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ON $[\varrho_1, \varrho_2]$-LOWER SUPERDENSE SETS

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ABSTRACT. The main goal of the paper is to characterize the families of $[\varrho_1, \varrho_2]$-lower superdense subsets of $\mathbb{R}$, which generalize well-known notions of density topology (in our paper, denoted by $L_1^{\blacklozenge}$) and T^* topology. Dense sets preserve density, i.e., if $E \in L_1^{\blacklozenge}$, then for every measurable $F \subset \mathbb{R}$, $E \cap F$ possesses the same lower and upper density as F at every $x \in E \cap F$. On the other hand, elements of T^* preserve positive lower density, i.e., if $E \in T^*$, F is measurable, $x \in E \cap F$ and $\underline{d}(F, x) > 0$, then $\underline{d}(E \cap F, x) > 0$. Taking arbitrary $0 \le \varrho_1 \le \varrho_2 \le 1$, $\varrho_2 - \varrho_1 < 1$, we can define subsets $E \subset \mathbb{R}$ which preserve $[\varrho_1, \varrho_2]$-lower density, i.e., if F is measurable, $x \in E \cap F$ and $\underline{d}(F, x)$ is greater or not less than ϱ_2, then $\underline{d}(E \cap F, x)$ is greater or not less than ϱ_1. We can define four types of superdense sets, but three of them are equal. Even though the definition and properties of $[\varrho_1, \varrho_2]$-lower superdensity and T^* topology are similar and all of them consist of very big sets, these families are essentially different. In the paper, we focus on basic properties, characterizations of superdense sets and relationships between $[\varrho_1, \varrho_2]$-lower superdense sets for different indices $[\varrho_1, \varrho_2]$. We apply the notion of $[\varrho_1, \varrho_2]$-lower superdensity to find adders of ϱ-lower continuous functions.

1. Introduction

Density of subsets of $\mathbb{R}$ is very old and very well-known property, see, e.g., [1–3, 5, 7, 8, 10, 12, 13]. Elements of density topology play essentially important role, i.e., the sets which lower density at every point is equal to 1. In [13], a very wide bibliography on density topology can be found. One of the most important property of elements of density topology $L_1^{\blacklozenge}$ is preserving lower and upper density, i.e., if $E \in L_1^{\blacklozenge}$, then intersection $E \cap F$ possesses the same lower and upper density as F for every measurable F and every $x \in E \cap F$. In [5, 9, 12],

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another type of sets preserving density was studied. Elements of T^* topology preserve positive lower density, i.e., if

$$E \in T^*, \quad F \text{ is measurable and } \quad \underline{d}\,(F, x) > 0,$$

then

$$\underline{d}\,(E \cap F, x) > 0 \quad \text{for every } \; x \in E \cap F.$$

Some types of generalized continuity connected with the notions of density were studied in [2,7,8,10]. Families of ϱ-lower continuous functions are examples of so-called $\mathcal{A}$-continuous functions and path continuous functions with respect to family $\mathcal{A}$. In [2,10], one can find many examples of generalized continuities and path continuities defined in this way. In the case of ϱ-lower continuous functions, the family $\mathcal{A}$ is defined in terms of lower density.

In the paper, we introduce families

$$L(\varrho_1, \varrho_2) \quad \text{and} \quad L^*(\varrho_1, \varrho_2) \quad \text{of } [\varrho_1, \varrho_2] \text{ and } [\varrho_1, \varrho_2]^*\text{-lower superdense sets.}$$

As in the case of density topology and T^* topology, the $[\varrho_1, \varrho_2]$-lower superdense sets have large lower density at every of their points and preserve some aspects of density if intersection with arbitray measurable set is taken. We also apply $[\varrho_1, \varrho_2]$-lower superdense sets to find adders for ϱ-lower continuous functions.

2. Preliminaries

The key concept of our paper is the density of subsets of $\mathbb{R}$. The density of a measurable set is a very old mathematical concept. Its intensive research began in the 1920's. Let λ be the Lebesgue measure in $\mathbb{R}$. By $\mathcal{S}$ we denote the class of Lebesgue measurable subsets of $\mathbb{R}$. Let $E \in \mathcal{S}$ and $x \in \mathbb{R}$. According to [2], the numbers

$$\underline{d}^+(E, x) = \liminf_{t \to 0^+} \frac{\lambda(E \cap [x, x+t])}{t}$$

and

$$\overline{d}^+(E, x) = \limsup_{t \to 0^+} \frac{\lambda(E \cap [x, x+t])}{t}$$

are called the right lower density of E at x and the right upper density of E at x, respectively. The left lower $\underline{d}^-(E, x)$ and the left upper $\overline{d}^-(E, x)$ densities of E at x are defined analogously. If

$$\underline{d}^+(E, x) = \overline{d}^+(E, x) \quad \text{or} \quad \underline{d}^-(E, x) = \overline{d}^-(E, x),$$

then we call these numbers the right density or the left density of E at x, respectively.

The numbers

$$\underline{d}\,(E,x) = \liminf_{t\to 0^+, k\to 0^+} \frac{\lambda(E\cap [x-t, x+k])}{k+t},$$

$$\overline{d}\,(E,x) = \limsup_{t\to 0^+, k\to 0^+} \frac{\lambda(E\cap [x-t, x+k])}{k+t}$$

are called the upper and the lower density of E at x, respectively. It is easy to check that

$$\overline{d}\,(E,x) = \max\{\overline{d}^{+}(E,x), \overline{d}^{-}(E,x)\}$$

and

$$\underline{d}\,(E,x) = \min\{\underline{d}^{+}(E,x), \underline{d}^{-}(E,x)\}.$$

If $\underline{d}\,(E,x) = \overline{d}\,(E,x)$, then we call this number the density of E at x and denote it by $d(E,x)$. If $d(E,x) = 1$, then we say that x is a point of density of E. For every $\varrho \in [0,1)$ let

$$L_\varrho = \{E \in \mathcal{S} \colon \underline{d}\,(E,x) > \varrho \quad \text{for every} \;\; x \in E\}.$$

Similarly, for every $\varrho \in (0,1]$ let

$$L_\varrho^\blacklozenge = \{E \in \mathcal{S} \colon \underline{d}\,(E,x) \geq \varrho \quad \text{for every} \;\; x \in E\}.$$

A particularly important role is played by $L_1^\blacklozenge$, which creates a topology called the density topology. Density topology has been the subject of intensive scientific research for many years (see [13]). As we mentioned before the definitions presented above come from Bruckner's monograph [2]. It is worth noting that the symmetric definition of lower and upper density is also considered in the literature.

In the paper, we consider ϱ-lower continuous functions for different $\varrho \in [0,1]$. The ϱ-lower continuity is a kind of generalized continuity. Families of ϱ-lower continuous functions are examples of so-called $\mathcal{A}$-continuous functions and path continuous functions with respect to the family $\mathcal{A} = \{A_x \colon x \in \mathbb{R}\}$, where A_x is a family of subsets of $\mathbb{R}$ containing x, for example, [6, 11]. Let I be an open interval. Then, continuity of the function $f \colon I \to \mathbb{R}$ at $x \in I$ means that the pre-image $f^{-1}\big((f(x)-\varepsilon, f(x)+\varepsilon)\big)$ contains an element of A_x for every $\varepsilon > 0$. Meanwhile, path continuity at $x \in I$ means that there is a set $E \in A_x$ such that f restricted to E is continuous in ordinary sense at x. Of course, path continuity with respect to $\mathcal{A}$ implies $\mathcal{A}$-continuity, and sometimes they are equivalent. In [2], one can find many examples of generalized continuities defined in this way. In the case of ϱ-lower continuous functions, the family $\mathcal{A}$ is defined in terms of the lower density.

In this paper, we study properties of $[\varrho_1, \varrho_2]$ and $[\varrho_1, \varrho_2]^*$-lower superdense sets. These concepts are close and symmetric to notions of density topology and T^* topology, i.e., $[\varrho_1, \varrho_2]$-lower superdense sets have large lower density at every of its points and preserve some values of lower density under taking an intersection of sets. Nevertheless, despite symmetric definitions and properties, there are huge differences between these families of sets. The families of

$$\begin{array}{rcll} [\varrho_1, \varrho_2] & \text{and} & [\varrho_1, \varrho_2]^*\text{-lower superdense sets} & \\ L(\varrho_1, \varrho_2) & \text{and} & L^*(\varrho_1, \varrho_2) & \text{for } 0 \le \varrho_1 \le \varrho_2 \le 1,\ \varrho_2 - \varrho_1 < 1 \end{array}$$

form monotone families of subsets of the real line, i.e.,

$$L(\varrho_1, \varrho_2) \subset L(\varrho_1', \varrho_2) \quad \text{and} \quad L^*(\varrho_1, \varrho_2) \subset L^*(\varrho_1', \varrho_2) \qquad \text{for every } 0 \le \varrho_1 < \varrho_1' \le \varrho_2 \le 1$$

and

$$L(\varrho_1, \varrho_2') \subset L(\varrho_1, \varrho_2) \quad \text{and} \quad L^*(\varrho_1, \varrho_2') \subset L^*(\varrho_1, \varrho_2) \qquad \text{for every } 0 \le \varrho_1 \le \varrho_2 < \varrho_2' \le 1.$$

In the final part of the paper, we use $[\varrho_1, \varrho_2]$-lower superdense sets to find adders of ϱ-lower continuous functions.

In Section 3, we recall basic properties of density. We also provide some important technical lemmas concerning constructions of supersets $A \subset B$ with the desired lower density of B.

The main part of the paper, Section 4, presents properties of $[\varrho_1, \varrho_2]$-lower superdense sets, i.e., sets whose intersection with each set with a lower density at least ϱ_2 at each point has a lower density at least ϱ_1 at each point. There are four types of $[\varrho_1, \varrho_2]$-lower superdense sets possible: depending on whether we choose sharp or weak inequalities. The very important Theorem 4.1 shows that three of these four types of superdensity are equivalent. The family of $[\varrho_1, \varrho_2]$-lower superdense sets for these three cases is denoted by $L(\varrho_1, \varrho_2)$. The case, where we are looking for sets whose intersection with every sets with a lower density at each point not less than ϱ_2 gives a set with a lower density at each point greater than ϱ_1 gives the second type of $[\varrho_1, \varrho_2]$-lower superdense sets, which we denote by $L^*(\varrho_1, \varrho_2)$. At the end of Section 4, we show some properties of the families $L(\varrho_1, \varrho_2)$ and $L^*(\varrho_1, \varrho_2)$. It turns out that the first one contains $L^{\blacklozenge}_{1-\varrho_2+\varrho_1}$ and the second one contains $L_{1-\varrho_2+\varrho_1}$. Unfortunately, in most cases there is no equality in these inclusions. Only if $\varrho_1 = \varrho_2$ or $\varrho_2 = 1$, we have equalities, which is proved in Theorem 4.6. We also present relationships between families of $[\varrho_1, \varrho_2]$-lower superdense sets for different values of ϱ_1 and ϱ_2.

Section 5 is devoted to adders of families of ϱ-lower continuous functions. First, we present a general definition of the concept of an adder $\mathcal{A}(X/Y)$ of the families of functions X and Y, i.e.,

$$\mathcal{A}(X/Y) = \{f\colon f + g \in X \text{ for every } g \in Y\}.$$

Then, we define and provide some basic properties of

$$\mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}), \quad \mathcal{A}(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}_{\varrho_2}),$$

$$\mathcal{A}(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}) \quad \text{and} \quad \mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}).$$

In Theorem 5.13, we characterize the adders

$$\mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}), \quad \mathcal{A}(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}_{\varrho_2}), \quad \mathcal{A}(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}),$$

and in Theorem 5.15, we characterize

$$\mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}).$$

It turns out that these adders consist of functions that are $L(\varrho_1, \varrho_2)$-continuous or $L^*(\varrho_1, \varrho_2)$-continuous. At the end of the section, in Remark 1, we show that multipliers of ϱ-lower continuous functions have a more complex structure than adders and they are their subsets, in most cases the proper ones.

The last section is devoted to adders of path continuous functions, where paths are taken from L_ϱ or $L^{\blacklozenge}_\varrho$. Given two families of real functions X and Y, we can consider all functions $g\colon \mathbb{R} \to \mathbb{R}$ such that the product $f \cdot g$ (the sum $f+g$) belongs to X whenever f belongs to Y. Such a function g is said to be a multiplier (an adder) of the set X over the set Y. Multipliers are a generalization of the maximal multiplicative classes and adders are a generalization of the notion of maximal additive classes of a given family of functions. In [4], one can find many interesting results concerning multipliers,

$$X/Y = \{f\colon f \cdot g \in X \text{ for every } g \in Y\}.$$

In Theorem 6.1, we characterize

$$\mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}) \quad \text{and} \quad \mathcal{A}(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}).$$

We do not know full characterizations of

$$\mathcal{A}(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}), \quad \mathcal{A}(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}) \quad \text{and} \quad \mathcal{A}(\mathcal{LPC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}).$$

In Theorem 6.2, we present some estimations for these adders.

It is worth mentioning that when we talk about measurability, we mean the Lebesgue measure. Moreover, we only consider real functions defined on an open (possible unbounded) interval I.

3. Properties of density

We will now introduce the basic properties of the lower density and introduce important technical lemmas needed later in the paper. The following properties of the lower density are immediate consequences of the definitions.

Proposition 3.1. *Let $E, F \in \mathcal{S}$. Then,*

(1) $\underline{d}^+(E \cap F, x) \geq \underline{d}^+(E, x) + \underline{d}^+(F, x) - 1$.

(2) $\underline{d}^+(E \cup F, x) \geq \underline{d}^+(E, x) + \underline{d}^+(F, x)$.

(3) $\underline{d}^+(E \cup F, x) \leq \underline{d}^+(E, x) + \overline{d}^+(F, x)$.

(4) $\underline{d}^+(E, x) = 1 - \overline{d}^+(\mathbb{R} \setminus E, x)$.

