle a = \langle b, x\rangle \big\langle \langle a, y\rangle b, x\big\rangle a = 0.$$

Then, semiprimeness of $\mathcal{M}$ implies that $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. $\square$

Lemma 2.2. *Let $\mathcal{M}$ be a Hilbert C^*-module. Then, the following conditions are equivalent:*

(i) *$\mathcal{M}$ is a prime Hilbert C^*-module.*

(ii) *For $a, b \in \mathcal{M}$, validity of $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$, implies that $a = 0$ or $b = 0$.*

(iii) *For $a, b \in \mathcal{M}$, validity of $\langle a, x\rangle a = \langle b, x\rangle b$ for all $x \in \mathcal{M}$, implies that $a = b$ or $a = -b$.*

Proof.

(i)⇒(ii): If $\mathcal{M}$ is a prime Hilbert C*-module and $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$, then by Lemma 2.1, $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$, and then by primeness of $\mathcal{M}$, $a = 0$ or $b = 0$.

(ii)⇒(i): Suppose that $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$. Then,

$$\big\langle \langle b, x\rangle a, y\big\rangle \langle b, x\rangle a = \langle b, x\rangle \langle a, y\rangle \langle b, x\rangle a = \langle b, x\rangle \big\langle \langle a, y\rangle b, x\big\rangle a = 0,$$

which implies that $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. Hence, $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$, and therefore, $a = 0$ or $b = 0$. Thus, $\mathcal{M}$ is a prime.

(ii)⇒(iii): Let $\langle a, x\rangle a = \langle b, x\rangle b$ for all $x \in \mathcal{M}$. Then,

$$\langle a - b, x\rangle (a + b) + \langle a + b, x\rangle (a - b) = 0,$$

for all $x \in \mathcal{M}$. Thus, $a - b = 0$ or $a + b = 0$.

(iii)⇒(ii): Let $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. Then,

$$\langle a - b, x\rangle (a - b) = \langle a + b, x\rangle (a + b),$$

for all $x \in \mathcal{M}$. Hence, $a - b = a + b$ or $a - b = -(a + b)$. That is, $a = 0$ or $b = 0$. □

Lemma 2.3. *Let $\mathcal{M}$ be a semiprime Hilbert C^*-module and $\Delta, \Omega : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be mappings which are additive in each variable, and $\Delta(a, b, a) = \Omega(a, b, a) = 0$ for all $a, b \in \mathcal{M}$. If*

$$\langle \Delta(a, b, c), x\rangle \Omega(a, b, c) = 0, \tag{4}$$

for all $a, b, c, x \in \mathcal{M}$, then $\langle \Delta(a, b, c), x\rangle \Omega(c, b, a) = 0$ for all $a, b, c, x \in \mathcal{M}$.

Proof. Suppose that $\langle \Delta(a, b, c), x\rangle \Omega(a, b, c) = 0$ for all $a, b, c, x \in \mathcal{M}$, then by Lemma 2.1, we get $\langle \Omega(a, b, c), x\rangle \Delta(a, b, c) = 0$ for all $a, b, c, x \in \mathcal{M}$.

Replacing a and c by $a + c$ in (4), we have

$$\langle \Delta(a + c, b, a + c), x\rangle \Omega(a + c, b, a + c) = 0,$$

which implies that

$$\langle \Delta(a, b, c), x\rangle \Omega(c, b, a) + \langle \Delta(c, b, a), x\rangle \Omega(a, b, c) = 0,$$

for all $a, b, c, x \in \mathcal{M}$. It follows from

$$\begin{aligned}
&\big\langle\langle\Delta(a,b,c),x\rangle\Omega(c,b,a),y\big\rangle\langle\Delta(a,b,c),x\rangle\Omega(c,b,a)\\
&= -\big\langle\langle\Delta(a,b,c),x\rangle\Omega(c,b,a),y\big\rangle\langle\Delta(c,b,a),x\rangle\Omega(a,b,c)\\
&= -\langle\Delta(a,b,c),x\rangle\langle\Omega(c,b,a),y\rangle\langle\Delta(c,b,a),x\rangle\Omega(a,b,c)\\
&= -\langle\Delta(a,b,c),x\rangle\big\langle\langle\Omega(c,b,a),y\rangle\Delta(c,b,a),x\rangle\Omega(a,b,c) = 0,
\end{aligned}$$

and semiprimeness of $\mathcal{M}$ that $\langle\Delta(a,b,c),x\rangle\Omega(c,b,a) = 0$ for all $a, b, c, x \in \mathcal{M}$. □

Lemma 2.4. *Let $\mathcal{M}$ be a Hilbert C^*-module. Then,*

$$\Big\langle a, \langle b, \langle c,x\rangle c\rangle b\Big\rangle a = \big\langle\langle a,b\rangle c, x\big\rangle\langle c,b\rangle a,$$

for all $a, b, c, x \in \mathcal{M}$.

