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APPROACH TO THE SUMMABLE IDEAL BY O'MALLEY DENSITY OF SPECIAL TESTING SETS ON THE REAL LINE

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ABSTRACT. The summable ideal for a harmonic series is described by the O'Malley density of subsets of the real line generated by sequences of reals. Since this approach leads to a special kind of summable ideals, we investigate the relationship between them and we find conditions under which summable ideals for specific series coincide with one for the harmonic series.

1. Introduction

It is known that the density ideal of subsets of the set $\mathbb{N}$ of all positive integers can be defined using the classical notion of density of subsets of the real line ([11]). An analogous relationship between f-density on $\mathbb{R}$ ([3]) and ideals of simple density on $\mathbb{N}$ defined in [1] was considered in [4]. In the same regard, summable ideals and O'Malley density were studied in [6] and in [4]. The common idea in these papers was as follows: a set $A \subset \mathbb{N}$ belongs to the considered ideal if and only if zero is a right-hand dispersion point in some sense of a specific interval subset of $\mathbb{R}$ defined for the sequence $1/n$, namely, of the set

$$\bigcup_{i\in A}\left[\frac{1}{i+1},\frac{1}{i}\right].$$

In [5], more general "testing sets"

$$D_{(x_n)}(A) := \bigcup_{i\in A}[x_{i+1}, x_i]$$

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defined for decreasing sequences (x_n) tending to zero were considered for the case of the density ideal. Conditions for the sequences (x_n) at which our approach gives the density ideal were found. The main aim of this paper is to undertake analogous considerations for the summable ideal for a harmonic series. In this case, the problem is reduced to investigating the relationship between different summable ideals and finding the conditions under which summable ideals for specific series coincide with one for the harmonic series.

2. Summable ideal and O'Malley density

Let us remind that for any divergent series of positive numbers $\sum_{n\in\mathbb{N}} a_n$, a family

$$\mathcal{I}_{a_n} := \left\{ A \subset \mathbb{N} : \sum_{n\in A} a_n < \infty \right\}$$

is a proper ideal on $\mathbb{N}$ called a summable ideal generated by the series $\sum_{n\in A} a_n$.

In this paper, we would like to focus our attention on a summable ideal for a harmonic series, i.e.,

$$\mathcal{I}_{\frac{1}{n}} := \left\{ A \subset \mathbb{N} : \sum_{n\in A} \frac{1}{n} < \infty \right\}.$$

Inspired by a result from [11] for the density ideal and classical density, in [6], the authors proved an analogous result for the ideal $\mathcal{I}_{\frac{1}{n}}$. It expresses a connection between this ideal and a special kind of "sparseness" near zero on the real line (in O'Malley sense [9]). Namely, the condition

$$\int_0^1 \frac{1}{t}\chi_B(t) < \infty,$$

where χ_B denotes a characteristic function of the set $B \subset \mathbb{R}$, indicates dispersion in O'Malley sense of the set B near 0.

To each set $A \subset \mathbb{N}$ we assign a testing set

$$D(A) := \bigcup_{i\in A} \left[\frac{1}{i+1}, \frac{1}{i} \right]$$

and we define an ideal

$$\mathcal{I}_{\mathrm{OM}} := \left\{ A \subset \mathbb{N} : \int_0^1 \frac{1}{t}\chi_{D(A)}(t)dt < \infty \right\}.$$

In [6], it has been shown that

$$\mathcal{I}_{\frac{1}{n}} = \mathcal{I}_{\mathrm{OM}}.$$

We are now interested in the effect of changing the testing set by using another sequence instead of $\left(\frac{1}{n}\right)$. Hence, for any decreasing sequence of real numbers (x_n) from $(0,1]$ tending to zero, define:

$$D_{(x_n)}(A) := \bigcup_{i\in A}[x_{i+1}, x_i]$$

and

$$\mathcal{I}_{\mathrm{OM}}^{(x_n)} := \left\{ A \subset \mathbb{N} : \int_0^1 \frac{1}{t}\chi_{D_{(x_n)}(A)}(t)dt < \infty \right\}.$$