(5) *If* $\underline{d}^+(F, x) = \overline{d}^+(F, x)$ *and* $E \subset F$, *then* $\underline{d}^+(F \setminus E, x) = \underline{d}^+(F, x) - \overline{d}^+(E, x)$.

(6) *If* $\underline{d}^+(E, x) = \overline{d}^+(E, x)$ *and* $E \subset F$, *then* $\underline{d}^+(F \setminus E, x) = \underline{d}^+(F, x) - \underline{d}^+(E, x)$.

Definition 3.2. A set $E \subset \mathbb{R}$ is called the interval set to the right of the point $x \in \mathbb{R}$ if it is of the form

$$E = \{x\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n],$$

where $x < \cdots < b_{n+1} < a_n < b_n < \cdots < b_1$ and $\lim_{n\to\infty} a_n = x$.

Lemma 3.3. *If $E = \{x\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n]$ is the interval set to the right of the point $x \in \mathbb{R}$, then*

$$\underline{d}^+(E, x) = \liminf_{n\to\infty} \frac{\sum_{k=n+1}^{\infty}(b_k - a_k)}{a_n - x}$$

and

$$\overline{d}^+(E, x) = \limsup_{n\to\infty} \frac{\sum_{k=n}^{\infty}(b_k - a_k)}{b_n - x}.$$

P r o o f. The formulas follow immediately from the fact that for $t \in [a_n, b_n]$ we have

$$\frac{\lambda(E \cap [x, a_n])}{a_n - x} \leq \frac{\lambda(E \cap [x, t])}{t - x} \leq \frac{\lambda(E \cap [x, b_n])}{b_n - x}$$

(we apply an obvious inequality $\frac{a+c}{b+c} \geq \frac{a}{b}$ for $0 < a \leq b$ and $c \geq 0$) and for $t \in [b_{n+1}, a_n]$ we have

$$\frac{\lambda(E \cap [x, a_n])}{a_n - x} \leq \frac{\lambda(E \cap [x, a_n])}{t - x} = \frac{\lambda(E \cap [x, t])}{t - x} = \frac{\lambda(E \cap [x, b_{n+1}])}{t - x}$$
$$\leq \frac{\lambda(E \cap [x, b_{n+1}])}{b_{n+1} - x}. \qquad \square$$

The following two propositions are obvious.

Proposition 3.4. *Let $E_1, E_2, F_1, F_2 \in \mathcal{S}$ and $\overline{d}\,(E_1 \setminus E_2, x) = \overline{d}\,(E_2 \setminus E_1, x) = 0 = \overline{d}\,(F_1 \setminus F_2, x) = \overline{d}\,(F_2 \setminus F_1, x)$ for some $x \in \mathbb{R}$. Then, $\underline{d}\,(E_1 \cup F_1, x) = \underline{d}\,(E_2 \cup F_2, x)$, $\underline{d}\,(E_1 \cap F_1, x) = \underline{d}\,(E_2 \cap F_2, x)$, $\overline{d}\,(E_1 \cup F_1, x) = \overline{d}\,(E_2 \cup F_2, x)$ and $\overline{d}\,(E_1 \cap F_1, x) = \overline{d}\,(E_2 \cap F_2, x)$.*

Proposition 3.5. *Let $E_1, E_2, F \in \mathcal{S}$, $\underline{d}^+(E_1, x) = 1$ and $\overline{d}^+(E_2, x) = 0$ for some $x \in \mathbb{R}$. Then, $\underline{d}^+(E_1 \cap F, x) = \underline{d}^+(F, x)$, $\overline{d}^+(E_1 \cap F, x) = \overline{d}^+(F, x)$, $\underline{d}^+(E_2 \cup F, x) = \underline{d}^+(F, x)$ and $\overline{d}^+(E_2 \cup F, x) = \overline{d}^+(F, x)$.*

Theorem 3.6. *For every $E \in \mathcal{S}$ and every $x \in E$ one can find a sequence of closed intervals $([a_n, b_n])_{n\geq 1}$ such that $x < \cdots < b_{n+1} < a_n < b_n < \cdots < b_1$, $\lim_{n\to\infty} a_n = x$ and $\overline{d}^+(\bigcup_{n=1}^{\infty}[a_n, b_n] \setminus E, x) = 0 = \overline{d}^+(E \setminus \bigcup_{n=1}^{\infty}[a_n, b_n], x)$. In particular,*

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n], x\right) = \overline{d}^+(E, x) \quad \textit{and} \quad \underline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n], x\right) = \underline{d}^+(E, x).$$

P r o o f. For every $n \geq 1$ let $E_n = E \cap [x + \frac{1}{n+1}, x + \frac{1}{n}]$, F_n be a closed subset of E_n such that $\lambda(E_n \setminus F_n) < \frac{1}{2^n}$ and let $((a_k^n, b_k^n))_{k\geq 1}$ be a set of connected components of $[x + \frac{1}{n+1}, x + \frac{1}{n}] \setminus F_n$. Take $n \geq 1$. Then,

$$\sum_{k=1}^{\infty}(b_k^n - a_k^n) = \lambda\left(\left[x + \frac{1}{n+1}, x + \frac{1}{n}\right] \setminus F_n\right).$$

We can find $k_n > 1$ for which $\sum_{k=k_n+1}^{\infty}(b_k^n - a_k^n) < \frac{1}{2^n}$. Let

$$G_n = \left[x + \frac{1}{n+1}, x + \frac{1}{n}\right] \Big\backslash \bigcup_{k=1}^{k_n}(a_k^n, b_k^n) \quad \text{and} \quad G = \{x\} \cup \bigcup_{n=1}^{\infty} G_n.$$

Obviously, G is the interval set to the right of x,

$$G \setminus E \subset \bigcup_{n=1}^{\infty} \bigcup_{k=k_n+1}^{\infty} (a_k^n, b_k^n)$$

and

$$(E \setminus G) \cap [x, x+1] \subset \bigcup_{n=1}^{\infty}(E_n \setminus F_n).$$

Therefore, for $t \in [x + \frac{1}{n+1}, x + \frac{1}{n}]$ we have

$$\frac{\lambda((G \setminus E) \cap [x, t])}{t - x} \leq \frac{\sum_{k=n}^{\infty} \frac{1}{2^k}}{\frac{1}{n+1}} = \frac{\frac{1}{2^{n-1}}}{\frac{1}{n+1}} = \frac{n+1}{2^{n-1}}$$

and

$$\frac{\lambda((E \setminus G) \cap [x, t])}{t - x} \leq \frac{\sum_{k=n}^{\infty} \frac{1}{2^k}}{\frac{1}{n+1}} = \frac{n+1}{2^{n-1}}.$$

Hence, $\overline{d}\,(G \setminus E, x) = \overline{d}\,(E \setminus G, x) = 0$. □

Lemma 3.7. *Let $E \in \mathcal{S}$ and $x \in E$. For every $\varrho \in (\underline{d}^+(E,x), 1]$ there exists $F \in \mathcal{S}$ such that $E \subset F$ and $\underline{d}^+(F,x) = \varrho$.*

Proof. By Theorem 3.6, we can find a sequence of closed intervals $([a_n, b_n])_{n\ge 1}$ such that $x < \dots < b_{n+1} < a_n < b_n < \dots < b_1$, $\lim_{n\to\infty} a_n = x$ and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n]\setminus E, x\right) = 0 = \overline{d}^+\left(E\setminus\bigcup_{n=1}^{\infty}[a_n,b_n], x\right).$$

Then,

$$\underline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n], x\right) = \liminf_{n\to\infty}\frac{\sum_{k=n+1}^{\infty}(b_k-a_k)}{a_n-x}.$$

Choose $\varrho \in \big(\underline{d}^+(E,x), 1\big]$ and let $(n_m)_{m\ge 1}$ be any increasing sequence of natural numbers for which $\lim_{m\to\infty}\frac{b_{n_{m+1}}-x}{a_{n_m}-x} = 0$ and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n], x\right) = \lim_{m\to\infty}\frac{\sum_{k=n_m+1}^{\infty}(b_k-a_k)}{a_{n_m}-x}.$$

Without any loss of generality, we may assume

$$\sum_{k=n_m+1}^{n_{m+1}}(b_k-a_k) < \varrho(a_{n_m}-a_{n_{m+1}}) \quad \text{for every } m.$$

Then, for every $m \ge 1$ we can find $x_m \in [a_{n_{m+1}}, a_{n_m}]$ such that

$$\lambda\left([a_{n_{m+1}}, x_m] \cup \bigcup_{k=n_m+1}^{n_{m+1}}[a_k,b_k]\right) = \varrho(a_{n_m}-a_{n_{m+1}}).$$

It is easy to see

$$\underline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n] \cup \bigcup_{m=1}^{\infty}[a_{n_{m+1}},x_m], x\right) = \varrho.$$

Hence, it is sufficient to take $F = E \cup \bigcup_{m=1}^{\infty}[a_{n_{m+1}}, x_m]$. $\square$

Lemma 3.8. *Let $E \in \mathcal{S}$ and $x \in E$. For every $\varrho \in (\underline{d}^+(E,x), 1]$ there exists $F \in \mathcal{S}$ such that*

$$\underline{d}^+(F,x) = \overline{d}^+(F,x) = \varrho\,,$$

$$\underline{d}^+(F\cap E,x) = \varrho\underline{d}^+(E,x)$$

and

$$\underline{d}^+(F\setminus E,x) = \varrho\big(1-\overline{d}^+(E,x)\big).$$

Proof. By Theorem 3.6, we can find a sequence of closed intervals $([a_n,b_n])_{n\geq 1}$ such that $x<\dots<b_{n+1}<a_n<b_n<\dots<b_1$, $\lim_{n\to\infty}a_n=x$,

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n]\setminus E,x\right)=0=\overline{d}^+\left(E\setminus\bigcup_{n=1}^{\infty}[a_n,b_n],x\right).$$

Then,

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n],x\right)=\overline{d}^+(E,x)=\limsup_{n\to\infty}\frac{\sum_{k=n}^{\infty}(b_k-a_k)}{b_n-x}.$$

Choose $\varrho\in(\underline{d}^+(E,x),1]$. For every $n\geq 1$ fix $m_n\in\mathbb{N}$ for which $\frac{b_n-a_n}{m_n}<\frac{a_n-x}{n}$ and $\frac{a_n-b_{n+1}}{m_n}<\frac{b_{n+1}-x}{n}$ and choose

$$(c_i^n)_{i=0}^{m_n},\quad (d_i^n)_{i=1}^{m_n},\quad (e_i^n)_{i=0}^{m_n},\quad (f_i^n)_{i=1}^{m_n}$$

such that

$$\begin{aligned}a_n=c_0^n<d_1^n<c_1^n &<\dots< d_{m_n}^n<c_{m_n}^n=b_n,\\ b_{n+1}=e_0^n<f_1^n<e_1^n &<\dots< f_{m_n}^n<e_{m_n}^n=a_n,\end{aligned}$$

$$\begin{aligned}c_{i+1}^n-c_i^n&=\tfrac{b_n-a_n}{m_n},\\ e_{i+1}^n-e_i^n&=\tfrac{a_n-b_{n+1}}{m_n},\\ d_{i+1}^n-c_i^n&=\varrho(c_{i+1}^n-c_i^n),\\ f_{i+1}^n-e_i^n&=\varrho(e_{i+1}^n-e_i^n)\end{aligned}$$

for all n and all proper i. Let

$$F=\{x\}\cup\bigcup_{n=1}^{\infty}\left(\bigcup_{i=0}^{m_n-1}[c_i^n,d_{i+1}^n]\cup\bigcup_{i=0}^{m_n-1}[e_i^n,f_{i+1}^n]\right).$$

Clearly,

$$\lambda(F\cap[x,c_i^n])=\varrho(c_i^n-x)\quad\text{for }\ n\geq 1\text{ and }i\in\{0,\dots,m_n\}.$$

Moreover,

$$\begin{aligned}\varrho-\frac{1}{n}\leq\varrho-\frac{\frac{a_n-x}{n}}{t-x}\leq\frac{\varrho(c_{i-1}^n-x)}{t-x}\leq\frac{\lambda(F\cap[x,t])}{t-x}\leq\frac{\varrho(c_i^n-x)}{t-x}\\ \leq\varrho+\frac{\frac{a_n-x}{n}}{t-x}\leq\varrho+\frac{1}{n}\end{aligned}$$

for every $t\in[c_{i-1}^n,c_i^n]$, $n\geq 1$ and $i\in\{1,2,\dots,m_n\}$. Similarly, $\lambda(F\cup[x,e_i^n])=\varrho(e_i^n-x)$ for $n\geq 1$ and $i\in\{0,\dots,m_n\}$ and

$$\begin{aligned}\varrho-\frac{1}{n}\leq\varrho-\frac{\frac{a_n-x}{n}}{t-x}\leq\frac{\varrho(e_{i-1}^n-x)}{t-x}\leq\frac{\lambda(F\cap[x,t])}{t-x}\leq\frac{\varrho(e_i^n-x)}{t-x}\\ \leq\varrho+\frac{\frac{a_n-x}{n}}{t-x}\leq\varrho+\frac{1}{n}\end{aligned}$$

for every $t\in[e_{i-1}^n,e_i^n]$, $n\geq 1$ and $i\in\{1,2,\dots,m_n\}$.

Hence,

$$\underline{d}^{+}(F,x)=\liminf_{t\to x^{+}}\frac{\lambda(F\cap[x,t])}{t-x}=\varrho \quad\text{and}\quad \overline{d}^{+}(F,x)=\limsup_{t\to x^{+}}\frac{\lambda(F\cap[x,t])}{t-x}=\varrho,$$

i.e., $d^{+}(F,x)=\varrho$.

