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NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS ANISOTROPIC SOBOLEV WEIGHTS AND NATURAL GROWTH TERMS

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ABSTRACT. The purpose of our paper is to prove the existence of the distributional solutions for anisotropic nonlinear elliptic equations with variable exponents, which contain lower order terms dependent on the gradient of the solution and on the solution itself. The terms are weighted, and the main results rely on the possibility of comparing the weights with each other, where the right-hand side is a sum of the natural growth term and the datum $f \in L^1(\Omega)$. Furthermore the weight function $\theta(\cdot)$ is in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, with $\theta(\cdot) > 0$ and connected with the coefficient $b(\cdot) \in L^1(\Omega)$ of the lower order term.

1. Introduction

Our aim here is to prove the existence of distributional solutions to the weighted anisotropic nonlinear elliptic equations whose model is

$$\begin{cases} -\sum_{i=1}^{N} D_i\big(\theta(x)|D_i u|^{p_i(x)-2} D_i u\big) + a(x)\sum_{i=1}^{N} |u|^{p_i(x)-2}u \\ \quad = b(x)\sum_{i=1}^{N} |D_i u|^{p_i(x)} + f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ a bounded open set with Lipschitz boundary $\partial\Omega$, $\theta(\cdot)$ is in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that, for some $\alpha > 0$

$$\theta(\cdot) \geq \alpha, \tag{2}$$

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f, $a(\cdot)$ and $b(\cdot)$ three $L^1(\Omega)$ functions such that, there exists $\beta > 0$, $\gamma > 0$:

$$|f(x)| \leq \beta a(x), \tag{3}$$

$$|b(x)| \leq \gamma \theta(x). \tag{4}$$

The existence results for a weighted elliptic equations it was given in the works [1, 3, 7, 10–12, 14–16].

The main result of the paper is to prove the existence of the distributional solution for the problem governed by equation (1), which is anisotropic and elliptic with variable exponents, and contains lower order terms dependent on the gradient of the solution and on the solution itself. The terms are weighted and the main results relies on the possibility of comparing the weights between each other, where the right-hand side is a sum of the (natural growth) lower order term and the datum $f \in L^1(\Omega)$, furthermore the weight function $\theta(\cdot)$ is in the anisotropic Sobolev space with variable exponents and zero boundary $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ under the assumption (2), and this is what distinguishes this work. The anisotropic differential operator and the (natural growth) lower order term, are connected by the fact that everyone $\theta(\cdot)$ and $b(\cdot)(\in L^1(\Omega))$ be singular and this is what appears through the assumption (4) where this assumption helped us find solutions in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, while the datum f had a relationship with $a(\cdot) \in L^1(\Omega)$ and it is what is manifested in the condition (3). The existence results in the isotropic scalar case (i.e., $p_i(x) = p$), is proven in [3].

The proof requires a priori estimates for a sequence of suitable approximate solutions (u_n), which in turn is proving its existence by Leray-Schauder's fixed point Theorem. After that we prove the strong convergence, then we pass to the limit in the weak formulation.

2. Preliminaries

In this section we need to provide some basics definitions and properties about isotropic and anisotropic variable exponent Lebesgue-Sobolev spaces (see [5, 6, 9]).

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open subset, we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto \mathbb{R}, \ \ / \ \ 1 < p^- \leq p^+ < \infty\},$$

where

$$p^+ = \max_{x \in \overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x \in \overline{\Omega}} p(x).$$

Let $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$. Then the following Young's inequality holds true for all $a, b \in \mathbb{R}$ and all $\varepsilon > 0$,

$$|ab| \leq \varepsilon |a|^{p(x)} + c(\varepsilon)|b|^{p'(x)}, \tag{5}$$

where, $p'(\cdot)$ denotes the Sobolev conjugate of $p(\cdot)$ (i.e., $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ in $\overline{\Omega}$). In addition, for any two real a, b $\big((a,b) \neq (0,0)\big)$

$$\big(|a|^{p(x)-2}a - |b|^{p(x)-2}b\big)(a-b) \geq \begin{cases} 2^{2-p^+}|a-b|^{p(x)}, & \text{if } p(x) \geq 2, \\ (p^- - 1)\frac{|a-b|^2}{(|a|+|b|)^{2-p(x)}}, & \text{if } 1 < p(x) < 2. \end{cases} \tag{6}$$