P r o o f. Let $a, b, c, x \in \mathcal{M}$. Then,

$$\begin{aligned}
\Big\langle a, \langle b, \langle c,x\rangle c\rangle b\Big\rangle a &= \langle a, \langle b,c\rangle\langle x,c\rangle b\rangle a = \big\langle a, \langle x,c\rangle b\big\rangle\langle c,b\rangle a\\
&= \langle a,b\rangle\langle c,x\rangle\langle c,b\rangle a = \big\langle\langle a,b\rangle c, x\big\rangle\langle c,b\rangle a.
\end{aligned}$$

□

Lemma 2.5. *Let $\mathcal{M}$ be a Hilbert C^*-module and $\psi : \mathcal{M} \to \mathcal{M}$ be a Hilbert C^*-module Jordan homomorphism on $\mathcal{M}$. Then,*

$$\psi(\langle a,b\rangle c + \langle c,b\rangle a) = \langle\psi(a),\psi(b)\rangle\psi(c) + \langle\psi(c),\psi(b)\rangle\psi(a), \tag{5}$$

for all $a, b, c \in \mathcal{M}$.

P r o o f. Replacing a by $a + c$ in (3), we get

$$\psi(\langle a+c,b\rangle(a+c)) = \langle\psi(a+c),\psi(b)\rangle\psi(a+c),$$

which implies that

$$\begin{aligned}
&\psi\big(\langle a,b\rangle a + \langle c,b\rangle a + \langle a,b\rangle c + \langle c,b\rangle c\big)\\
&= \langle\psi(a),\psi(b)\rangle\psi(a) + \langle\psi(a),\psi(b)\rangle\psi(c) + \langle\psi(c),\psi(b)\rangle\psi(a) + \langle\psi(c),\psi(b)\rangle\psi(c),
\end{aligned}$$

for all $a, b, c \in \mathcal{M}$. Canceling like terms from both sides of the above equation, we get equation (5). □

Theorem 2.6. *Let $\mathcal{M}$ be a prime Hilbert C^*-module. Then, every Hilbert C^*--module Jordan homomorphism from $\mathcal{M}$ onto $\mathcal{M}$ is either a Hilbert C^*-module homomorphism or a Hilbert C^*-module anti-homomorphism on $\mathcal{M}$.*

P r o o f.
Let ψ be a Hilbert C*-module Jordan homomorphism on $\mathcal{M}$ and $a, b, c \in \mathcal{M}$. Define

$$\Delta(a,b,c) := \psi(\langle a,b\rangle c) - \langle \psi(a), \psi(b)\rangle \psi(c), \tag{6}$$

$$\Omega(a,b,c) := \psi(\langle a,b\rangle c) - \langle \psi(c), \psi(b)\rangle \psi(a). \tag{7}$$

Trivially, we have

$$\Delta(a,b,a) = \Omega(a,b,a) = 0, \quad \Delta(a,b,c) + \Delta(c,b,a) = 0$$

and

$$\Omega(a,b,c) + \Omega(c,b,a) = 0.$$

We have

$$\begin{aligned}
S &= \psi\Big(\Big\langle a, \langle b, \langle c,x\rangle c\rangle b\Big\rangle a + \Big\langle c, \langle b, \langle a,x\rangle a\rangle b\Big\rangle c\Big)\\
&= \Big\langle \psi(a), \psi(\langle b, \langle c,x\rangle c\rangle b)\Big\rangle \psi(a) + \Big\langle \psi(c), \psi(\langle b, \langle a,x\rangle a\rangle b)\Big\rangle \psi(c)\\
&= \Big\langle \psi(a), \langle \psi(b), \psi(\langle c,x\rangle c)\rangle \psi(b)\Big\rangle \psi(a) + \Big\langle \psi(c), \langle \psi(b), \psi(\langle a,x\rangle a)\rangle \psi(b)\Big\rangle \psi(c)\\
&= \Big\langle \psi(a), \langle \psi(b), \langle \psi(c), \psi(x)\rangle \psi(c)\rangle \psi(b)\Big\rangle \psi(a)\\
&\quad + \Big\langle \psi(c), \langle \psi(b), \langle \psi(a), \psi(x)\rangle \psi(a)\rangle \psi(b)\Big\rangle \psi(c),
\end{aligned}$$