Let $A=\bigcup_{n=1}^{\infty}[a_n,b_n]$. Obviously,

$$\lambda(F\cap A\cap[x,c_i^n])=\varrho\lambda(A\cap[x,c_i^n])$$

for every $n\geq 1$ and $i\in\{0,\dots,m_n\}$. Then,

$$\frac{\lambda(F\cap A\cap[x,t])}{t-x}\leq\frac{\varrho\lambda(A\cap[x,c_i^n])}{t-x}\leq\frac{\varrho\lambda(A\cap[x,t])}{t-x}+\frac{\frac{a_n-x}{n}}{t-x}$$
$$\leq\varrho\frac{\lambda(A\cap[x,t])}{t-x}+\frac{1}{n}$$

and

$$\frac{\lambda(F\cap A\cap[x,t])}{t-x}\geq\frac{\varrho\lambda(A\cap[x,c_{i-1}^n])}{t-x}\geq\frac{\varrho\lambda(A\cap[x,t])}{t-x}-\frac{\frac{a_n-x}{n}}{t-x}$$
$$\geq\varrho\frac{\lambda(A\cap[x,t])}{t-x}-\frac{1}{n}$$

for $t\in[c_{i-1}^n,c_i^n]$, $n\geq 1$, $i\in\{1,2,\dots,m_n\}$. Analogously,

$$\lambda(F\cap A\cap[x,e_i^n])=\varrho\lambda(A\cap[x,e_i^n])\quad\text{for}\ \ n\geq 1\ \ \text{and}\ \ i\in\{0,\dots,m_n\}.$$

Moreover,

$$\frac{\lambda(F\cap A\cap[x,t])}{t-x}\leq\frac{\varrho\lambda(A\cap[x,e_i^n])}{t-x}\leq\frac{\varrho\lambda(A\cap[x,t])}{t-x}+\frac{\frac{a_n-x}{n}}{t-x}$$
$$\leq\varrho\frac{\lambda(A\cap[x,t])}{t-x}+\frac{1}{n}$$

and

$$\frac{\lambda(F\cap A\cap[x,t])}{t-x}\geq\frac{\varrho\lambda(A\cap[x,e_{i-1}^n])}{t-x}\geq\frac{\varrho\lambda(A\cap[x,t])}{t-x}-\frac{\frac{a_n-x}{n}}{t-x}$$
$$\geq\varrho\frac{\lambda(A\cap[x,t])}{t-x}-\frac{1}{n}$$

for $t\in[e_{i-1}^n,e_i^n]$, $n\geq 1$, $i\in\{1,2,\dots,m_n\}$. Then,

$$\underline{d}^{+}(F\cap A,x)=\liminf_{t\to x^{+}}\frac{\lambda(F\cap A\cap[x,t])}{t-x}=\varrho\underline{d}^{+}(A,x).$$

Since $\overline{d}^{+}(A\setminus E,x)=\overline{d}^{+}(E\setminus A,x)=0$, we obtain

$$\underline{d}^{+}(F\cap E,x)=\underline{d}^{+}(F\cap A,x)=\varrho\underline{d}^{+}(A,x)=\varrho\underline{d}^{+}(E,x).$$

Now, we take

$$B = [x, b_1] \setminus A = \bigcup_{n=1}^{\infty} (b_{n+1}, a_n).$$

Obviously,

$$\overline{d}^+\big(B \setminus (\mathbb{R} \setminus E), x\big) = \overline{d}^+\big((\mathbb{R} \setminus E) \setminus B, x\big) = 0.$$

Clearly,

$$\lambda(F \cap B \cap [x, c_i^n]) = \varrho\lambda(B \cap [x, c_i^n]) \quad \text{for every } \ n \geq 1 \text{ and } i \in \{0, \dots, m_n\}.$$

Moreover,

$$\frac{\lambda(F \cap B \cap [x,t])}{t-x} \leq \frac{\varrho\lambda(B \cap [x, c_i^n])}{t-x} \leq \frac{\varrho\lambda(B \cap [x,t])}{t-x} + \frac{\frac{a_n - x}{n}}{t-x} \leq \varrho\frac{\lambda(B \cap [x,t])}{t-x} + \frac{1}{n}$$

and

$$\frac{\lambda(F \cap B \cap [x,t])}{t-x} \geq \frac{\varrho\lambda(B \cap [x, c_{i-1}^n])}{t-x} \geq \frac{\varrho\lambda(B \cap [x,t])}{t-x} - \frac{\frac{a_n - x}{n}}{t-x} \geq \varrho\frac{\lambda(B \cap [x,t])}{t-x} - \frac{1}{n}$$

for $t \in [c_{i-1}^n, c_i^n]$, $n \geq 1$, $i \in \{1, 2, \dots, m_n\}$. Analogously,

$$\lambda(F \cap B \cap [x, e_i^n]) = \varrho\lambda(B \cap [x, e_i^n]) \quad \text{for every } \ n \geq 1 \ \text{ and } \ i \in \{0, \dots, m_n\}.$$

Moreover,

$$\frac{\lambda(F \cap B \cap [x,t])}{t-x} \leq \frac{\varrho\lambda(B \cap [x, e_i^n])}{t-x} \leq \frac{\varrho\lambda(B \cap [x,t])}{t-x} + \frac{\frac{a_n - x}{n}}{t-x} \leq \varrho\frac{\lambda(B \cap [x,t])}{t-x} + \frac{1}{n}$$

and

$$\frac{\lambda(F \cap B \cap [x,t])}{t-x} \geq \frac{\varrho\lambda(B \cap [x, e_{i-1}^n])}{t-x} \geq \frac{\varrho\lambda(B \cap [x,t])}{t-x} - \frac{\frac{a_n - x}{n}}{t-x} \geq \varrho\frac{\lambda(B \cap [x,t])}{t-x} - \frac{1}{n}$$

for $t \in [e_{i-1}^n, e_i^n]$, $n \geq 1$, $i \in \{1, 2, \dots, m_n\}$. Then,

$$\underline{d}^+(F \cap B, x) = \liminf_{t \to x^+} \frac{\lambda(F \cap B \cap [x,t])}{t-x} = \varrho\underline{d}^+(B, x) = \varrho\underline{d}^+(\mathbb{R} \setminus A, x) = \varrho\underline{d}^+(\mathbb{R} \setminus E, x).$$

Finally,

$$\begin{aligned}\underline{d}^{+}(F\setminus E,x)&=\underline{d}^{+}(F\setminus A,x)\\&=\underline{d}^{+}(F\cap B,x)=\varrho\underline{d}^{+}(\mathbb{R}\setminus E,x)=\varrho\big(1-\overline{d}^{+}(E,x)\big),\end{aligned}$$

by Proposition 3.1. The proof is completed. □

Definition 3.9. For $\varrho\in[0,1)$ let

$$L_{\varrho}=\{E\in\mathcal{S}\colon \underline{d}\,(E,x)>\varrho \quad \text{for every } \ x\in E\}$$

and

$$U_{\varrho}=\{E\in\mathcal{S}\colon \min\{\overline{d}^{+}(E,x),\overline{d}^{-}(E,x)\}>\varrho \quad \text{for every } \ x\in E\}.$$

Similarly, for $\varrho\in(0,1]$ let

$$L_{\varrho}^{\blacklozenge}=\{E\in\mathcal{S}\colon \underline{d}\,(E,x)\geq\varrho \quad \text{for every } \ x\in E\}$$

and

$$U_{\varrho}^{\blacklozenge}=\big\{E\in\mathcal{S}\colon \min\{\overline{d}^{+}(E,x),\overline{d}^{-}(E,x)\}\geq\varrho \quad \text{for every } \ x\in E\big\}.$$

4. $[\varrho_1,\varrho_2]$-lower superdensity

We now introduce the notion of $[\varrho_1,\varrho_2]$-lower superdensity and present its basic properties. To this end, we start with an important theorem showing the equivalence of certain conditions.

Theorem 4.1. *Let $0<\varrho_1\leq\varrho_2<1$, $E\in\mathcal{S}$ and $x\in E$. The following properties are equivalent:*

(1) *For every $F\in\mathcal{S}$ containing x, if $\underline{d}\,(F,x)>\varrho_2$, then $\underline{d}(E\cap F,x)>\varrho_1$.*

(2) *For every $F\in\mathcal{S}$ containing x, if $\underline{d}\,(F,x)\geq\varrho_2$, then $\underline{d}(E\cap F,x)\geq\varrho_1$.*

(3) *For every $F\in\mathcal{S}$ containing x, if $\underline{d}\,(F,x)>\varrho_2$, then $\underline{d}(E\cap F,x)\geq\varrho_1$.*

Proof.

(1)⇒(3) Implication is obvious.

(2)⇒(3) Assume that condition (2) is fulfilled. Take $F\in\mathcal{S}$ such that $\underline{d}\,(F,x)>\varrho_2$. Obviously, $\underline{d}\,(F,x)\geq\varrho_2$. By condition (2), we obtain $\underline{d}\,(E\cap F,x)\geq\varrho_1$.

(3)⇒(1) Suppose, on the contrary, that there exist $E\in\mathcal{S}$ and $x\in E$ that satisfy (3) but do not satisfy (1). Then, we can find $F\in\mathcal{S}$ such that

$$x\in F,\quad \underline{d}\,(F,x)>\varrho_2 \quad\text{and}\quad \underline{d}(E\cap F,x)\leq\varrho_1.$$

Without loss of generality, we may assume that $\underline{d}^{+}(E\cap F,x)\leq\varrho_1$.

By Theorem 3.6, we can find a sequence of closed intervals

$$([a_n, b_n])_{n\geq 1}$$

such that

$$x < \cdots < b_{n+1} < a_n < b_n < \cdots < b_1, \quad \lim_{n\to\infty} a_n = x$$

and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n] \setminus (E\cap F), x\right) = 0 = \overline{d}^+\left((E\cap F)\setminus \bigcup_{n=1}^{\infty}[a_n, b_n], x\right).$$

Take any $\alpha \in (0, \underline{d}\,(F,x) - \varrho_2)$. For every $n \geq 1$ let $c_n = (1-\alpha)b_n + \alpha a_n$, i.e.,

$$b_n - c_n = \alpha(b_n - a_n) \quad \text{and} \quad c_n - a_n = (1-\alpha)(b_n - a_n).$$

Define

$$G_\alpha = \bigcup_{n=1}^{\infty}[a_n, c_n], \quad H_\alpha = \bigcup_{n=1}^{\infty}[c_n, b_n]$$

and

$$F_\alpha = F \setminus H_\alpha.$$

Then,

$$\begin{aligned}\underline{d}^+(G_\alpha, x) &= \liminf_{n\to\infty} \frac{\sum_{k=n+1}^{\infty}(c_k - a_k)}{a_n - x} = \liminf_{n\to\infty}(1-\alpha)\frac{\sum_{k=n+1}^{\infty}(b_k - a_k)}{a_n - x}\\ &= (1-\alpha)\underline{d}^+(E\cap F, x)\end{aligned}$$

and

$$\begin{aligned}\overline{d}^+(H_\alpha, x) &= \limsup_{n\to\infty} \frac{\sum_{k=n}^{\infty}(b_k - c_k)}{b_n - x} = \limsup_{n\to\infty} \alpha\frac{\sum_{k=n}^{\infty}(b_k - a_k)}{b_n - x}\\ &= \alpha\overline{d}^+(E\cap F, x) \leq \alpha.\end{aligned}$$

Hence,

$$\underline{d}^+(F_\alpha, x) \geq \underline{d}^+(F,x) - \overline{d}^+(H_\alpha, x) \geq \underline{d}^+(F,x) - \alpha > \varrho_2$$

and

$$\underline{d}^+(E\cap F_\alpha, x) = \underline{d}^+(G_\alpha, x) = (1-\alpha)\underline{d}^+(E\cap F, x) < \underline{d}^+(E\cap F, x) \leq \varrho_1.$$

The set F_α witnesses that (3) does not hold for E and x, a contradiction.

(3) $\Rightarrow$ (2) Suppose, on the contrary that there exist $E \in \mathcal{S}$ and $x \in E$ that satisfy (3) but do not satisfy (2). Then, we can find $F\in\mathcal{S}$ such that $x\in F$,

$$\underline{d}\,(F,x) \geq \varrho_2 \quad \text{and} \quad \underline{d}(E\cap F, x) < \varrho_1.$$

Without any loss of generality, we may assume that

$$\underline{d}^+(E\cap F, x) < \varrho_1.$$

By Theorem 3.6, we can find a sequence of closed intervals $([a_n, b_n])_{n\geq 1}$ such that

$$x < \cdots < b_{n+1} < a_n < b_n < \cdots < b_1, \quad \lim_{n\to\infty} a_n = x$$

and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n]\setminus F, x\right) = 0 = \overline{d}^+\left(F\setminus \bigcup_{n=1}^{\infty}[a_n,b_n], x\right).$$

In particular,

$$\varrho_2 \leq \underline{d}^+(F,x) = \liminf_{n\to\infty} \frac{\sum_{k=n}^{\infty}(b_k-a_k)}{a_{n-1}-x}.$$

Therefore, we may assume

$$b_n - x > \sum_{k=n}^{\infty}(b_k-a_k) > \frac{\varrho_2}{2}(a_{n-1}-x) \quad \text{and} \quad \frac{b_n-x}{a_{n-1}-x} > \frac{\varrho_2}{2} \quad \text{for every } n>1.$$

Fix any positive $\alpha < \frac{\varrho_2}{2}(\varrho_1 - \underline{d}^+(E\cap F, x))$. For every $n\geq 2$ let

$$c_n = (1-\alpha)b_n + \alpha a_{n-1} \quad \text{and} \quad d_n = \alpha a_n + (1-\alpha)b_n,$$

i.e., $c_n - b_n = \alpha(a_{n-1}-b_n)$, $b_n - d_n = \alpha(b_n-a_n)$ and $c_n - d_n = \alpha(a_{n-1}-a_n)$.