An easy calculation shows that these ideals are in fact summable ideals for certain series:

$$\mathcal{I}_{\mathrm{OM}}^{(x_n)} = \left\{ A \subset \mathbb{N} : \sum_{n\in A} \ln \frac{x_n}{x_{n+1}} < +\infty \right\},$$

i.e.,

$$\mathcal{I}_{\mathrm{OM}}^{(x_n)} = \mathcal{I}_{\ln \frac{x_n}{x_{n+1}}}.$$

Hence, obviously $\mathcal{I}_{\mathrm{OM}}^{(x_n)} \supset \mathcal{F}\mathrm{in}$, where $\mathcal{F}\mathrm{in}$ denotes the ideal of finite subsets of $\mathbb{N}$.

The question is for which sequences $\mathcal{I}_{\mathrm{OM}}^{(x_n)}$ is a summable ideal for the harmonic series $\mathcal{I}_{\frac{1}{n}}$.

3. Main results

Theorem 1. *If* $\liminf_{n\to\infty} \frac{x_n}{x_{n+1}} > 1$, *then*

$$\mathcal{I}_{\mathrm{OM}}^{(x_n)} = \mathcal{F}\mathrm{in} \subsetneq \mathcal{I}_{\frac{1}{n}}.$$

Proof. If $\liminf_{n\to\infty} \frac{x_n}{x_{n+1}} > 1$, then there exists $\beta > 1$ and $n_0 \in \mathbb{N}$ such that for every $n > n_0$ we have

$$\frac{x_n}{x_{n+1}} > \beta, \quad \text{so} \quad \ln\frac{x_n}{x_{n+1}} > \ln\beta > 0.$$

Hence for any infinite set $A \subset \mathbb{N}$,

$$\sum_{n\in A} \ln\frac{x_n}{x_{n+1}} = +\infty, \quad \text{so} \quad A \notin \mathcal{I}_{\mathrm{OM}}^{(x_n)}. \qquad \square$$

Theorem 2. *If* $\limsup_{n\to\infty} \frac{x_n}{x_{n+1}} > 1$, *then* $\mathcal{I}_{\frac{1}{n}} \setminus \mathcal{I}_{\mathrm{OM}}^{(x_n)} \neq \emptyset$.

Proof. Let $\limsup_{n\to\infty} \frac{x_n}{x_{n+1}} = \alpha > 1$. Denote by (x_{n_i}) a subsequence of (x_n) such that $\frac{x_{n_i}}{x_{n_i+1}} \to \alpha$. As in the proof of the previous theorem, we can show that for any subsequence $\left(x_{n_{i_k}}\right)$ of the sequence (x_{n_i}), the series

$$\sum_{k\in\mathbb{N}} \ln \frac{x_{n_{i_k}}}{x_{n_{i_k}+1}} \quad \text{is divergent.}$$

Since the summable ideal has the "tall" Property ([2]), which means that any infinite set $B \subset \mathbb{N}$ contains an infinite subset belonging to $\mathcal{I}_{\frac{1}{n}}$, we can find an infinite subset A of the set of indices of the (x_{n_i}), i.e., $A \subset \{n_i : i \in N\}$ belonging to $\mathcal{I}_{\frac{1}{n}}$. Then, $(x_t)_{t\in A}$ is a subsequence of (x_{n_i}), so $A \notin \mathcal{I}_{\mathrm{OM}}^{(x_n)}$. $\square$

Of course, different sequences can generate the same summable ideal. In [6], a simple and useful sufficient condition is given for this.