Variable exponent Lebesgue space with $L^{p(\cdot)}(\Omega)$ defined by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R}; \rho_{p(\cdot)}(u) < \infty\},$$

where

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x, \quad \text{the convex modular.}$$

It is a Banach space, and reflexive if $p^- > 1$, under the Luxemburg norm

$$u \mapsto \|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \mid \rho_{p(\cdot)}(u/\lambda) \leq 1\right\}.$$

The Hölder type inequality

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

holds true, such that p_i' denotes the Sobolev congugate of p_i

$$(\text{i.e., } \tfrac{1}{p(x)} + \tfrac{1}{p'(x)} = 1).$$

The variable exponents Sobolev space $W^{1,p(\cdot)}(\Omega)$ defined as fellows

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |Du| \in L^{p(\cdot)}(\Omega)\right\},$$

it becomes a Banach space when equipped with the norm

$$u \mapsto \|u\|_{W^{1,p(\cdot)}(\Omega)} := \|Du\|_{p(\cdot)}.$$

The Banach space $W_0^{1,p(\cdot)}(\Omega)$ defined by

$$W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)},$$

under the norm

$$u \mapsto \|u\|_{W_0^{1,p(\cdot)}(\Omega)} := \|u\|_{W^{1,p(\cdot)}(\Omega)}.$$

Moreover, is reflexive and separable if $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$.

The following results came in [5,6]. If (u_n), $u \in L^{p(\cdot)}(\Omega)$, then we have

$$\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right), \tag{7}$$

$$\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right). \tag{8}$$

Now, we will go to introduce the variable exponents anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Let $p_i(\cdot) \in C\big(\overline{\Omega}, [1, +\infty)\big)$, $i = 1, \dots, N$, and we set for every x in $\overline{\Omega}$

$$\overrightarrow{p}(x) = \big(p_1(x), \dots, p_N(x)\big),$$

$$p_+(x) = \max_{1\le i\le N} p_i(x), \quad p_-(x) = \min_{1\le i\le N} p_i(x),$$

$$p_-^- = \min_{x\in\overline{\Omega}} p_-(x), \qquad p_+^+ = \max_{x\in\overline{\Omega}} p_+(x),$$

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i(x)}}, \qquad \overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N, \\ +\infty, & \text{for } \overline{p}(x) \ge N. \end{cases}$$

The Banach space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega), D_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \dots, N\right\},$$

under the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}.$$

The spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ are defined as follow

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)},$$
$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

The following embedding results given in [8, 9].

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in \big(\mathcal{C}_+(\overline{\Omega})\big)^N$.

(i) If $r \in \mathcal{C}_+(\overline{\Omega})$ and $\forall x \in \overline{\Omega}$, $r(x) < \max\big(p_+(x),\ \overline{p}^\star(x)\big)$. Then the embedding

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \quad \text{is compact.} \tag{9}$$

(ii) If we have

$$\forall x \in \overline{\Omega},\ p_+(x) < \overline{p}^\star(x). \tag{10}$$

Then the following inequality holds

$$\|u\|_{p_+(\cdot)} \le C \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{11}$$

where $C > 0$ independent of u. Thus,

$$u \mapsto \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)} \text{ is an equivalent norm on } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{12}$$

3. Statement of results and proof

Definition 3.1. The function u is a distributional solution for (1) if and only if $u \in W_0^{1,1}(\Omega)$, and for every $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \sum_{i=1}^N \theta(x)|D_i u|^{p_i(x)-2} D_i u D_i \varphi \,\mathrm{d}x + \int_\Omega a(x) \sum_{i=1}^N |u|^{p_i(x)-2} u\varphi \,\mathrm{d}x = \int_\Omega b(x) \sum_{i=1}^N |D_i u|^{p_i(x)} \varphi \,\mathrm{d}x + \int_\Omega f(x)\varphi \,\mathrm{d}x.$$

The following is our main results.