Define

$$G_\alpha = \bigcup_{n=2}^{\infty}[b_n,c_n] \quad \text{and} \quad F_\alpha = F\cup G_\alpha \cup (-\infty, x).$$

Then, $\underline{d}^-(F_\alpha, x) = 1$ and

$$\begin{aligned}
\underline{d}^+(F_\alpha,x) &= \liminf_{n\to\infty}\frac{\sum_{k=n+1}^{\infty}(c_k-a_k)}{a_n-x} \\
&= \liminf_{n\to\infty}\frac{\sum_{k=n+1}^{\infty}(d_k-a_k)+\sum_{k=n+1}^{\infty}(c_k-d_k)}{a_n-x} \\
&\geq (1-\alpha)\liminf_{n\to\infty}\frac{\sum_{k=n+1}^{\infty}(b_k-a_k)}{a_n-x} + \alpha\liminf_{n\to\infty}\frac{\sum_{k=n+1}^{\infty}(a_{k-1}-a_k)}{a_n-x} \\
&= (1-\alpha)\underline{d}^+(F,x)+\alpha = \underline{d}^+(F,x)+\alpha\big(1-\underline{d}^+(F,x)\big) > \underline{d}^+(F,x).
\end{aligned}$$

Therefore, $\underline{d}\,(F_\alpha,x) > \underline{d}\,(F,x) \geq \varrho_2$. On the other hand,

$$\begin{aligned}
\overline{d}^+(G_\alpha,x) &= \limsup_{n\to\infty}\frac{\sum_{k=n}^{\infty}(c_k-b_k)}{c_n-x} = \limsup_{n\to\infty}\left[\alpha\frac{\sum_{k=n}^{\infty}(a_{k-1}-b_k)}{a_{n-1}-x}\cdot\frac{a_{n-1}-x}{c_n-x}\right] \\
&\leq \alpha\underline{d}^+(\mathbb{R}\setminus F,x)\frac{2}{\varrho_2} \leq \alpha\frac{2}{\gamma_2} < \varrho_1 - \underline{d}^+(E\cap F,x)
\end{aligned}$$

and

$$\underline{d}^+(E\cap F_\alpha,x) \leq \underline{d}^+(E\cap F,x)+\overline{d}^+(G_\alpha,x) < \underline{d}^+(E\cap F,x)+\varrho_1-\underline{d}^+(E\cap F,x) = \varrho_1.$$

Thus,

$$\underline{d}\,(F_\alpha,x) > \varrho_2 \quad \text{and} \quad \underline{d}^+(E\cap F_\alpha,x) < \varrho_1.$$

A contradiction, because E satisfies condition (3). The proof is completed. $\square$

Definition 4.2. Let $0 \le \varrho_1 \le \varrho_2 \le 1$ and $\varrho_2 - \varrho_1 < 1$. For $0 < \varrho_1 \le \varrho_2 < 1$ we say that a set $E \in \mathcal{S}$ is $[\varrho_1,\varrho_2]$-lower superdense at $x \in E$ if it satisfies one of the conditions of Theorem 4.1. For $0 \le \varrho_2 < 1$ we say that $E \in \mathcal{S}$ is $[0,\varrho_2]$-lower superdense at $x \in E$ if it satisfies the first condition of Theorem 4.1. For $0 < \varrho_1 \le 1$ we say that $E \in \mathcal{S}$ is $[\varrho_1, 1]$-lower superdense at $x \in E$ if it satisfies the second condition of Theorem 4.1. We say that $E \in \mathcal{S}$ is $[\varrho_1,\varrho_2]$-lower superdense if it is $[\varrho_1,\varrho_2]$-lower superdense at each of its point. We denote the set of all $[\varrho_1,\varrho_2]$-lower superdense subsets of $\mathbb{R}$ by $L(\varrho_1,\varrho_2)$.

Corollary 4.3. *If $E \in \mathcal{S}$ and $x \in E$ is the density point of E, then E is $[\varrho_1,\varrho_2]$-lower superdense at x for every $0 \le \varrho_1 \le \varrho_2 \le 1$ and $\varrho_2 - \varrho_1 < 1$.*

Corollary 4.4. *Let $0 < \varrho_1 \le \varrho_2 < 1$ and $E \in \mathcal{S}$. If $E \notin L(\varrho_1,\varrho_2)$, then there exist $x \in E$ and $F \in \mathcal{S}$ such that $\underline{d}\,(F,x) > \varrho_2$ and $\underline{d}\,(E \cap F, x) < \varrho_1$.*

Corollary 4.5. *Let $0 < \varrho_1 \le \varrho_2 < 1$ and $E \in \mathcal{S}$ be $[\varrho_1,\varrho_2]$-lower superdense at some $x \in E$. There exists $F \subset E$ such that $x \in F$ and F belongs to $L(\varrho_1,\varrho_2)$.*

P r o o f. It suffices to take F to be the union of $\{x\}$ and the set of the density points of E. $\square$

Theorem 4.6. $L^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subset L(\varrho_1,\varrho_2)$ *for every* $0 \le \varrho_1 \le \varrho_2 \le 1$ *and* $\varrho_2 - \varrho_1 < 1$. *Moreover, $L^{\blacklozenge}_1 = L(\varrho,\varrho)$ for $\varrho \in (0,1)$ and $L^{\blacklozenge}_{\varrho} = L(\varrho,1)$ for $\varrho \in (0,1]$. Otherwise, the inclusion $L^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subset L(\varrho_1,\varrho_2)$ is proper.*

P r o o f. First, we show $L^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subset L(\varrho_1,\varrho_2)$ for $0 \le \varrho_1 \le \varrho_2 < 1$. Let $E \in L^{\blacklozenge}_{1-\varrho_2+\varrho_1}$ and $x \in E$. Then, $\underline{d}\,(E,x) \ge 1 - \varrho_2 + \varrho_1$. Choose $F \in \mathcal{S}$ such that $\underline{d}\,(F,x) > \varrho_2$. Then,

$$\underline{d}\,(E \cap F, x) \ge \underline{d}\,(E,x) + \underline{d}\,(F,x) - 1 > 1 - \varrho_2 + \varrho_1 + \varrho_2 - 1 = \varrho_1,$$

by Proposition 3.1 (1). Since F and x are arbitrary, $E \in L(\varrho_1,\varrho_2)$. Therefore,

$$L^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subset L(\varrho_1,\varrho_2) \quad \mathit{for} \quad 0 \le \varrho_1 \le \varrho_2 < 1.$$

Now, we show $L^{\blacklozenge}_{\varrho} \subset L(\varrho,1)$ for $\varrho \in (0,1]$. If $F \in \mathcal{S}$ and $\underline{d}\,(F,x) = 1$, then $\underline{d}\,(E \cap F, x) = \underline{d}\,(E,x) \ge \varrho$ for every $E \in L^{\blacklozenge}_{\varrho}$. Hence,

$$L^{\blacklozenge}_{\varrho} \subset L(\varrho,1) \quad \text{for every } \varrho \in (0,1].$$

Take any $\varrho \in (0,1)$ and let $E \notin L^{\blacklozenge}_1$, i.e., $\underline{d}\,(E,x) = 1 - \overline{d}\,(\mathbb{R} \setminus E) < 1$ for some $x \in E$. We may assume $\underline{d}^{+}(E,x) < 1$.

By Lemma 3.8, applying for $\mathbb{R} \setminus E$, we can find $F_1 \in \mathcal{S}$ such that

$$\overline{d}^{+}(F_1,x) = \underline{d}^{+}(F_1,x) = \varrho$$

and

$$\underline{d}^{+}\big(F_1 \setminus (\mathbb{R} \setminus E), x\big) = \underline{d}^{+}(F_1 \cap E, x) = \varrho\big(1 - \overline{d}^{+}(\mathbb{R} \setminus E, x)\big) < \varrho.$$

Let $F = (-\infty, x] \cup F_1$. Then,

$$\underline{d}\,(F, x) = \varrho \quad \text{and} \quad \underline{d}^{+}(F \cap E, x) < \varrho.$$

Hence, $E \notin L(\varrho, \varrho)$ and the equality $L_1^{\blacklozenge} = L(\varrho, \varrho)$ for $\varrho \in (0, 1)$ is proved.

Now, take any $\varrho \in (0, 1]$ and let $E \notin L_\varrho^{\blacklozenge}$, i.e., $\underline{d}\,(E, x) < \varrho$ for some $x \in E$. Let $F = \mathbb{R}$. Then, $\underline{d}\,(F, x) = 1$ and $\underline{d}\,(F \cap E, x) = \underline{d}\,(E, x) < \varrho$. Hence, $E \notin L(\varrho, 1)$ and the equality $L_\varrho^{\blacklozenge} = L(\varrho, 1)$ for $\varrho \in (0, 1]$ is proved.

Finally, we show that for $0 \leq \varrho_1 < \varrho_2 < 1$ the inclusion is proper. Take any $\eta \in (0, 1)$ and let

$$E_\eta = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} (a_n, b_n),$$

where $0 < \cdots < b_{n+1} < a_n < b_n < \cdots$, $\lim_{n\to\infty} a_n = 0$, $b_{n+1} - a_{n+1} = \eta(a_n - a_{n+1})$ for every $n \geq 1$ and $\lim_{n\to\infty} \frac{b_n - a_n}{a_n} = \infty$. Since every $x \in E_\eta$, $x \neq 0$, is the density point of E_η, E_η is $[\varrho_1, \varrho_2]$-lower superdense at every $x \neq 0$. Moreover,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty} \frac{a_{n+1}}{b_{n+1} - a_{n+1}} = 0,$$

$$\lim_{n\to\infty} \frac{b_{n+1}}{a_n} = \lim_{n\to\infty} \left(\eta \left(1 - \frac{a_{n+1}}{a_n} \right) + \frac{a_{n+1}}{a_n} \right) = \eta$$

and

$$\underline{d}^{+}(E_\eta, 0) = \liminf_{n\to\infty} \frac{\sum_{k=n+1}^{\infty} (b_k - a_k)}{a_n} = \liminf_{n\to\infty} \frac{\eta a_n}{a_n} = \eta.$$

Let

$$\gamma = \frac{1 - \eta + \eta\varrho_2}{\varrho_2} = \frac{(1-\eta)(1-\varrho_2) + \varrho_2}{\varrho_2} \quad \text{and} \quad c_n = \gamma a_n \quad \text{for every } \ n \geq 1$$

(since $\lim_{n\to\infty} \frac{b_n - a_n}{a_n} = \infty$, we may assume $\frac{b_n}{a_n} > \gamma$ for all n and then $a_n < c_n < b_n$ for every $n \geq 1$).

Let us take $F \in \mathcal{S}$ such that $\underline{d}\,(F, 0) \geq \varrho_2$. We estimate $\underline{d}\,(F \cap E_\eta, 0)$. Obviously, $\underline{d}^{-}(F \cap E_\eta, 0) = \underline{d}^{-}(F, 0) \geq \varrho_2 > \varrho_1$. Fix $n \geq 1$ and take any $x \in [b_{n+1}, b_n]$. If $x \in [b_{n+1}, c_n]$, then

$$\begin{aligned}
\frac{\lambda((F \cap E_\eta) \cap [0, x])}{x} &\geq \frac{\lambda((F \cap E_\eta) \cap [0, b_{n+1}])}{c_n} \geq \frac{\lambda(F \cap [0, b_{n+1}]) - a_{n+1}}{c_n} \\
&\geq \frac{\lambda(F \cap [0, b_{n+1}])}{b_{n+1}} \frac{b_{n+1}}{c_n} - \frac{a_{n+1}}{a_n} \\
&= \frac{\lambda(F \cap [0, b_{n+1}])}{b_{n+1}} \frac{b_{n+1}}{\gamma a_n} - \frac{a_{n+1}}{a_n}
\end{aligned}$$

and if $x \in [c_n, b_n]$, then

$$\begin{aligned}\frac{\lambda((F\cap E_\eta)\cap[0,x])}{x} &\ge \frac{\lambda(F\cap[0,x]) - \sum_{k=n}^{\infty}(a_k - b_{k+1})}{x}\\ &\ge \frac{\lambda(F\cap[0,x])}{x} - \frac{\lambda((\mathbb{R}\setminus E_\eta)\cap[0,a_n])}{a_n}\cdot\frac{a_n}{c_n}\\ &= \frac{\lambda(F\cap[0,x])}{x} - \frac{\lambda((\mathbb{R}\setminus E_\eta)\cap[0,a_n])}{a_n}\frac{1}{\gamma}.\end{aligned}$$

Since

$$\limsup_{n\to\infty}\frac{\lambda((\mathbb{R}\setminus E_\eta)\cap[0,a_n])}{a_n} \le \overline{d}^+(\mathbb{R}\setminus E_\eta, 0) = 1 - \underline{d}^+(E_\eta,0) = 1-\eta,$$

$$\liminf_{n\to\infty}\frac{\lambda(F\cap[0,b_{n+1}])}{b_{n+1}} \ge \underline{d}^+(F,0) \ge \varrho_2,$$

$$\lim_{n\to\infty}\frac{b_{n+1}}{a_n} = \eta, \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 \quad \text{and} \quad \liminf_{x\to0^+}\frac{\lambda(F\cap[0,x])}{x} = \underline{d}^+(F,0) \ge \varrho_2,$$

we conclude

$$\underline{d}\,(F\cap E_\eta, 0) \ge \min\left\{\frac{\varrho_2\eta}{\gamma}, \varrho_2 - \frac{1-\eta}{\gamma}\right\}.$$

We check equality $\frac{\varrho_2\eta}{\gamma} = \varrho_2 - \frac{1-\eta}{\gamma}$. Actually, $\varrho_2\eta = \gamma\varrho_2 - 1 + \eta$ if and only if $\gamma\varrho_2 = 1 - \eta + \eta\varrho_2$ if and only if $\gamma = \frac{1-\eta+\eta\varrho_2}{\varrho_2}$, i.e., $\frac{1}{\gamma} = \frac{\varrho_2}{1-\eta+\eta\varrho_2}$. Therefore,

$$\underline{d}\,(F\cap E_\eta, 0) \ge \frac{\eta\varrho_2^2}{1-\eta+\eta\varrho_2}.$$

In particular, $\underline{d}\,(F\cap E_\eta,0) > 0$ and $E_\eta \in L(0,\varrho)\setminus L^{\blacklozenge}_{1-\varrho}$ for every $\varrho\in(0,1)$ and every $\eta\in(0,1-\varrho)$. Hence, the inclusion $L^{\blacklozenge}_{1-\varrho}\subset L(0,\varrho)$ is proper.