Lemma 3. *Let $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ be divergent series. If*

$$0 < \liminf_{n\to\infty} \frac{a_n}{b_n} \le \limsup_{n\to\infty} \frac{a_n}{b_n} < \infty, \quad \textit{then} \quad \mathcal{I}_{a_n} = \mathcal{I}_{b_n}.$$

According to the above lemma, we can easily formulate the sufficient condition for obtaining $\mathcal{I}_{\frac{1}{n}}$ using different testing set to define $\mathcal{I}_{\mathrm{OM}}^{(x_n)}$, i.e., to get $\mathcal{I}_{a_n} = \mathcal{I}_{\frac{1}{n}}$, where $a_n = \ln \frac{x_n}{x_{n+1}}$.

Theorem 4. *Let (x_n) be a decreasing sequence of real numbers from $(0,1]$ tending to zero. If*

$$0 < \liminf_{n\to\infty} n \ln \frac{x_n}{x_{n+1}} \le \limsup_{n\to\infty} n \ln \frac{x_n}{x_{n+1}} < \infty,$$

then

$$\mathcal{I}_{\frac{1}{n}} = \mathcal{I}_{\mathrm{OM}}^{(x_n)}.$$

Let us note that our considerations have led us to a more general question concerning summable ideals. We want to find a sufficient and necessary condition for positive terms of a divergent series $\sum_{n\in\mathbb{N}} a_n$ to have $\mathcal{I}_{\frac{1}{n}} = \mathcal{I}_{a_n}$.

Remark 5. Suppose that series $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ are divergent. If

$$K \in \mathcal{I}_{a_n} \quad \text{and} \quad \lim \sup_{n\in\mathbb{N}\setminus K} \frac{a_n}{b_n} < \infty,$$

then

$$\mathcal{I}_{b_n} \subset \mathcal{I}_{a_n}.$$

Indeed, by assumption there exists a positive number M such that $a_n < Mb_n$ for $n \in \mathbb{N} \setminus K$ big enough. Hence, for any $B \in \mathcal{I}_{b_n}$,

$$\sum_{n\in B} a_n = \sum_{n\in B\setminus K} a_n + \sum_{n\in B\cap K} a_n < M \sum_{n\in B} b_n + \sum_{n\in K} a_n < \infty.$$

Observe that if $K = \emptyset$, we get that if $\limsup_{n\in\mathbb{N}} \frac{a_n}{b_n} < \infty$, then $\mathcal{I}_{b_n} \subset \mathcal{I}_{a_n}$.

Proposition 6. *Let* $\lim_{n\to\infty} b_n = 0$ *and* $\sum_{n\in\mathbb{N}} b_n$ *be divergent. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty, \quad \textit{then} \quad \mathcal{I}a_n \subsetneqq \mathcal{I}_{b_n}.$$

P r o o f. It is clear that from the assumptions we immediately have

$$\sum_{n\in\mathbb{N}} a_n = \infty \quad \text{and} \quad \mathcal{I}_{a_n} \subset \mathcal{I}_{b_n}.$$

We inductively construct the set

$$B \in \mathcal{I}_{b_n} \setminus \mathcal{I}_{a_n}.$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, there exists $l_1 \in \mathbb{N}$ such that

$$\frac{a_n}{b_n} > 2 \quad \text{for every} \quad n \geq l_1.$$

Since $\lim_{n\to\infty} b_n = 0$, there exist numbers $m_1 > l_1$ and $k_1 \in \mathbb{N}$ such that

$$\frac{1}{2} \leq \sum_{i=m_1}^{m_1+k_1} b_i \leq 1.$$

Define

$$B_1 := \{m_1, m_1+1, \ldots, m_1+k_1\}.$$

For every $i \in B_1$, we have $a_i > 2b_i$, hence

$$\sum_{i\in B_1} a_i > 1.$$

Suppose we have defined the finite sequences $\{l_1, \ldots, l_{p-1}\}$, $\{m_1, \ldots, m_{p-1}\}$ and $\{k_1, \ldots, k_{p-1}\}$ such that for any $j = 1, \ldots, p-1$

$$m_j > l_j,$$

$$\frac{a_n}{b_n} > 2^j \quad \text{for every} \quad n \geq l_j$$

and

$$\frac{1}{2^j} \leq \sum_{i=m_j}^{m_j+k_j} b_i \leq \frac{1}{2^{j-1}}.$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, there exists $l_p > m_{p-1} + k_{p-1}$ such that