Theorem 3.2. *Let $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$ such that $\overline{p}(x) < N$ for all $x \in \overline{\Omega}$ and (10) holds, and let f, $a(\cdot)$ and $b(\cdot)$ are in $L^1(\Omega)$, and $\theta(\cdot)$ is in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that (2), (3), and (4) holds. Then the problem (1) has at least one solution $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the sense of distributions.*

3.1. Existence of approximate solutions

We define

$$\begin{aligned} f_n(x) &= \frac{f(x)}{1+\frac{|f(x)|}{n}}, \qquad & a_n(x) &= \frac{a(x)}{1+\frac{\beta|a(x)|}{n}}, \\ b_n(x) &= \frac{b(x)}{1+\frac{|b(x)|}{n\gamma}}, \qquad & \theta_n(x) &= \frac{\theta(x)}{1+\frac{\theta(x)}{n}}. \end{aligned} \tag{13}$$

We must first notice that: Since $\Theta(x) = \frac{x}{1+\frac{x}{n}}$ is increasing, we deduce by (3) and (4) that

$$|f_n(x)| \leq \frac{\beta a(x)}{1+\frac{\beta}{n}a(x)} = \beta a_n(x). \tag{14}$$

$$|b_n(x)| \leq \frac{\gamma\theta(x)}{1+\frac{\gamma}{n\gamma}\theta(x)} = \gamma\theta_n(x). \tag{15}$$

And note that, thanks to (2), that for all $x \in \overline{\Omega}$, we have

$$\frac{\alpha}{1+\alpha} \leq \theta_n(x) \leq n. \tag{16}$$

Also, since for all $x \in \overline{\Omega}$

$$D_i\theta_n(x) = \frac{D_i\theta(x)}{\left(1+\frac{\theta(x)}{n}\right)^2}, \qquad i = 1, \ldots, N.$$

Then, we get that

$$|D_i\theta_n(x)| \leq |D_i\theta(x)|.$$

So,

$$\theta_n(\cdot) \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

And since it is clear that $0 \leq \theta_n(x) \leq \theta(x)$, we get

$$\theta_n \quad \text{is bounded in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{17}$$

and

$$\theta_n \quad \text{strongly converges to } \theta \text{ in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{18}$$

Remark 1. The assumption (13) implies that, for all $x \in \overline{\Omega}$

$$f_n(x) \leq f(x), \qquad b_n(x) \leq b(x),$$

$$a_n(x) \leq a(x), \qquad \theta_n(x) \leq \theta(x).$$

Lemma 3.3. *Let $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$ such that $\overline{p} < N$ and (10) holds, and let f, $a(\cdot)$ and $b(\cdot)$ are in $L^1(\Omega)$, and $\theta(\cdot)$ is in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that (2), (3), and (4) holds. Then, there exists at least one weak solution $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ to the approximated problems*

$$\begin{cases} -\sum_{i=1}^{N} D_i\big(\theta_n(x)|D_i u_n|^{p_i(x)-2} D_i u_n\big) + a_n(x) u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2} \\ \quad = b_n(x) \sum_{i=1}^{N} |D_i u_n|^{p_i(x)} + f_n & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{19}$$

in the sense that

$$\sum_{i=1}^{N} \int_{\Omega} \theta_n(x)|D_i u_n|^{p_i(x)-2} D_i u_n D_i \varphi \, \mathrm{d}x + \int_{\Omega} \sum_{i=1}^{N} a_n(x)|u_n|^{p_i(x)-2} u_n \varphi \, \mathrm{d}x = \int_{\Omega} \sum_{i=1}^{N} b_n(x)|D_i u_n|^{p_i(x)} \varphi \, \mathrm{d}x + \int_{\Omega} f_n \varphi \, \mathrm{d}x, \tag{20}$$

for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Moreover,

$$\sum_{i=1}^{N} |u_n|^{p_i(x)-1} \leq \beta. \tag{21}$$

P r o o f. We consider for $X = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ the operator

$$\Psi : X \times [0,1] \longrightarrow X,$$

$$(v, \sigma) \longmapsto u = \Psi(v, \sigma),$$

where u is the only weak solution of the problem

$$\begin{cases} -\sum_{i=1}^{N} D_i\left(\theta_n(x)|D_i u|^{p_i(x)-2} D_i u\right) \\ \quad = \sigma\left(b_n(x)\sum_{i=1}^{N}|D_i v|^{p_i(x)} + f_n - a_n(x)v\sum_{i=1}^{N}|v|^{p_i(x)-2}\right) & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{22}$$

verify, for all $\varphi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, the weak formulation

$$\sum_{i=1}^{N}\int_{\Omega}\theta_n(x)|D_i u|^{p_i(x)-2} D_i u D_i\varphi \,\mathrm{d}x = \\ \sigma\int_{\Omega}\left(b_n(x)\sum_{i=1}^{N}|D_i v|^{p_i(x)} + f_n - a_n(x)v\sum_{i=1}^{N}|v|^{p_i(x)-2}\right)\varphi\,\mathrm{d}x. \tag{23}$$