Now, we assume $\varrho_1 > 0$. Define function $\varphi\colon (0,1)\to(0,\infty)$ by

$$\varphi(t) = \frac{t\varrho_2^2}{1-t+t\varrho_2} = \frac{\varrho_2^2}{\frac{1}{t} - 1 + \varrho_2}.$$

Then, φ is strictly increasing, $\underline{d}^+(F\cap E_\eta, 0) \ge \varphi(\eta)$ and

$$\varphi\left(\frac{\varrho_1}{\varrho_2^2+\varrho_1-\varrho_1\varrho_2}\right) = \frac{\varrho_2^2}{\frac{\varrho_2^2+\varrho_1-\varrho_1\varrho_2}{\varrho_1} - 1 + \varrho_2} = \varrho_1.$$

It remains only to show the inequality

$$\frac{\varrho_1}{\varrho_2^2+\varrho_1-\varrho_1\varrho_2} < 1 - \varrho_2 + \varrho_1.$$

Rearranging this inequality, we obtain

$$\varrho_1 < \varrho_2(\varrho_2 - \varrho_1) + \varrho_1 - \varrho_2(\varrho_2 - \varrho_1)^2 - \varrho_1(\varrho_2 - \varrho_1),$$

$$0 < (\varrho_2 - \varrho_1)^2 - \varrho_2(\varrho_2 - \varrho_1)^2$$

and finally,

$$0 < (\varrho_2 - \varrho_1)^2(1 - \varrho_2).$$

The last inequality is obvious. We have showed that the inequality

$$\underline{d}\,(F \cap E_\eta, 0) > \varrho_1$$

holds for every $\eta \in \left(\frac{\varrho_1}{\varrho_2^2+\varrho_1-\varrho_1\varrho_2}, 1-\varrho_2+\varrho_1\right)$ and every $F \in \mathcal{S}$ satisfying $\underline{d}\,(F,0) \geq \varrho_2$. Therefore,

$$E_\eta \in L(\varrho_1, \varrho_2) \setminus L^{\blacklozenge}_{1-\varrho_2+\varrho_1} \quad \text{for } \eta \in \left(\frac{\varrho_1}{\varrho_2^2 + \varrho_1 - \varrho_1\varrho_2}, 1 - \varrho_2 + \varrho_1\right)$$

and the inclusion $L^{\blacklozenge}_{1-\varrho_2+\varrho_1} \subset L(\varrho_1, \varrho_2)$ is proper for $0 \leq \varrho_1 < \varrho_2 < 1$. □

Theorem 4.7. *Let $0 \leq \varrho_1 \leq \varrho_2 \leq 1$, $\varrho_2 - \varrho_1 < 1$. Then,*

(1) $L(\varrho_1, \varrho_2) \subsetneqq U^{\blacklozenge}_{1-\varrho_2+\varrho_1}$.

(2) *If $E \in L(\varrho_1, \varrho_2)$, $x \in E$ and $\underline{d}\,(E,x) = \overline{d}\,(E,x) = \alpha$ then $\alpha \geq 1 - \varrho_2 + \varrho_1$.*

Proof.

(1) Let $E \in \mathcal{S}$ and $E \notin U^{\blacklozenge}_{1-\varrho_2+\varrho_1}$. Then,

$$\min\{\overline{d}^-(E,x), \overline{d}^+(E,x)\} < 1 - \varrho_2 + \varrho_1 \quad \text{at some } x \in E.$$

We may assume that $\overline{d}^+(E,x) < 1 - \varrho_2 + \varrho_1$. Let

$$F = (-\infty, x] \cup (\mathbb{R} \setminus E).$$

Obviously,

$$E \cap F \cap (x, \infty) = \emptyset \quad \text{and} \quad \underline{d}^+(F,x) = 1 - \overline{d}^+(E,x) > \varrho_2 - \varrho_1.$$

If $\underline{d}^+(F,x) \geq \varrho_2$, then we immediately get $E \notin L(\varrho_1, \varrho_2)$, because $\underline{d}^+(E \cap F, x) = 0$. If $\underline{d}^+(F,x) < \varrho_2$, then, by Lemma 3.7, we can find $F' \in \mathcal{S}$ such that

$$\underline{d}^+(F',x) = \varrho_2 \quad \text{and} \quad F \subset F'.$$

Then,

$$\underline{d}^+(E \cap F', x) = \underline{d}^+(F' \setminus F, x) \leq \underline{d}^+(F',x) - \underline{d}^+(F,x) < \varrho_2 - (\varrho_2 - \varrho_1) = \varrho_1,$$

which witnesses $E \notin L(\varrho_1, \varrho_2)$. Hence, $L(\varrho_1, \varrho_2) \subset U^{\blacklozenge}_{1-\varrho_2+\varrho_1}$.

Let $E = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{(2n+1)!}, \frac{1}{(2n)!}\right)$. Obviously,

$$d(E,x) = 1 \text{ for every } x \in E \setminus \{0\}, \ \underline{d}^+(E,0) = 0 \text{ and } \overline{d}^-(E,0) = \overline{d}^+(E,0) = 1.$$

Hence,

$$E \in U^{\blacklozenge}_{1-\varrho_2+\varrho_1}.$$

Let us take $F=\mathbb{R}$. Then, $\underline{d}(F,0)=1$ and $\underline{d}^{+}(E\cap F,0)=\underline{d}^{+}(E,0)=0$. Thus, $E\notin L(\varrho_1,\varrho_2)$, and the inclusion $L(\varrho_1,\varrho_2)\subset U^{\blacklozenge}_{1-\varrho_2+\varrho_1}$ is proper.

(2) It follows immediately from (1) and Theorem 4.6. □

Definition 4.8. Let $0\le\varrho_1<\varrho_2\le 1$. We say that $E\in\mathcal{S}$ is $[\varrho_1,\varrho_2]^*$-lower superdense at $x\in E$ if for every $F\in\mathcal{S}$ containing x such that $\underline{d}(F,x)\ge\varrho_2$ implies $\underline{d}(E\cap F,x)>\varrho_1$. We say that E is $[\varrho_1,\varrho_2]^*$-lower superdense if it is $[\varrho_1,\varrho_2]^*$-lower superdense at each of its point. By $L^*(\varrho_1,\varrho_2)$, we denote the set of all $[\varrho_1,\varrho_2]^*$-lower superdense subsets of $\mathbb{R}$.

Theorem 4.9. *Let $0\le\varrho_1<\varrho_2\le 1$. Then, $L_{1-\varrho_2+\varrho_1}\subset L^*(\varrho_1,\varrho_2)$. Moreover, $L_\varrho=L^*(\varrho,1)$ for $\varrho\in(0,1)$ and $L_{1-\varrho_2+\varrho_1}\subsetneqq L^*(\varrho_1,\varrho_2)$, if $\varrho_2\ne 1$.*

Proof. (Proof of this theorem is very similar to the proof of Theorem 4.6). First, we show $L_{1-\varrho_2+\varrho_1}\subset L^*(\varrho_1,\varrho_2)$ for $0\le\varrho_1<\varrho_2\le 1$. Let $E\in L_{1-\varrho_2+\varrho_1}$ and $x\in E$. Then, $\underline{d}(E,x)>1-\varrho_2+\varrho_1$. Choose any $F\in\mathcal{S}$ such that $\underline{d}(F,x)\ge\varrho_2$. Then,

$$\underline{d}(E\cap F,x)\ge\underline{d}(E,x)+\underline{d}(F,x)-1>1-\varrho_2+\varrho_1+\varrho_2-1=\varrho_1,$$

by Proposition 3.1 (1). Therefore,

$$L_{1-\varrho_2+\varrho_1}\subset L^*(\varrho_1,\varrho_2)\quad\text{for }\ 0\le\varrho_1<\varrho_2\le 1.$$

Now, take any $\varrho\in(0,1)$ and let $E\notin L_\varrho$, i.e., $\underline{d}(E,x)\le\varrho$ for some $x\in E$. Let $F=\mathbb{R}$. Then, $\underline{d}(F,x)=1$ and $\underline{d}(F\cap E,x)=\underline{d}(E,x)\le\varrho$. Hence, $E\notin L^*(\varrho,1)$, and the equality $L_\varrho=L^*(\varrho,1)$ for $\varrho\in(0,1)$ is proved.

Finally, we show that for $0\le\varrho_1<\varrho_2<1$ the inclusion is proper. In the proof of Theorem 4.6, we constructed for every $\eta\in(0,1)$ a set $E_\eta\in\mathcal{S}$ such that every $x\in E_\eta$, $x\ne 0$, is a point of the density of E_η, $\underline{d}^{+}(E_\eta,0)=\eta$ such that for every $F\in\mathcal{S}$ satisfying $\underline{d}(F,0)\ge\varrho_2$, we have $\underline{d}(F\cap E_\eta,0)>\varrho_1$ for every $\eta\in\left(\frac{\varrho_1}{\varrho_2^2+\varrho_1-\varrho_1\varrho_2},1-\varrho_2+\varrho_1\right)$. Therefore,

$$E_\eta\in L^*(\varrho_1,\varrho_2)\setminus L_{1-\varrho_2+\varrho_1}\quad\text{for }\ \eta\in\left(\frac{\varrho_1}{\varrho_2^2+\varrho_1-\varrho_1\varrho_2},1-\varrho_2+\varrho_1\right)$$

and the inclusion $L_{1-\varrho_2+\varrho_1}\subset L^*(\varrho_1,\varrho_2)$ is proper for $0\le\varrho_1<\varrho_2<1$. □

Theorem 4.10. *Let $0\le\varrho_1<\varrho_2\le 1$. Then,*

(1) *$L^*(\varrho_1,\varrho_2)\subsetneqq U_{1-\varrho_2+\varrho_1}$.*

(2) *If $E\in L^*(\varrho_1,\varrho_2)$, $x\in E$ and $\underline{d}(E,x)=\overline{d}(E,x)=\alpha$, then $\alpha>1-\varrho_2+\varrho_1$.*

Proof of this theorem is very similar to the proof of Theorem 4.7 and we omit it.

Theorem 4.11. *For every* $0 \le \varrho_1 < \varrho_2 \le 1$*,* $\varrho_2 - \varrho_1 < 1$*, we have* $L^*(\varrho_1, \varrho_2) \subsetneqq L(\varrho_1, \varrho_2)$*.*

P r o o f. The inclusion follows directly from the definitions. Let $E = (-\infty, 0] \cup \bigcup_{n=1}^{\infty}(a_n, b_n)$, where $0 < \cdots < b_{n+1} < a_n < b_n < \cdots$, $\lim_{n\to\infty} a_n = 0$ and $\underline{d}^+(E, 0) = \overline{d}^+(E, 0) = 1 - \varrho_2 + \varrho_1$. Then, $E \notin L^*(\varrho_1, \varrho_2)$ and $E \in L(\varrho_1, \varrho_2)$, by Theorem 4.10 (2) and Theorem 4.7 (2). □

Theorem 4.12.

(1) *For every* $0 \le \varrho_1 < \varrho_1' \le \varrho_2 \le 1$*,* $\varrho_2 - \varrho_1 < 1$*, we have* $L(\varrho_1, \varrho_2) \supsetneqq L(\varrho_1', \varrho_2)$*.*
(2) *For every* $0 \le \varrho_1 < \varrho_1' < \varrho_2 \le 1$*, we have* $L^*(\varrho_1, \varrho_2) \supsetneqq L^*(\varrho_1', \varrho_2)$*.*
(3) *For every* $0 \le \varrho_1 \le \varrho_2 < \varrho_2' \le 1$*,* $\varrho_2' - \varrho_1 < 1$*, we have* $L(\varrho_1, \varrho_2) \subsetneqq L(\varrho_1, \varrho_2')$*.*
(4) *For every* $0 \le \varrho_1 < \varrho_2 < \varrho_2' \le 1$*, we have* $L^*(\varrho_1, \varrho_2) \subsetneqq L^*(\varrho_1, \varrho_2')$*.*

P r o o f. All inclusions follow directly from the definitions. For every $c \in (0, 1)$ let

$$E_c = (-\infty, 0] \cup \bigcup_{n=1}^{\infty}(a_n, b_n),$$

where

$$0 < \cdots < b_{n+1} < a_n < b_n < \cdots, \ \lim_{n\to\infty} a_n = 0 \ \text{ and } \ \underline{d}^+(E_c, 0) = \overline{d}^+(E_c, 0) = c.$$

If $c = 1 - \varrho_2 + \frac{\varrho_1 + \varrho_1'}{2}$, then $E_c \in L(\varrho_1, \varrho_2) \cap L^*(\varrho_1, \varrho_2)$ by Theorem 4.6 and Theorem 4.9, and $E_c \notin L(\varrho_1', \varrho_2) \cup L^*(\varrho_1', \varrho_2)$ by Theorem 4.7 and Theorem 4.10. Similarly, if $c = 1 - \frac{\varrho_2 + \varrho_2'}{2} + \varrho_1$, then $E_c \in L(\varrho_1, \varrho_2') \cap L^*(\varrho_1, \varrho_2')$ by Theorem 4.6 and Theorem 4.9, and $E_c \notin L(\varrho_1, \varrho_2) \cup L^*(\varrho_1, \varrho_2)$ by Theorem 4.7 and Theorem 4.10. □

Theorem 4.13. *Let* $0 \le \varrho_1 < \varrho_2 < 1$ *and* $0 < \varrho_1' < \varrho_2' \le 1$ *be such that* $\varrho_1 < \varrho_1'$ *and* $\varrho_2 < \varrho_2'$*. If* $1 - \varrho_2 + \varrho_1 < 1 - \varrho_2' + \varrho_1'$*, then* $L(\varrho_1, \varrho_2) \not\subset L(\varrho_1', \varrho_2')$ *and* $L^*(\varrho_1, \varrho_2) \not\subset L^*(\varrho_1', \varrho_2')$*.*

P r o o f. For every $c \in (0, 1)$ let E_c be the set constructed in the proof of the previous theorem. If $c = \frac{1 - \varrho_2 + \varrho_1 + 1 - \varrho_2' + \varrho_1'}{2}$, then $E_c \in L(\varrho_1, \varrho_2) \cap L^*(\varrho_1, \varrho_2)$ and $E_c \notin L(\varrho_1', \varrho_2') \cup L^*(\varrho_1', \varrho_2')$. □

Question 4.14. *Let* $0 \le \varrho_1 < \varrho_2 < 1$ *and* $0 < \varrho_1' < \varrho_2' \le 1$ *be such that* $\varrho_1 < \varrho_1'$*,* $\varrho_2 < \varrho_2'$ *and* $1 - \varrho_2 + \varrho_1 < 1 - \varrho_2' + \varrho_1'$*. Do the following inclusions* $L(\varrho_1', \varrho_2') \subset L(\varrho_1, \varrho_2)$ *and* $L^*(\varrho_1', \varrho_2') \subset L^*(\varrho_1, \varrho_2)$ *hold?*

5. Adder spaces of family of functions determined by lower density

Given two real function classes X and Y, can we identify all functions $g\colon I \to \mathbb{R}$ such that the sum $f + g$ belongs to X whenever f belongs to Y? Such a function g is said to be an adder of the set X over the set Y.