$$\frac{a_n}{b_n} > 2^p \quad \text{for every} \quad n \geq l_p.$$

Since $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=l_p}^{\infty} b_n = \infty$, there exist numbers $m_p > l_p$ and $k_p \in \mathbb{N}$ such that

$$\frac{1}{2^p} \leq \sum_{i=m_p}^{m_p+k_p} b_i \leq \frac{1}{2^{p-1}}.$$

For $B_p := \{m_p, m_p + 1, \ldots, m_p + k_p\}$, we have

$$\sum_{i\in B_p} a_i > 1.$$

Define

$$B = \bigcup_{p=1}^{\infty} B_p.$$

Since $\sum_{i\in B} b_i \leq 2$, we have $B \in \mathcal{I}_{b_n}$. On the other hand, $\sum_{i\in B} a_i = \infty$, hence $B \notin \mathcal{I}_{a_n}$. □

Corollary 7. *Let $\lim_{n\to\infty} b_n = 0$. If there exists a set $M \subset \mathbb{N}$ such that $\sum_{n\in M} b_n$ is divergent and $\lim_{n\in M} \frac{a_n}{b_n} = \infty$, then*

$$\mathcal{I}_{b_n} \not\subseteq \mathcal{I}_{a_n}.$$

Proof. Arrange the elements of M in an increasing sequence $M = (m_1, m_2, \ldots)$. Sequences (d_n) and (c_n) defined as follows

$$d_n := b_{m_n} \quad \text{and} \quad c_n := a_{m_n}$$

fulfil the assumptions of the previous proposition. Therefore, there exists $D \subset \mathbb{N}$ such that

$$\sum_{i\in D} d_i < \infty \quad \text{and} \quad \sum_{i\in D} c_i = \infty.$$

Hence,

$$B := \{m_n : n \in D\} \in \mathcal{I}_{b_n} \setminus \mathcal{I}_{a_n}.$$ □

Lemma 8. *If $(a_n) \searrow 0$, and $\limsup_{n\to\infty} n a_n = \infty$, then there exists a set $A \subset \mathbb{N}$ such that $\lim_{n\in A} n a_n = \infty$ and $\sum_{n\in A} \frac{1}{n} = \infty$.*

Proof. According to Olivier's theorem ([8], v. 80, Theorem), from the assumptions we have $\sum_{n=1}^{\infty} a_n = \infty$.

We construct the necessary set inductively. Since

$$\limsup_{n\to\infty} n a_n = \infty, \quad \text{there exists} \quad k_1 > 4 \quad \text{such that} \quad k_1 \cdot a_{k_1} > 2^2.$$

Define

$$A_1 := \left\{ \left\lceil \frac{k_1}{2} \right\rceil, \dots, k_1 \right\}.$$

Then,

$$\sum_{i\in A_1} \frac{1}{i} > \frac{1}{k_1}\left(k_1 - \left(\frac{k_1}{2}+1\right)\right) = \frac{1}{2} - \frac{1}{k_1}$$

and for any $i \in A_1$

$$ia_i \geq \frac{k_1}{2} a_{k_1} > \frac{1}{2}\cdot 4\,.$$

Suppose we have already defined a finite increasing sequence $(k_1, \dots, k_{p-1})$ and finite sets of indices $A_1, \ \dots, A_{p-1}$ such that for every number $l \in \{1, \dots, p-1\}$

(1) $\min A_l > \max A_{l-1}$,

(2) $a_{k_l} \cdot k_l > 2^{l+1}$,

(3) $\sum_{i\in A_l} \frac{1}{i} > \frac{1}{2} - \frac{1}{k_l}$,

(4) $ia_i \geq a_{k_l} \cdot \frac{k_l}{2} > 2^l$ for every $i \in A_l$.