for all $x \in \mathcal{M}$. Thus, using Lemma 2.4, we have

$$\begin{aligned}
S &= \Big\langle \langle \psi(a), \psi(b)\rangle \psi(c), \psi(x)\Big\rangle \langle \psi(c), \psi(b)\rangle \psi(a)\\
&\quad + \Big\langle \langle \psi(c), \psi(b)\rangle \psi(a), \psi(x)\Big\rangle \langle \psi(a), \psi(b)\rangle \psi(c),
\end{aligned}$$

for all $x \in \mathcal{M}$. On the other hand, using Lemmas 2.4 and 2.5, we get

$$\begin{aligned}
S &= \psi\Big(\big\langle \langle a,b\rangle c, x\big\rangle \langle c,b\rangle a + \big\langle \langle c,b\rangle a, x\big\rangle \langle a,b\rangle c\Big)\\
&= \big\langle \psi(\langle a,b\rangle c), \psi(x)\big\rangle \psi(\langle c,b\rangle a) + \big\langle \psi(\langle c,b\rangle a), \psi(x)\big\rangle \psi(\langle a,b\rangle c),
\end{aligned}$$

for all $x \in \mathcal{M}$. It follows from the above equations that

$$\begin{aligned}
&\big\langle \psi(\langle a,b\rangle c), \psi(x)\big\rangle \psi(\langle c,b\rangle a) + \big\langle \psi(\langle c,b\rangle a), \psi(x)\big\rangle \psi(\langle a,b\rangle c) \qquad (8)\\
&= \Big\langle \langle \psi(a), \psi(b)\rangle \psi(c), \psi(x)\Big\rangle \langle \psi(c), \psi(b)\rangle \psi(a)\\
&\quad + \Big\langle \langle \psi(c), \psi(b)\rangle \psi(a), \psi(x)\Big\rangle \langle \psi(a), \psi(b)\rangle \psi(c),
\end{aligned}$$

for all $x \in \mathcal{M}$.

Subtracting

$$\Big\langle \langle\psi(a),\psi(b)\rangle\psi(c),\psi(x)\Big\rangle\psi(\langle c,b\rangle a)+\Big\langle \langle\psi(c),\psi(b)\rangle\psi(a),\psi(x)\Big\rangle\psi(\langle a,b\rangle c)$$

from both sides of the above equation, we have

$$\begin{aligned}&\Big\langle \Delta(a,b,c),\psi(x)\Big\rangle\psi(\langle c,b\rangle a)+\Big\langle \Delta(c,b,a),\psi(x)\Big\rangle\psi(\langle a,b\rangle c)\\ &+\Big\langle \langle\psi(a),\psi(b)\rangle\psi(c),\psi(x)\Big\rangle\Delta(c,b,a)+\Big\langle \langle\psi(c),\psi(b)\rangle\psi(a),\psi(x)\Big\rangle\Delta(a,b,c)=0,\end{aligned}$$

for all $x\in\mathcal{M}$. Since $\Delta(c,b,a)=-\Delta(a,b,c)$, from the above equation, we get that

$$\begin{aligned}&\Big\langle \Delta(a,b,c),\psi(x)\Big\rangle\Big(\psi(\langle c,b\rangle a)-\psi(\langle a,b\rangle c)\Big)\\ &+\Big\langle \langle\psi(c),\psi(b)\rangle\psi(a)-\langle\psi(a),\psi(b)\rangle\psi(c),\psi(x)\Big\rangle\Delta(a,b,c)=0,\end{aligned}$$

for all $x\in\mathcal{M}$. Using the above equation and the equations

$$\begin{aligned}\psi(\langle c,b\rangle a)-\psi(\langle a,b\rangle c)&=\Omega(c,b,a)-\Delta(a,b,c),\\ \langle\psi(c),\psi(b)\rangle\psi(a)-\langle\psi(a),\psi(b)\rangle\psi(c)&=\Delta(a,b,c)-\Omega(a,b,c),\end{aligned}$$