Definition 5.1. Let X and Y be two sets of functions from I to $\mathbb{R}$. We call the set

$$\mathcal{A}\,(X/Y) = \{g\colon I \to \mathbb{R}\colon f + g \in X \text{ for all } f \in Y\}$$

the adder of X over Y.

The following properties of $\mathcal{A}\,(X/Y)$ are immediate consequences of the definition.

Proposition 5.2. *Let X, X_1, X_2, Y, Y_1, Y_2 be sets of real-valued functions defined on I. The following statements are true:*

(1) *If $X_1 \subset X_2$, then $\mathcal{A}\,(X_1/Y) \subset \mathcal{A}\,(X_2/Y)$.*
(2) *If $Y_1 \subset Y_2$, then $\mathcal{A}\,(X/Y_2) \subset \mathcal{A}\,(X/Y_1)$.*
(3) *If $0 \in Y$, then $\mathcal{A}\,(X/Y) \subset X$.*
(4) *If $Y \subset X$ and X is closed under addition, then $X \subset \mathcal{A}\,(X/Y)$.*
(5) *If $Y \subset X$ and Y is closed under addition, then $Y \subset \mathcal{A}\,(X/Y)$.*
(6) *If $0 \in X \cap Y$ and X is closed under addition, then $\mathcal{A}\,(X/Y) = X$ if and only if $Y \subset X$.*

Now, we find the adders of the ϱ-lower continuous functions. We start with definitions of these classes of ϱ-lower continuous functions and their basic properties.

Definition 5.3. Let $\varrho \in [0, 1)$. We say that $f\colon I \to \mathbb{R}$ is LC_ϱ-continuous at $x \in I$ if for every $\varepsilon > 0$ there exists a measurable set $E \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ such that $\underline{d}\,(E, x) > \varrho$. The set of points at which f is LC_ϱ-continuous is denoted by $LC_\varrho(f)$ and $\mathcal{LC}_\varrho$ denotes the set of functions $f\colon I \to \mathbb{R}$ which are LC_ϱ-continuous at every $x \in I$.

Definition 5.4. Let $\varrho \in (0, 1]$. We say that $f\colon I \to \mathbb{R}$ is $LC_\varrho^\blacklozenge$-continuous at $x \in I$ if for every $\varepsilon > 0$ there exists a measurable $E \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ such that $\underline{d}\,(E, x) \geq \varrho$. The set of points at which f is $LC_\varrho^\blacklozenge$-continuous is denoted by $LC_\varrho^\blacklozenge(f)$ and $\mathcal{LC}_\varrho^\blacklozenge$ denotes the set of $f\colon I \to \mathbb{R}$ which are $LC_\varrho^\blacklozenge$-continuous at every $x \in I$.

In particular, $\mathcal{LC}_1^\blacklozenge$ consists of approximate continuous functions.

Lemma 5.5. *Let $f\colon I \to \mathbb{R}$. If for every $x \in I$ and every $\varepsilon > 0$ there exists a measurable set $A \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ for which $\overline{d}\,(A, x) > 0$, then f is measurable.*

P r o o f. Let us suppose, on the contrary, that there exists nonmeasurable f that satisfies the assumption. Then, we can find $\alpha \in \mathbb{R}$ such that

$$E = \{x \in I \colon f(x) < \alpha\}$$

(or $E = \{x \in I \colon f(x) > \alpha\}$, but, in this case, the proof is analogous) is nonmeasurable. Then, $F = I \setminus E = \{x \in I \colon f(x) \geq \alpha\}$ is also nonmeasurable. There exist measurable $E' \subset E$ and $F' \subset F$ such that $E \setminus E'$ and $F \setminus F'$ contain no set of positive measure. Then, $E' \cup F'$ and $G = (E \setminus E') \cup (F \setminus F')$ are measurable. Since $E \setminus E'$ is nonmeasurable, it has positive outer measure. By the Lebesgue Density Theorem, we can find $x_0 \in E \setminus E'$, which is a density point of G. Since $x_0 \in E$, $f(x_0) < \alpha$. Let $\varepsilon = \alpha - f(x_0)$. By assumption, we can find a measurable set $A \subset \{y \in I \colon |f(x_0) - f(y)| < \varepsilon\}$ for which $\overline{d}(A, x_0) > 0$. Obviously, $A \subset E$. Since $E \setminus E'$ contains no subset of positive measure, $\overline{d}(A \cap E', x_0) = \overline{d}(A, x_0) > 0$. Moreover, $\lambda\big((A \cap E') \cap G\big) = 0$. Therefore,

$$1 = \underline{d}(G, x_0) = 1 - \overline{d}(\mathbb{R} \setminus G, x_0) \leq 1 - \overline{d}(A \cap E', x_0) < 1,$$

which is a contradiction. □

Corollary 5.6. *$\mathcal{LC}_\varrho^\blacklozenge$ for $\varrho \in (0,1]$ and $\mathcal{LC}_\varrho$ for $\varrho \in [0,1)$ consist of measurable functions.*

Theorem 5.7. *Let $\varrho \in (0,1]$, $f \colon I \to \mathbb{R}$ be measurable and $x \in I$. Then, f is $\mathcal{LC}_\varrho^\blacklozenge$-continuous at $x \in I$ if and only if there exists a measurable set $E \subset I$ such that $x \in E$, f restricted to E is continuous at x and $\underline{d}(E, x) \geq \varrho$.*

P r o o f. First, let us assume the existence of a measurable set $E \subset I$ such that $x \in E$, f restricted to E is continuous at x and $\underline{d}(E, x) \geq \varrho$. Choose any $\varepsilon > 0$. By continuity of f restricted to E, there is $\delta > 0$ such that

$$E \cap (x - \delta, x + \delta) \subset \{y \in I \colon |f(x) - f(y)| < \varepsilon\}.$$

Therefore,

$$\underline{d}\big(\{y \in I \colon |f(x) - f(y)| < \varepsilon\}, x\big) \geq \underline{d}\big(E \cap (x - \delta, x + \delta), x\big) = \underline{d}(E, x) \geq \varrho$$

for every $\varepsilon > 0$, which means $LC_\varrho^\blacklozenge$-continuity of f at x.

Now, assume that f is $LC_\varrho^\blacklozenge$-continuous at x. Let

$$E_n = \left\{y \in I \colon |f(x) - f(y)| < \frac{1}{n}\right\}.$$

Then, $\underline{d}^+(E_n, x) \geq \varrho$ for $n \geq 1$. Hence, we can find a decreasing sequence $(x_n)_{n \geq 1}$ converging to x such that $\frac{\lambda(E_n \cap [x,t])}{t-x} > \varrho - \frac{1}{n}$ for every $t \in (x, x_n]$. Let $E^+ = \{x\} \cup \bigcup_{n=1}^{\infty} (E_n \cap [x + \frac{x_{n+1}-x}{n}, x_n])$. Obviously, f restricted to E^+ is continuous at x. Moreover,

$$\frac{\lambda(E^+ \cap [x,t])}{t-x} \geq \frac{\lambda(E_n \cap [x + \frac{x_{n+1}-x}{n}, t])}{t-x} \geq \frac{\lambda(E_n \cap [x,t]) - \frac{x_{n+1}-x}{n}}{t-x} > \varrho - \frac{2}{n}$$

for every $t \in [x_{n+1}, x_n]$. This proves $\underline{d}^{+}(E^{+}, x) \geq \varrho$. In a similar way, we can construct a measurable $E^{-} \subset (-\infty, x]$ such that $x \in E^{-}$, f restricted to E^{-} is continuous at x and $\underline{d}^{-}(E^{-}, x) \geq \varrho$. Then, $E = E^{+} \cup E^{-}$ witnesses $\mathcal{LC}_{\varrho}^{\blacklozenge}$-continuity of f at x. $\square$

Definition 5.8. Let $\varrho \in [0, 1)$. We say that $f\colon I \to \mathbb{R}$ is LPC_{ϱ}-continuous at $x \in I$ if for every $\varepsilon > 0$ there exists a measurable set

$$E_{\varepsilon} \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$$

such that $\lim_{\varepsilon \to 0^{+}} \underline{d}\,(E_{\varepsilon}, x) > \varrho$. The set of points at which f is LPC_{ϱ}-continuous is denoted by $LPC_{\varrho}(f)$ and $\mathcal{LPC}_{\varrho}$ denotes the set of $f\colon I \to \mathbb{R}$ which are LPC_{ϱ}-continuous at every $x \in I$.

Theorem 5.9. *Let $\varrho \in (0, 1]$, $f\colon I \to \mathbb{R}$ and $x \in I$. Then, f is LPC_{ϱ}-continuous at $x \in I$ if and only if there exists a measurable set $E \subset I$ such that $x \in E$, f restricted to E is continuous at x and $\underline{d}\,(E, x) > \varrho$.*

Proof. Let a measurable $E \subset I$ be such that $x \in E$, f restricted to E is continuous at x and $\underline{d}\,(E, x) > \varrho$. Choose any $\varepsilon > 0$. There exists $\delta > 0$ such that

$$E \cap (x - \delta, x + \delta) \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}.$$

Hence,

$$\underline{d}\big(\{y \in I\colon |f(x) - f(y)| < \varepsilon\}, x\big) \geq \underline{d}\big(E \cap (x - \delta, x + \delta), x\big) = \underline{d}\,(E, x),$$

and

$$\lim_{\varepsilon \to 0^{+}} \underline{d}\big(\{y \in I\colon |f(x) - f(y)| < \varepsilon\}, x\big) \geq \underline{d}\,(E, x) > \varrho,$$

which means LPC_{ϱ}-continuity of f at x.

Now, assume f is LPC_{ϱ}-continuous at x and let

$$E_n = \left\{y \in I\colon |f(x) - f(y)| < \frac{1}{n}\right\} \quad \text{for } n \geq 1.$$

Then, $\lim_{n \to \infty} \underline{d}^{+}(E_n, x) = c > \varrho$. Hence, we can find a decreasing sequence $(x_n)_{n \geq 1}$ converging to x such that $\frac{\lambda(E_n \cap [x,t])}{t - x} > c - \frac{1}{n}$ for every $t \in (x, x_n]$.

Let

$$E^{+} = \{x\} \cup \bigcup_{n=1}^{\infty} \left(E_n \cap \left[x + \frac{x_{n+1} - x}{n}, x_n\right]\right).$$

Obviously, f restricted to E^{+} is continuous at x. Moreover,

$$\frac{\lambda(E^{+} \cap [x, t])}{t - x} \geq \frac{\lambda(E_n \cap [x + \frac{x_{n+1} - x}{n}, t])}{t - x} \geq \frac{\lambda(E_n \cap [x, t]) - \frac{x_{n+1} - x}{n}}{t - x} > c - \frac{2}{n}$$

for every $t \in [x_{n+1}, x_n]$. This proves $\underline{d}^{+}(E^{+}, x) \geq c > \varrho$. Similarly, we can construct a measurable set $E^{-} \subset (-\infty, x]$ such that $x \in E^{-}$, f restricted to E^{-}

is continuous at x and $\underline{d}^-(E^-, x) \geq c > \varrho$. Then, $E = E^+ \cup E^-$ possesses all required properties. $\square$

Proposition 5.10. *Let* $0 < \varrho_1 < \varrho_2 < 1$. *Then,*

$$\mathcal{LC}_1^\blacklozenge \subset \mathcal{LPC}_{\varrho_2} \subset \mathcal{LC}_{\varrho_2} \subset \mathcal{LC}_{\varrho_2}^\blacklozenge \subset \mathcal{LPC}_{\varrho_1} \subset \mathcal{LC}_{\varrho_1} \subset \mathcal{LC}_{\varrho_1}^\blacklozenge \subset \mathcal{LPC}_0 \subset \mathcal{LC}_0.$$

Definition 5.11. Let $0 \leq \varrho_1 \leq \varrho_2 \leq 1$, $\varrho_2 - \varrho_1 < 1$. We say that $f\colon I \to \mathbb{R}$ is L_{ϱ_1,ϱ_2}-continuous at $x \in I$ if for every $\varepsilon > 0$ there exists a measurable $E \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$ which is $[\varrho_1, \varrho_2]$-lower superdense at x. The set of points at which f is L_{ϱ_1,ϱ_2}-continuous is denoted by $L_{\varrho_1,\varrho_2}(f)$ and $\mathcal{L}_{\varrho_1,\varrho_2}$ denotes the set of functions $f\colon I \to \mathbb{R}$ which are L_{ϱ_1,ϱ_2}-continuous at every $x \in I$.