Since $\limsup_{n\to\infty} na_n = \infty$, there exists $k_p > 2k_{p-1}$ such that $k_p \cdot a_{k_p} > 2^{p+1}$. Let

$$A_p := \left\{ \left\lceil \frac{k_p}{2} \right\rceil, \dots, k_p \right\}.$$

Then, $\min A_p > \max A_{p-1}$ and

$$\sum_{i\in A_p} \frac{1}{i} > \frac{1}{k_p}\left(k_p - \left(\frac{k_p}{2}+1\right)\right) = \frac{1}{2} - \frac{1}{k_p}.$$

Moreover, for every $i \in A_p$

$$ia_i \geq \frac{k_p}{2} a_{k_p} > 2^p.$$

Finally, defining

$$A = \bigcup_{p=1}^{\infty} A_p,$$

we obtain

$$\sum_{i\in A} \frac{1}{i} > \sum_{p=1}^{\infty}\left(\frac{1}{2} - \frac{1}{k_p}\right) > \sum_{p=1}^{\infty} \frac{1}{4} = \infty.$$

Moreover,

$$\lim_{n\in A} na_n = \infty. \qquad \square$$

Corollary 9. *If* $(a_n) \searrow 0$ *and* $\limsup_{n\to\infty} na_n = \infty$, *then*

$$\mathcal{I}_{\frac{1}{n}} \not\subseteq \mathcal{I}_{a_n}.$$

Note that from the above and Remark 5 for $(a_n) \searrow 0$, we have an equivalence

$$\limsup_{n\to\infty} na_n < \infty \;\Leftrightarrow\; \mathcal{I}_{\frac{1}{n}} \subset \mathcal{I}_{a_n}.$$

Using an analogous procedure, we obtain the following results

Proposition 10. *Suppose that* $\lim_{n\to\infty} b_n = 0$ *and* $\sum_{n\in\mathbb{N}} b_n$ *is divergent. If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, *then*

$$\mathcal{I}_{b_n} \subsetneq \mathcal{I}_{a_n}.$$

Corollary 11. *Let* $\lim_{n\to\infty} b_n = 0$. *If there exists a set* $M \subset \mathbb{N}$ *such that* $\sum_{n\in M} b_n$ *is divergent and* $\lim_{n\in M} \frac{a_n}{b_n} = 0$, *then*

$$\mathcal{I}_{a_n} \not\subseteq \mathcal{I}_{b_n}.$$

Lemma 12. *Let* $(a_n) \searrow 0$, $\sum_{n=1}^{\infty} a_n$ *be divergent and* $\liminf_{n\to\infty} na_n = 0$. *Then, there exists a set* $A \subset \mathbb{N}$ *such that*

$$\lim_{n\in A} na_n = 0 \quad \textit{and} \quad \sum_{n\in A} \frac{1}{n} = \infty.$$

Corollary 13. *If* $(a_n) \searrow 0$, $\sum_{n=1}^{\infty} a_n$ *is divergent and* $\liminf_{n\to\infty} na_n = 0$, *then*

$$\mathcal{I}_{a_n} \subsetneq \mathcal{I}_{\frac{1}{n}}.$$

We have shown that if $(a_n) \searrow 0$, then

$$\mathcal{I}_{a_n} = \mathcal{I}_{\frac{1}{n}}$$

implies

$$0 < \liminf_{n\to\infty} na_n \le \limsup_{n\to\infty} na_n < \infty.$$

Therefore, we can add a new result to Theorem 4.