we have

$$\begin{aligned}&\Big\langle \Delta(a,b,c),\psi(x)\Big\rangle\Big(\Omega(c,b,a)-\Delta(a,b,c)\Big)\\ &+\Big\langle \Delta(a,b,c)-\Omega(a,b,c),\psi(x)\Big\rangle\Delta(a,b,c)=0,\end{aligned}$$

for all $x\in\mathcal{M}$. From the equation $\Omega(c,b,a)=-\Omega(a,b,c)$ and the above equation, it follows that

$$\Big\langle \Delta(a,b,c),\psi(x)\Big\rangle\Omega(a,b,c)+\Big\langle \Omega(a,b,c),\psi(x)\Big\rangle\Delta(a,b,c)=0,$$

for all $x\in\mathcal{M}$. Since ψ is onto, we have

$$\Big\langle \Delta(a,b,c),y\Big\rangle\Omega(a,b,c)+\Big\langle \Omega(a,b,c),y\Big\rangle\Delta(a,b,c)=0,$$

for all $y\in\mathcal{M}$. Since $\mathcal{M}$ is semiprime, from Lemma 2.1 it follows that

$$\big\langle \Delta(a,b,c),y\big\rangle\Omega(a,b,c)=\big\langle \Omega(a,b,c),y\big\rangle\Delta(a,b,c)=0,$$

for all $y\in\mathcal{M}$. Since $\mathcal{M}$ is prime, from Lemma 2.2 it follows that $\Delta(a,b,c)=0$ or $\Omega(a,b,c)=0$. If $\Delta(a,b,c)=0$, then we get $\psi(\langle a,b\rangle c)=\langle\psi(a),\psi(b)\rangle\psi(c)$ and so, ψ is a Hilbert C*-module homomorphism. If $\Omega(a,b,c)=0$, then $\psi(\langle a,b\rangle c)=\langle\psi(c),\psi(b)\rangle\psi(a)$ and so, ψ is a Hilbert C*-module anti-homomorphism. $\square$

3. Generalized Hilbert C*-module Jordan homomorphisms

Let $\mathcal{M}$ be a Hilbert C*-module. A linear mapping $\varphi : \mathcal{M} \to \mathcal{M}$ is said to be a

(i) *generalized Hilbert C*-module homomorphism* if there exists a Hilbert C*-module homomorphism $\psi : \mathcal{M} \to \mathcal{M}$ such that

$$\varphi(\langle a, b\rangle c) = \langle \varphi(a), \psi(b)\rangle \psi(c), \tag{9}$$

for all $a, b, c \in \mathcal{M}$;

(ii) *generalized Hilbert C*-module anti-homomorphism* if there exists a Hilbert C*-module anti-homomorphism $\psi : \mathcal{M} \to \mathcal{M}$ such that

$$\varphi(\langle a, b\rangle c) = \langle \varphi(c), \psi(b)\rangle \psi(a), \tag{10}$$

for all $a, b, c \in \mathcal{M}$;

(iii) *generalized Hilbert C*-module Jordan homomorphism* if there exists a Hilbert C*-module Jordan homomorphism $\psi : \mathcal{M} \to \mathcal{M}$ such that

$$\varphi(\langle a, b\rangle a) = \langle \varphi(a), \psi(b)\rangle \psi(a), \tag{11}$$

for all $a, b \in \mathcal{M}$.

In this section, we prove that on a prime Hilbert C*-module $\mathcal{M}$ every generalized Hilbert C*-module Jordan homomorphism on $\mathcal{M}$ is a generalized Hilbert C*-module homomorphism when the related Hilbert C*-module Jordan homomorphism ψ is onto.

Lemma 3.1. *Let $\mathcal{M}$ be a Hilbert C*-module and $\varphi : \mathcal{M} \to \mathcal{M}$ be a generalized Hilbert C*-module Jordan homomorphism on $\mathcal{M}$ with related Hilbert C*-module Jordan homomorphism ψ. Then,*

$$\varphi(\langle a, b\rangle c + \langle c, b\rangle a) = \langle \varphi(a), \psi(b)\rangle \psi(c) + \langle \varphi(c), \psi(b)\rangle \psi(a), \tag{12}$$

for all $a, b, c \in \mathcal{M}$.

Proof of Lemma 3.1 is similar to the proof of Lemma 2.5.