Definition 5.12. Let $0 \leq \varrho_1 < \varrho_2 \leq 1$. We say that $f\colon I \to \mathbb{R}$ is $L^*_{\varrho_1,\varrho_2}$--continuous at $x \in I$ if for every $\varepsilon > 0$ there exists a measurable set

$$E \subset \{y \in I\colon |f(x) - f(y)| < \varepsilon\}$$

which is $[\varrho_1, \varrho_2]^*$-lower superdense at x. The set of points at which f is $L^*_{\varrho_1,\varrho_2}$-continuous is denoted by $L^*_{\varrho_1,\varrho_2}(f)$ and $\mathcal{L}^*_{\varrho_1,\varrho_2}$ denotes the set of functions $f\colon I \to \mathbb{R}$ which are $L^*_{\varrho_1,\varrho_2}$-continuous at every $x \in I$.

Obviously, $\mathcal{L}_{\varrho_1,\varrho_2}$ and $\mathcal{L}^*_{\varrho_1,\varrho_2}$ consist of measurable functions.

Theorem 5.13.

(1) $\mathcal{L}_{\varrho_1,\varrho_2} = \mathcal{A}\left(\mathcal{LC}_{\varrho_1}^\blacklozenge / \mathcal{LC}_{\varrho_2}^\blacklozenge\right)$ *for* $0 < \varrho_1 \leq \varrho_2 \leq 1$.

(2) $\mathcal{L}_{\varrho_1,\varrho_2} = \mathcal{A}\left(\mathcal{LC}_{\varrho_1}^\blacklozenge / \mathcal{LC}_{\varrho_2}\right)$ *for* $0 < \varrho_1 \leq \varrho_2 < 1$.

(3) $\mathcal{L}_{\varrho_1,\varrho_2} = \mathcal{A}\left(\mathcal{LC}_{\varrho_1} / \mathcal{LC}_{\varrho_2}\right)$ *for* $0 \leq \varrho_1 \leq \varrho_2 < 1$.

(4) *If the function* $f\colon I \to \mathbb{R}$ *is measurable and* $f \notin \mathcal{L}_{\varrho_1,\varrho_2}$, *then there exists* $g \in \mathcal{LPC}_{\varrho_2}$ *such that*
   - $f + g \notin \mathcal{LC}_{\varrho_1}^\blacklozenge$ *for* $0 < \varrho_1 \leq \varrho_2 < 1$;
   - $f + g \notin \mathcal{LC}_0$ *for* $0 = \varrho_1 \leq \varrho_2 < 1$.

P r o o f. First, we show that $\mathcal{L}_{\varrho_1,\varrho_2}$ is contained in all considered adders. Choose $0 \leq \varrho_1 \leq \varrho_2 \leq 1$, $\varrho_2 - \varrho_1 < 1$, and $f \in \mathcal{L}_{\varrho_1,\varrho_2}$. Take any $g_1 \in \mathcal{LC}_{\varrho_2}^\blacklozenge$ (only for $\varrho_1 > 0$), $g_2 \in \mathcal{LC}_{\varrho_2}$ (only for $\varrho_2 \neq 1$), $x \in I$ and $\varepsilon > 0$. Then,

$$\underline{d}\Big(\Big\{y \in I\colon |g_1(x) - g_1(y)| < \frac{\varepsilon}{2}\Big\}, x\Big) \geq \varrho_2,$$
$$\underline{d}\Big(\Big\{y \in I\colon |g_2(x) - g_2(y)| < \frac{\varepsilon}{2}\Big\}, x\Big) > \varrho_2$$

and

$$E = \Big\{y \in I\colon |f(x) - f(y)| < \frac{\varepsilon}{2}\Big\}$$

is $[\varrho_1, \varrho_2]$-lower superdense at x.

Let
$$F_1=\{y\in I\colon |(f+g_1)(x)-(f+g_1)(y)|<\varepsilon\}$$
and
$$F_2=\{y\in I\colon |(f+g_2)(x)-(f+g_2)(y)|<\varepsilon\}.$$
Then,
$$F_1\supset E\cap\left\{y\in I\colon |g_1(x)-g_1(y)|<\frac{\varepsilon}{2}\right\}$$
and
$$F_2\supset E\cap\left\{y\in I\colon |g_2(x)-g_2(y)|<\frac{\varepsilon}{2}\right\}.$$
Since E is $[\varrho_1,\varrho_2]$-lower superdense at x, $\underline{d}\,(F_1,x)\geq\varrho_1$ and $\underline{d}\,(F_2,x)>\varrho_1$. Since the choice of $x\in I$ is arbitrary and $\varepsilon>0$ is arbitrarily chosen, we conclude
$$f+g_1\in\mathcal{LC}^{\blacklozenge}_{\varrho_1}\quad\text{and}\quad f+g_2\in\mathcal{LC}_{\varrho_1}.$$
Therefore, $\mathcal{L}_{\varrho_1,\varrho_2}$ is a subset of all considered adders.

Now, take any measurable $f\notin\mathcal{L}_{\varrho_1,\varrho_2}$. We can find $x\in I$, $\varepsilon>0$ and $E\in\mathcal{S}$ such that
$$\underline{d}\,(E,x)>\varrho_2\quad(\text{or }\underline{d}\,(E,x)=1\text{ if }\varrho_2=1)$$
and
$$\underline{d}\big(E\cap\{y\in I\colon |f(x)-f(y)|<\varepsilon\},x\big)<\varrho_1$$
(or $\underline{d}\big(E\cap\{y{\in}I\colon |f(x)-f(y)|<\varepsilon\},x\big){=}0$ if $\varrho_1=0$). Without any loss of generality, we may assume
$$\underline{d}^+\big(E\cap\{y\in I\colon |f(x)-f(y)|<\varepsilon\},x\big)<\varrho_1$$
(or $\underline{d}^+(E\cap\{y\in I\colon |f(x)-f(y)|<\varepsilon\},x)=0$ if $\varrho_1=0$).

By Theorem 3.6, there exists a sequence of closed intervals $([a_n,b_n])_{n\geq1}$ such that
$$x<\cdots<b_{n+1}<a_n<b_n<\cdots<b_1,\qquad\lim_{n\to\infty}a_n=x,$$
$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n]\setminus E,x\right)=0=\overline{d}^+\left(E\setminus\bigcup_{n=1}^{\infty}[a_n,b_n],x\right).$$
Hence,
$$\underline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n],x\right)=\underline{d}^+(E,x)>\varrho_2\quad\text{or}\quad\underline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n],x\right)=1\text{ if }\varrho_2=1.$$
We can easily find a sequence of closed intervals $([c_n,d_n])_{n\geq1}$ such that
$$x<\cdots<d_{n+1}<c_n<a_n<b_n<d_n<c_{n-1}<\cdots<d_1$$
and
$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}([c_n,a_n]\cup[b_n,d_n]),x\right)=0.$$

For every $n \geq 1$ there exists $\alpha_n \in \mathbb{R}$ such that

$$\lambda(\{y \in [d_{n+1}, c_n] \colon |f(y)| \geq \alpha_n - \varepsilon\}) < \frac{a_{n+1} - x}{2^n}.$$

Hence,

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}\{y \in [d_{n+1}, c_n] \colon |f(y)| \geq \alpha_n - \varepsilon\}, x\right) = 0.$$

Define $g \colon I \to \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & \text{for } t \in (I \cap (-\infty, x]) \cup \bigcup_{n=1}^{\infty}[a_n, b_n] \cup (I \cap [d_1, \infty)), \\ \alpha_n & \text{for } t \in \bigcup_{n=1}^{\infty}[d_{n+1}, c_n], \\ \text{linear} & \text{on every interval } [c_n, a_n],\ [b_n, d_n], \quad n = 1, 2, \dots \end{cases}$$

Obviously, g is continuous at every point except x. Moreover,

$$\{y \in I \colon g(y) = g(x) = 0\} \supset \big((-\infty, x] \cap I\big) \cup \bigcup_{n=1}^{\infty}[a_n, b_n]$$

and

$$\underline{d}\big(\{y \in I \colon g(y) = g(x) = 0\}, x\big) = \underline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n], x\right) = \underline{d}^+(E, x) > \varrho_2$$

(or $\underline{d}\,(\{y \in I \colon g(y) = g(x) = 0\}, x) = 1$ if $\varrho_2 = 1$). Hence, $g \in \mathcal{LPC}_{\varrho_2}$ (or $g \in \mathcal{LC}_1^{\blacklozenge}$ if $\varrho_2 = 1$).

On the other hand, let

$$A = \bigcup_{n=1}^{\infty}[a_n, b_n] \setminus E,$$

$$B = \bigcup_{n=1}^{\infty}([c_n, a_n] \cup [b_n, d_n])$$

and

$$C = \bigcup_{n=1}^{\infty}\{y \in [d_{n+1}, c_n] \colon |f(y)| \geq \alpha_n - \varepsilon\}$$

Then,

$$\{y \geq x \colon |(f+g)(y) - (f+g)(x)| < \varepsilon\} \subset \big(E \cap \{y \geq x \colon |f(x) - f(y)| < \varepsilon\}\big) \cup A \cup B \cup C$$

and $\overline{d}^+(A, x) = \overline{d}^+(B, x) = \overline{d}^+(C, x) = 0$. Therefore,

$$\begin{aligned} &\underline{d}^+\big(\{y \in I \colon |(f+g)(y) - (f+g)(x)| < \varepsilon\}, x\big) \\ &\qquad \leq \underline{d}^+\big((E \cap \{y \in I \colon |f(x) - f(y)| < \varepsilon\}), x\big) < \varrho_1 \end{aligned}$$

(or $\underline{d}^+\big(\{y \in I \colon |(f+g)(y) - (f+g)(x)| < \varepsilon\}, x\big) = 0$ if $\varrho_1 = 0$).

This shows $f+g\notin \mathcal{LC}^{\blacklozenge}_{\varrho_1}$ (or $f+g\notin\mathcal{LC}_0$ if $\varrho_1=0$). Hence,

$$f\notin\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right)\cup\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)\cup\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)$$

for the appropriate ϱ_1,ϱ_2. The proof is completed. □

Corollary 5.14.

(1) $\mathcal{LC}^{\blacklozenge}_1=\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho}/\mathcal{LC}^{\blacklozenge}_{\varrho}\right)$ *for* $\varrho\in(0,1]$, $\mathcal{LC}^{\blacklozenge}_{\varrho}=\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho}/\mathcal{LC}^{\blacklozenge}_{1}\right)$ *for* $\varrho\in(0,1)$ *and* $\mathcal{LC}^{\blacklozenge}_{1-\varrho_2+\varrho_1}\subsetneqq\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right)$ *for* $0<\varrho_1<\varrho_2<1$.

(2) $\mathcal{LC}^{\blacklozenge}_1=\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho}/\mathcal{LC}_{\varrho}\right)$ *for* $\varrho\in(0,1)$ *and* $\mathcal{LC}^{\blacklozenge}_{1-\varrho_2+\varrho_1}\subsetneqq\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)$ *for* $0<\varrho_1<\varrho_2<1$.

(3) $\mathcal{LC}^{\blacklozenge}_1=\mathcal{A}\left(\mathcal{LC}_{\varrho}/\mathcal{LC}_{\varrho}\right)$ *for* $\varrho\in(0,1]$ *and* $\mathcal{LC}^{\blacklozenge}_{1-\varrho_2+\varrho_1}\subsetneqq\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)$ *for* $0\le\varrho_1<\varrho_2<1$.

Theorem 5.15. *Let* $0\le\varrho_1<\varrho_2\le 1$. *Then,* $\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right)=\mathcal{L}^*_{\varrho_1,\varrho_2}$.

Proof. (The proof is very similar to the proof of Theorem 5.13.) Take any $f\in\mathcal{L}^*_{\varrho_1,\varrho_2}$, $g\in\mathcal{LC}^{\blacklozenge}_{\varrho_2}$, $x\in I$ and $\varepsilon>0$. Then,

$$\underline{d}\Big(\Big\{y\in I\colon |g(x)-g(y)|<\frac{\varepsilon}{2}\Big\},x\Big)\ge\varrho_2\quad\text{and}\quad E=\Big\{y\in I\colon |f(x)-f(y)|<\frac{\varepsilon}{2}\Big\}$$

is $[\varrho_1,\varrho_2]^*$-lower superdense at x.