Theorem 14. *Let* (x_n) *be a decreasing sequence of real numbers from* $(0,1]$ *tending to zero such that* $\left(\ln \frac{x_n}{x_{n+1}}\right) \searrow 0$. *If*

$$\mathcal{I}_{\frac{1}{n}} = \mathcal{I}_{\mathrm{OM}}^{(x_n)},$$

then

$$0 < \liminf_{n\to\infty} n \ln \frac{x_n}{x_{n+1}} \le \limsup_{n\to\infty} n \ln \frac{x_n}{x_{n+1}} < \infty.$$

At this point it is worth emphasising that the implication

$$\limsup_{n\to\infty} n \ln \frac{x_n}{x_{n+1}} < \infty \;\Rightarrow\; \mathcal{I}_{\frac{1}{n}} \subset \mathcal{I}_{\mathrm{OM}}^{(x_n)} = \mathcal{I}_{\ln \frac{x_n}{x_{n+1}}}$$

obtained in Theorem 4 for any decreasing sequence (x_n) of real numbers from $(0,1]$ tending to zero cannot be reversed. Namely, there exists a decreasing sequence (x_n) of real numbers from $(0,1]$ tending to zero such that

$$\mathcal{I}_{\frac{1}{n}} \subset \mathcal{I}_{\ln \frac{x_n}{x_{n+1}}} \quad \text{and} \quad \limsup_{n\to\infty} n \ln \frac{x_n}{x_{n+1}} = \infty.$$

Indeed, we start with an observation that for any sequence (b_n) of positive reals such that $\sum_{n=1}^{\infty} b_n = \infty$, there exists a decreasing sequence (x_n) tending to zero such that $b_n = \ln \frac{x_n}{x_{n+1}}$. It is suffices to define

$$x_1 := 1 \quad \text{and} \quad x_n := e^{-\sum_{i=1}^{n-1} b_i} \quad \text{for} \quad n \geq 2\,.$$

Now, we want to define a sequence (b_n) of positive reals such that

$$\sum_{n=1}^{\infty} b_n = \infty, \quad \limsup_{n\to\infty} n b_n = \infty \quad \text{and} \quad \mathcal{I}_{\frac{1}{n}} \subset \mathcal{I}_{b_n}.$$

Let $A \subset \mathbb{N}$ be an infinite set such that $\sum_{n\in A} \frac{1}{\sqrt{n}} < \infty$. Let

$$b_n := \begin{cases} \frac{1}{\sqrt{n}} & \text{if} \quad n \in A, \\ \frac{1}{n} & \text{if} \quad n \notin A. \end{cases}$$

Since $nb_n = \sqrt{n}$ for $n \in A$, we have $\limsup_{n\to\infty} nb_n = \infty$. Consider now any $B \notin \mathcal{I}_{b_n}$. It means that

$$\sum_{n\in B} b_n = \infty, \quad \text{but} \quad \sum_{n\in B} b_n = \sum_{n\in A\cap B} b_n + \sum_{n\in B\setminus A} b_n.$$

Since the first sum is finite,

$$\sum_{n\in B\setminus A} b_n = \infty, \quad \text{so} \quad \sum_{n\in B\setminus A} \frac{1}{n} = \infty.$$

Thus, $B \setminus A \notin \mathcal{I}_{\frac{1}{n}}$, and consequently, $B \notin \mathcal{I}_{\frac{1}{n}}$. We conclude that $\mathcal{I}_{\frac{1}{n}} \subset \mathcal{I}_{b_n}$.

Of course, if we add an assumption that $\left(\ln \frac{x_n}{x_{n+1}}\right) \searrow 0$, the considered implication "$\Leftarrow$" holds (compare Corollary 9).

It is interesting that the condition from Theorem 4 and Theorem 14 is equivalent to that one from Theorem 5 in [5] concerning the density ideal. To show this, we need an auxiliary lemma.

Lemma 15. *Let (x_n) be a decreasing sequence of real numbers from $(0,1]$ tending to zero and let $a_n := n\frac{x_n - x_{n+1}}{x_n}$ and $b_n := n \ln \frac{x_n}{x_{n+1}}$ for $n \in \mathbb{N}$. If one of the sequences (a_n), (b_n) is convergent to a finite positive limit, then the other one is also convergent and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.*

P r o o f. Observe that
$$a_n = n\left(1 - \frac{x_{n+1}}{x_n}\right) \quad \text{and} \quad \frac{x_n}{x_{n+1}} = \frac{n}{n - a_n},$$
so
$$b_n = n \ln \frac{n}{n - a_n} \quad \text{and} \quad a_n = n\left(1 - e^{-\frac{b_n}{n}}\right).$$