Theorem 3.2. *Let $\mathcal{M}$ be a prime Hilbert C*-module and φ be a generalized Hilbert C*-module Jordan homomorphism on $\mathcal{M}$ with the related Hilbert C*-module Jordan homomorphism ψ. If ψ is onto, then φ is either a generalized Hilbert C*-module homomorphism or a generalized Hilbert C*-module anti-homomorphism on $\mathcal{M}$.*

P r o o f. Let $a, b, c \in \mathcal{M}$ and define

$$\Delta(a,b,c) := \psi(\langle a,b\rangle c) - \langle \psi(a), \psi(b)\rangle \psi(c), \tag{13}$$

$$\Omega(a,b,c) := \psi(\langle a,b\rangle c) - \langle \psi(c), \psi(b)\rangle \psi(a), \tag{14}$$

$$\Delta'(a,b,c) := \varphi(\langle a,b\rangle c) - \langle \varphi(a), \psi(b)\rangle \psi(c), \tag{15}$$

$$\Omega'(a,b,c) := \varphi(\langle a,b\rangle c) - \langle \varphi(c), \psi(b)\rangle \psi(a). \tag{16}$$

Trivially,

$$\Delta(a,b,a) = \Omega(a,b,a) = \Delta'(a,b,a) = \Omega'(a,b,a) = 0,$$
$$\Delta(a,b,c) + \Delta(c,b,a) = 0, \qquad \Delta'(a,b,c) + \Delta'(c,b,a) = 0,$$
$$\Omega(a,b,c) + \Omega(c,b,a) = 0 \quad \text{and} \quad \Omega'(a,b,c) + \Omega'(c,b,a) = 0.$$

We have

$$\begin{aligned}
S &= \varphi\Big(\Big\langle a, \langle b, \langle c,x\rangle c\rangle b\Big\rangle a + \Big\langle c, \langle b, \langle a,x\rangle a\rangle b\Big\rangle c\Big)\\
&= \Big\langle \varphi(a), \psi(\langle b, \langle c,x\rangle c\rangle b)\Big\rangle \psi(a) + \Big\langle \varphi(c), \psi(\langle b, \langle a,x\rangle a\rangle b)\Big\rangle \psi(c)\\
&= \Big\langle \varphi(a), \langle \psi(b), \psi(\langle c,x\rangle c)\rangle \psi(b)\Big\rangle \psi(a) + \Big\langle \varphi(c), \langle \psi(b), \psi(\langle a,x\rangle a)\rangle \psi(b)\Big\rangle \psi(c)\\
&= \Big\langle \varphi(a), \langle \psi(b), \langle \psi(c), \psi(x)\rangle \psi(c)\rangle \psi(b)\Big\rangle \psi(a)\\
&\quad + \Big\langle \varphi(c), \langle \psi(b), \langle \psi(a), \psi(x)\rangle \psi(a)\rangle \psi(b)\Big\rangle \psi(c),
\end{aligned}$$

for all $x \in \mathcal{M}$. Thus, using Lemma 2.4, we have

$$\begin{aligned}
S &= \Big\langle \langle \varphi(a), \psi(b)\rangle \psi(c), \psi(x)\Big\rangle \langle \psi(c), \psi(b)\rangle \psi(a)\\
&\quad + \Big\langle \langle \varphi(c), \psi(b)\rangle \psi(a), \psi(x)\Big\rangle \langle \psi(a), \psi(b)\rangle \psi(c),
\end{aligned}$$

for all $x \in \mathcal{M}$. On the other hand, using Lemmas 2.4 and 2.5, we get

$$\begin{aligned}
S &= \varphi\Big(\big\langle \langle a,b\rangle c, x\big\rangle \langle c,b\rangle a + \big\langle \langle c,b\rangle a, x\big\rangle \langle a,b\rangle c\Big)\\
&= \big\langle \varphi(\langle a,b\rangle c), \psi(x)\big\rangle \psi(\langle c,b\rangle a) + \big\langle \varphi(\langle c,b\rangle a), \psi(x)\big\rangle \psi(\langle a,b\rangle c),
\end{aligned}$$

for all $x \in \mathcal{M}$. From the above equations, it follows that

$$\begin{aligned}
&\big\langle \varphi(\langle a,b\rangle c), \psi(x)\big\rangle \psi(\langle c,b\rangle a) + \big\langle \varphi(\langle c,b\rangle a), \psi(x)\big\rangle \psi(\langle a,b\rangle c) \qquad (17)\\
&= \Big\langle \langle \varphi(a), \psi(b)\rangle \psi(c), \psi(x)\Big\rangle \langle \psi(c), \psi(b)\rangle \psi(a)\\
&\quad + \Big\langle \langle \varphi(c), \psi(b)\rangle \psi(a), \psi(x)\Big\rangle \langle \psi(a), \psi(b)\rangle \psi(c),
\end{aligned}$$

for all $x \in \mathcal{M}$.