Let $F=\{y\in I\colon |(f+g)(x)-(f+g)(y)|<\varepsilon\}$. Then,

$$F\supset E\cap\Big\{y\in I\colon |g(x)-g(y)|<\frac{\varepsilon}{2}\Big\}.$$

Since E is $[\varrho_1,\varrho_2]^*$-lower superdense at x, $\underline{d}\,(F,x)>\varrho_1$. Therefore,

$$\mathcal{L}^*_{\varrho_1,\varrho_2}\subset\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right).$$

Now, take any measurable $f\notin\mathcal{L}^*_{\varrho_1,\varrho_2}$. We can find $x\in I$, $\varepsilon>0$ and $E\in\mathcal{S}$ such that

$$\underline{d}\,(E,x)\ge\varrho_2\quad\text{and}\quad\underline{d}^+\big(E\cap\{y\in I\colon |f(x)-f(y)|\le\varepsilon\},x\big)\le\varrho_1.$$

By Theorem 3.6, there exists a sequence of closed intervals $([a_n,b_n])_{n\ge1}$ such that

$$x<\cdots<b_{n+1}<a_n<b_n<\cdots<b_1,\quad\lim_{n\to\infty}a_n=x,$$

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n,b_n]\setminus E,x\right)=0=\overline{d}^+\left(E\setminus\bigcup_{n=1}^{\infty}[a_n,b_n],x\right).$$

Hence,

$$\underline{d}^{+}\left(\bigcup_{n=1}^{\infty}[a_n,b_n],x\right)=\underline{d}\,(E,x)\geq\varrho_2.$$

Let $([c_n,d_n])_{n\geq1}$ be a sequence of closed intervals such that

$$x<\dots<d_{n+1}<c_n<a_n<b_n<d_n<\dots<d_1$$

and

$$\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}([c_n,a_n]\cup[b_n,d_n]),x\right)=0.$$

For each $n\geq1$ let $\alpha_n\in\mathbb{R}$ be such that

$$\lambda\big(\{y\in[d_{n+1},c_n]\colon|f(y)|\geq\alpha_n-\varepsilon\}\big)<\frac{a_n-x}{2^n}$$

and

$$\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}\{y\in[d_{n+1},c_n]\colon|f(y)|\geq\alpha_n-\varepsilon\},x\right)=0.$$

Define $g\colon I\to\mathbb{R}$ by

$$g(t)=\begin{cases}0 & \text{for } t\in(I\cap(-\infty,x])\cup\bigcup_{n=1}^{\infty}[a_n,b_n]\cup(I\cap[d_1,\infty)),\\ \alpha_n & \text{for } t\in\bigcup_{n=1}^{\infty}[d_{n+1},c_n],\\ \text{linear} & \text{on every interval } [c_n,a_n],\ [b_n,d_n],\quad n=1,2,\dots\end{cases}$$

Similarly, as in the proof of Theorem 5.13, we show $g\in\mathcal{LC}_{\varrho_2}^{\blacklozenge}$ and

$$\begin{aligned}&\underline{d}\big(\{y\in I\colon|(f+g)(y)-(f+g)(x)|<\varepsilon\},x\big)\\&\qquad\leq\underline{d}\big(E\cap\{y\in I\colon|f(x)-f(y)|<\varepsilon\},x\big)\leq\varrho_1.\end{aligned}$$

This shows $f+g\notin\mathcal{LC}_{\varrho_1}$ and $f\notin\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}^{\blacklozenge}\right)$. The proof is completed. $\square$

Corollary 5.16.

$$\mathcal{LC}_{\varrho}=\mathcal{A}\left(\mathcal{LC}_{\varrho}/\mathcal{LC}_{1}^{\blacklozenge}\right)\quad\textit{for}\quad\varrho\in[0,1)$$

and

$$\mathcal{LC}_{1-\varrho_2+\varrho_1}\subsetneqq\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}^{\blacklozenge}\right)\quad\textit{for}\quad 0\leq\varrho_1<\varrho_2<1.$$

Remark 1. Multipliers of ϱ-lower continuous functions have more complicated structure than adders. For example, we will show that for $0<\varrho_1<\varrho_2<1$ we have

$$\mathcal{A}\left(\mathcal{LC}_{\varrho_1}^{\blacklozenge}/\mathcal{LC}_{\varrho_2}^{\blacklozenge}\right)=\mathcal{L}_{\varrho_1,\varrho_2}\subsetneqq\mathcal{LC}_{\varrho_1}^{\blacklozenge}/\mathcal{LC}_{\varrho_2}^{\blacklozenge}.$$

P r o o f. Repeating arguments from the proof of Theorem 5.13, we can easily show the considered inclusion. Let $([a_n,b_n])_{n\geq1}$, $([c_n,d_n])_{n\geq1}$ be two sequences of closed intervals such that

$$0<\dots<d_{n+1}<c_n<a_n<b_n<d_n<\dots<d_1,$$

$$\lim_{n\to\infty} a_n = 0, \quad \overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n], 0\right) = \underline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n], 0\right) = \varrho_1$$

and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[c_n, d_n], 0\right) = \underline{d}^+\left(\bigcup_{n=1}^{\infty}[c_n, d_n], 0\right) = \varrho_1.$$

Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0] \cup \bigcup_{n=1}^{\infty}[a_n, b_n], \\ 1 & \text{for } x \in [\, d_1,\ \infty) \cup \bigcup_{n=1}^{\infty}[d_{n+1}, c_n], \\ \text{affine} & \text{on every interval } [c_n, a_n], [b_n, d_n], n \geq 1. \end{cases}$$

Clearly, f is continuous at every point except 0. Moreover,

$$\underline{d}\big(\{x \in \mathbb{R}\colon |f(x)| < \varepsilon\}, 0\big) = \varrho_1 \quad \text{for every } \varepsilon \in (0, 1).$$

Therefore,

$$f \in \mathcal{LC}^{\blacklozenge}_{\varrho_1} \quad \text{and} \quad f \notin \mathcal{LC}^{\blacklozenge}_{\varrho} \quad \text{for any } \ \varrho > \varrho_1.$$

Take any $g \in \mathcal{LC}^{\blacklozenge}_{\varrho_2}$. Obviously, $f \cdot g$ is $LC^{\blacklozenge}_{\varrho_1}$-continuous at every $x \neq 0$. Moreover,

$$\underline{d}\big(\{x \in I\colon (f \cdot g)(x) = (f \cdot g)(0) = 0\}, 0\big) \geq \underline{d}\big(\{x \in I\colon f(x) = 0\}, 0\big) = \varrho_1.$$

Therefore, $f \cdot g$ is $LC^{\blacklozenge}_{\varrho_1}$-continuous at 0,

$$f \cdot g \in \mathcal{LC}^{\blacklozenge}_{\varrho_1} \quad \text{and} \quad f \in \mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2} \quad \text{for every } \ \varrho_2 \in [\varrho_1, 1).$$

On the other hand, $\varrho_1 < 1 - \varrho_2 + \varrho_1$ and, by Theorem 4.7, $f \notin \mathcal{L}_{\varrho_1,\varrho_2}$. $\square$

6. Adder spaces of family of path continuous functions

In the last section of the paper, we study adders of ϱ-lower path continuous functions.

Theorem 6.1.

(1) $\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) = \mathcal{L}_{\varrho_1,\varrho_2}$ *for every* $0 < \varrho_1 \leq \varrho_2 < 1$.

(2) $\mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) = \mathcal{L}_{\varrho_1,\varrho_2}$ *for every* $0 \leq \varrho_1 \leq \varrho_2 < 1$.

Proof. Since $\mathcal{LPC}_{\varrho_2} \subset \mathcal{LC}^{\blacklozenge}_{\varrho_2}$, we obtain

$$\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right) \subset \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right)$$

according to (2) of Proposition 5.2.

By Theorem 5.13,

$$\mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right) = \mathcal{L}_{\varrho_1,\varrho_2}.$$

Finally,

$$\mathcal{L}_{\varrho_1,\varrho_2} \subset \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) \quad \text{for } 0 < \varrho_1 \le \varrho_2 < 1.$$

Similarly, since

$$\mathcal{LPC}_{\varrho_2} \subset \mathcal{LC}_{\varrho_2}, \quad \text{we get} \quad \mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}) \subset \mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2})$$

according to (2) of Proposition 5.2. By Theorem 5.13,

$$\mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}) = \mathcal{L}_{\varrho_1,\varrho_2}.$$

Finally,

$$\mathcal{L}_{\varrho_1,\varrho_2} \subset \mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}) \quad \text{for } 0 \le \varrho_1 \le \varrho_2 < 1.$$

Let $f \notin \mathcal{L}_{\varrho_1,\varrho_2}$ (for corresponding ϱ's). Then, by (4) of Theorem 5.13, there exists a function g such that

$$f+g \notin \mathcal{LC}^{\blacklozenge}_{\varrho_1} \text{ for } 0<\varrho_1\le\varrho_2<1 \text{ or } f+g \notin \mathcal{LC}_0 \text{ for } 0\le!\varrho_1\le\varrho_2<1.$$

Hence,

$$f\notin \mathcal{A}(\mathcal{LC}^{\blacklozenge}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}) \text{ for } 0<\varrho_1\le\varrho_2<1 \text{ or } f\notin\mathcal{A}(\mathcal{LC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}) \text{ for } 0\le\varrho_1\le\varrho_2<1.$$

□

Theorem 6.2.

(1) $\bigcup_{\varrho\in(\varrho_1,\varrho_2)} \mathcal{L}_{\varrho,\varrho_2}\subset\mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right)\subset\mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right)\subset\mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right)\subset\mathcal{L}_{\varrho_1,\varrho_2}$ *for every* $0 \le \varrho_1 < \varrho_2 < 1$.

(2) $\mathcal{A}\left(\mathcal{LPC}_{\varrho}/\mathcal{LPC}_{\varrho}\right) = \mathcal{L}_{\varrho,\varrho} = \mathcal{LC}^{\blacklozenge}_1$ *and* $\mathcal{A}\left(\mathcal{LPC}_{\varrho}/\mathcal{LC}^{\blacklozenge}_1\right) = \mathcal{LPC}_{\varrho}$ *for* $\varrho \in [0,1)$.

P r o o f.

(1) According to the first two points of Proposition 5.2,

$$\mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right) \subset \mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}_{\varrho_2}\right) \subset \mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) \subset \mathcal{A}\left(\mathcal{LC}_{\varrho_1}/\mathcal{LPC}_{\varrho_2}\right) \subset \mathcal{L}_{\varrho_1,\varrho_2} \quad \text{for } 0 \le \varrho_1 < \varrho_2 < 1.$$

Take any $f\in\bigcup_{\varrho\in(\varrho_1,\varrho_2)}\mathcal{L}_{\varrho,\varrho_2}$. There exist $\varrho_0\in(\varrho_1,\varrho_2)$ for which $f\in\mathcal{L}_{\varrho_0,\varrho_2}$. Since $\mathcal{LC}^{\blacklozenge}_{\varrho_0} \subset \mathcal{LPC}_{\varrho_1}$, we obtain

$$\mathcal{L}_{\varrho_0,\varrho_2} = \mathcal{A}\left(\mathcal{LC}^{\blacklozenge}_{\varrho_0}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right) \subset \mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right),$$

by Proposition 5.2. Therefore,

$$f \in \mathcal{A}\left(\mathcal{LPC}_{\varrho_1}/\mathcal{LC}^{\blacklozenge}_{\varrho_2}\right).$$

(2) The equality $\mathcal{L}_{\varrho,\varrho} = \mathcal{LC}_1^{\blacklozenge}$ follows directly from Corollary 5.14.

The inclusion $\mathcal{LC}_1^{\blacklozenge} \subset \mathcal{A}\,(\mathcal{LPC}_\varrho/\mathcal{LPC}_\varrho)$ for $\varrho \in [0,1)$ is obvious.

The inclusion $\mathcal{A}\,(\mathcal{LPC}_\varrho/\mathcal{LPC}_\varrho) \subset \mathcal{L}_{\varrho,\varrho}$ follows from (4) of Theorem 5.13.

Obviously,
$$\mathcal{LPC}_\varrho \supset \mathcal{A}\left(\mathcal{LPC}_\varrho/\mathcal{LC}_1^{\blacklozenge}\right) \quad \text{for} \quad \varrho \in [0,1).$$
Take $\varepsilon > 0$, measurable $g\colon I \to \mathbb{R}$, $f \in \mathcal{LC}_1^{\blacklozenge}$ and $x \in I$. Then,
$$\begin{aligned}&\{t \in I\colon |(f+g)(t) - (f+g)(x)| < \varepsilon\}\\ &\qquad\qquad \supset \left\{t \in I\colon |f(t) - f(x)| < \frac{\varepsilon}{2}\right\} \cap \left\{t \in I\colon |g(t) - g(x)| < \frac{\varepsilon}{2}\right\}.\end{aligned}$$
Since $f \in \mathcal{LC}_1^{\blacklozenge}$, we obtain
$$\underline{d}\left(\left\{t \in I\colon |f(t) - f(x)| < \frac{\varepsilon}{2}\right\}, x\right) = 1.$$
Then, by (1) of Proposition 3.1, we obtain
$$\begin{aligned}&\underline{d}\big(\{t \in I\colon |(f+g)(t) - (f+g)(x)| < \varepsilon\}, x\big)\\ &\geq \underline{d}\left(\left\{t \in I\colon |f(t) - f(x)| < \frac{\varepsilon}{2}\right\} \cap \left\{t \in I\colon |g(t) - g(x)| < \frac{\varepsilon}{2}\right\}, x\right)\\ &\geq \underline{d}\left(\left\{t \in I\colon |f(t) - f(x)| < \frac{\varepsilon}{2}\right\}, x\right) + \underline{d}\left(\left\{t \in I\colon |g(t) - g(x)| < \frac{\varepsilon}{2}\right\}, x\right) - 1\\ &= \underline{d}\left(\left\{t \in I\colon |g(t) - g(x)| < \frac{\varepsilon}{2}\right\}, x\right).\end{aligned}$$
Therefore,
$$\mathcal{LPC}_\varrho \subset \mathcal{A}\left(\mathcal{LPC}_\varrho/\mathcal{LC}_1^{\blacklozenge}\right) \quad \text{and finally,} \quad \mathcal{LPC}_\varrho = \mathcal{A}\left(\mathcal{LPC}_\varrho/\mathcal{LC}_1^{\blacklozenge}\right)$$
for every $\varrho \in [0,1)$. □

Question 6.3. *Let $f \in \mathcal{LC}_{1-\varrho_2+\varrho_1}^{\blacklozenge}$ be continuous at every point except some $x \in I$,*
$$d\big(\{t \in I\colon f(x) = f(t)\}, x\big) = 1 - \varrho_2 + \varrho_1$$
and
$$d\big(\{t \in I\colon |f(x) - f(t)| > \varepsilon\}, x\big) = \varrho_2 - \varrho_1$$
for every $\varepsilon > 0$. Then,
$$f \in \bigcup_{\varrho \in (\varrho_1,\varrho_2)} \mathcal{L}_{\varrho,\varrho_2} \setminus \mathcal{L}_{\varrho_1,\varrho_2} \qquad \textit{and} \qquad \mathcal{L}_{\varrho_1,\varrho_2} \subsetneqq \bigcup_{\varrho \in (\varrho_1,\varrho_2)} \mathcal{L}_{\varrho,\varrho_2}.$$
Is there any equality in (1) of Theorem 6.2?

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