Let us now assume that $\lim_{n\to\infty} a_n = g > 0$. Let $g > \varepsilon > 0$. Then, there exists n_0 such that for any $n > n_0$ we have $g - \varepsilon < a_n < g + \varepsilon$. Hence,
$$n \ln \frac{n}{n - (g - \varepsilon)} < b_n < n \ln \frac{n}{n - (g + \varepsilon)}.$$
For any $c > 0$ we have $\lim_{x\to\infty} x \ln \frac{x}{x-c} = c$.

Indeed,
$$\lim_{x\to\infty} x \ln \frac{x}{x - c} = \lim_{x\to\infty} \frac{\ln \frac{x}{x-c}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{cx}{x - c} = c.$$
Therefore, there exists $n_1 > n_0$ such that for any $n > n_1$ we have
$$g - 2\varepsilon < b_n < g + 2\varepsilon.$$
By arbitrariness of ε, we have $\lim_{n\to\infty} b_n = g$.

Now suppose that $\lim_{n\to\infty} b_n = g > 0$. Let $g > \varepsilon > 0$. Then, there exists n_0 such that for any $n > n_0$ we have
$$g - \varepsilon < b_n < g + \varepsilon.$$
Hence,
$$n\left(1 - e^{-\frac{(g-\varepsilon)}{n}}\right) < a_n < n\left(1 - e^{-\frac{(g+\varepsilon)}{n}}\right).$$
For any $a > 0$ we have
$$\lim_{x\to\infty} x\left(1 - e^{-\frac{a}{x}}\right) = a.$$

Indeed,
$$\lim_{x\to\infty} x\left(1 - e^{-\frac{a}{x}}\right) = \lim_{x\to\infty} \frac{\left(1 - e^{-\frac{a}{x}}\right)}{\frac{1}{x}} = \lim_{x\to\infty} e^{-\frac{a}{x}} a = a.$$
Therefore, there exists $n_1 > n_0$ such that for any $n > n_1$ we have
$$g - 2\varepsilon < a_n < g + 2\varepsilon.$$
By arbitrariness of ε, we have $\lim_{n\to\infty} a_n = g$. $\square$

Proposition 16. *Let (x_n) be a decreasing sequence of real numbers from $(0,1]$ tending to zero and let*

$$a_n := n\frac{x_n - x_{n+1}}{x_n} \quad \textit{and} \quad b_n := n\ln\frac{x_n}{x_{n+1}} \quad \textit{for} \quad n \in \mathbb{N}.$$

Then,

$$0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n < \infty$$

if and only if

$$0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < \infty.$$

Proof. Assume that $\liminf_{n\to\infty} a_n > 0$ and suppose that $\liminf_{n\to\infty} b_n = 0$. Then, there exists a subsequence (b_{n_k}) such that $\lim_{k\to\infty} b_{n_k} = 0$ and by assumption, $\liminf_{k\to\infty} a_{n_k} > 0$, so eventually choosing a subsequence, we have

$$\lim_{k\to\infty} a_{n_k} = g > 0.$$

By Lemma 15, $\lim_{k\to\infty} b_{n_k} = g > 0$, which is a contradiction.

Analogously, we will show that if

$$\limsup_{n\to\infty} a_n < \infty, \quad \text{then} \quad \limsup_{n\to\infty} b_n < \infty.$$

Now, it is easy to see that the inverse implication holds. □

We have thus proved

Theorem 17. *Let (x_n) be a decreasing sequence of real numbers from $(0,1]$ tending to zero. If*

$$0 < \liminf_{n\to\infty} n\frac{x_n - x_{n+1}}{x_n} \le \limsup_{n\to\infty} n\frac{x_n - x_{n+1}}{x_n} < \infty,$$

then

$$\mathcal{I}_{\frac{1}{n}} = \mathcal{I}_{\mathrm{OM}}^{(x_n)}.$$

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