Subtracting

$$\Big\langle \langle\varphi(a),\psi(b)\rangle\psi(c),\psi(x)\Big\rangle\psi(\langle c,b\rangle a)+\Big\langle \langle\varphi(c),\psi(b)\rangle\psi(a),\psi(x)\Big\rangle\psi(\langle a,b\rangle c)$$

from both sides of the above equation, we have

$$\begin{aligned}&\Big\langle \Delta'(a,b,c),\psi(x)\Big\rangle\psi(\langle c,b\rangle a)+\Big\langle \Delta'(c,b,a),\psi(x)\Big\rangle\psi(\langle a,b\rangle c)\\&+\Big\langle \langle\varphi(a),\psi(b)\rangle\psi(c),\psi(x)\Big\rangle\Delta(c,b,a)+\Big\langle \langle\varphi(c),\psi(b)\rangle\psi(a),\psi(x)\Big\rangle\Delta(a,b,c)=0,\end{aligned}$$

for all $x\in\mathcal{M}$. Since $\Delta(c,b,a)=-\Delta(a,b,c)$ and $\Delta'(c,b,a)=-\Delta'(a,b,c)$, from the above equation, we get that

$$\begin{aligned}&\Big\langle \Delta'(a,b,c),\psi(x)\Big\rangle\Big(\psi(\langle c,b\rangle a)-\psi(\langle a,b\rangle c)\Big)\\&+\Big\langle \langle\varphi(c),\psi(b)\rangle\psi(a)-\langle\varphi(a),\psi(b)\rangle\psi(c),\psi(x)\Big\rangle\Delta(a,b,c)=0,\end{aligned}$$

for all $x\in\mathcal{M}$. Using the above equation and the equations

$$\begin{aligned}\psi(\langle c,b\rangle a)-\psi(\langle a,b\rangle c)&=\Omega(c,b,a)-\Delta(a,b,c)\\ \langle\varphi(c),\psi(b)\rangle\psi(a)-\langle\varphi(a),\psi(b)\rangle\psi(c)&=\Delta'(a,b,c)-\Omega'(a,b,c),\end{aligned}$$

we have

$$\begin{aligned}&\Big\langle \Delta'(a,b,c),\psi(x)\Big\rangle\Big(\Omega(c,b,a)-\Delta(a,b,c)\Big)\\&+\Big\langle \Delta'(a,b,c)-\Omega'(a,b,c),\psi(x)\Big\rangle\Delta(a,b,c)=0,\end{aligned}$$

for all $x\in\mathcal{M}$. It follows from $\Omega(c,b,a)=-\Omega(a,b,c)$ and the above equation that

$$\Big\langle \Delta'(a,b,c),\psi(x)\Big\rangle\Omega(a,b,c)+\Big\langle \Omega'(a,b,c),\psi(x)\Big\rangle\Delta(a,b,c)=0,$$

for all $x\in\mathcal{M}$. Since ψ is onto, we get

$$\Big\langle \Delta'(a,b,c),y\Big\rangle\Omega(a,b,c)+\Big\langle \Omega'(a,b,c),y\Big\rangle\Delta(a,b,c)=0,$$

for all $y\in\mathcal{M}$. Now, by Theorem 2.6, ψ is either a Hilbert C*-module homomorphism or a Hilbert C*-module anti-homomorphism on $\mathcal{M}$. If ψ is a Hilbert C*-module homomorphism, then $\Delta(a,b,c)=0$ and so, for all $y\in\mathcal{M}$, we have

$$\Big\langle \Delta'(a,b,c),y\Big\rangle\Omega(a,b,c)=0.$$

Since $\mathcal{M}$ is prime, from Lemma 2.2, it follows that either $\Delta'(a,b,c)=0$ or $\Omega(a,b,c)=0$. Since ψ is a Hilbert C*-module homomorphism, $\Omega(a,b,c)\neq 0$ and hence, $\Delta'(a,b,c)=0$ which implies that φ is a generalized Hilbert C*-module homomorphism. If ψ is a Hilbert C*-module anti-homomorphism, then a similar argument shows that φ is a generalized Hilbert C*-module anti-homomorphism. □

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