After putting for all $(v,\sigma) \in X \times [0,1]$,

$$g(x,v,Dv) = \sigma\left(b_n(x)\sum_{i=1}^{N}|D_i v|^{p_i(x)} + f_n - a_n(x)v\sum_{i=1}^{N}|v|^{p_i(x)-2}\right),$$

we note that, since $v \in X$ and by using Young's inequality we have for all $v \in X$ and all $\varepsilon > 0$,

$$\int_{\Omega} a_n(x)\sum_{i=1}^{N}|v|^{p_i(x)-1}\,\mathrm{d}x \le n\sum_{i=1}^{N}\left(C(\varepsilon) + \varepsilon\int_{\Omega}|v|^{p_i(x)}\,\mathrm{d}x\right) \le c(\varepsilon),$$

this implies that

$$a_n(x)v\sum_{i=1}^{N}|v|^{p_i(x)-2} \in L^1(\Omega). \tag{24}$$

And we also have through (13) ((the hypotheses on f_n)) and since $v \in X$, we can get

$$\left(b_n(x)\sum_{i=1}^{N}|D_i v|^{p_i(x)} + f_n\right) \in L^1(\Omega). \tag{25}$$

From (24) and (25) we conclude that the right-hand side $g(x,v,Dv)$ of (22) belongs to $L^1(\Omega)$, then the existence of the weak solution u of the problem (22) in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ is directly produced by the main Theorem on monotone operators (see [4,13]). Let us prove the uniqueness of this solution.

Let $u_1,\, u_2 \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ be two weak solutions of (22). Considering the weak formulation of u_1 and u_2, by choosing $\varphi = u_1 - u_2$ as a test function, we have

$$\sum_{i=1}^{N} \int_{\Omega} \theta_n |D_i u_1|^{p_i(x)-2} D_i u_1 D_i(u_1 - u_2)\,\mathrm{d}x = \int_{\Omega} g(x, v, Dv)(u_1 - u_2)\,\mathrm{d}x, \quad (26)$$

and

$$\sum_{i=1}^{N} \int_{\Omega} \theta_n |D_i u_2|^{p_i(x)-2} D_i u_2 D_i(u_1 - u_2)\,\mathrm{d}x = \int_{\Omega} g(x, v, Dv)(u_1 - u_2)\,\mathrm{d}x. \quad (27)$$

By subtracting (27) from (26), we get that

$$\sum_{i=1}^{N} \int_{\Omega} \theta_n \Big(|D_i u_1|^{p_i(x)-2} D_i u_1 - |D_i u_2|^{p_i(x)-2} D_i u_2 \Big) D_i(u_1 - u_2)\,\mathrm{d}x = 0. \quad (28)$$

Putting for all $i = 1, \ldots, N$,

$$I_i = \int_{\Omega} \Big(|D_i u_1|^{p_i(x)-2} D_i u_1 - |D_i u_2|^{p_i(x)-2} D_i u_2 \Big) (D_i u_1 - D_i u_2)\,\mathrm{d}x.$$

Then, by using (16), (28), and the fact that(due (6))

$$\Big(|D_i u_1|^{p_i(x)-2} D_i u_1 - |D_i u_2|^{p_i(x)-2} D_i u_2 \Big) (D_i u_1 - D_i u_2) \geq 0,$$

we get for all $i = 1, \ldots, N$,

$$I_i = 0. \quad (29)$$

Right now, we put for all $i = 1, \ldots, N$,

$$\Omega_i^1 = \{x \in \Omega, p_i(x) \geq 2\},$$

and

$$\Omega_i^2 = \{x \in \Omega, 1 < p_i(x) < 2\}.$$

Then, by (6) we have, for all $i = 1, \ldots, N$

$$I_i \geq 2^{2-p_+^+} \int_{\Omega_i^1} |D_i(u_1 - u_2)|^{p_i(x)}. \quad (30)$$

On the other hand, by Hölder inequality, (6), (7), and since $u_1,\, u_2 \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have

$$
\begin{aligned}
&\int_{\Omega_i^2} |D_i(u_1-u_2)|^{p_i(x)}\,\mathrm{d}x \qquad (31)\\
&\le 2\left\| \frac{|D_i(u_1-u_2)|^{p_i(x)}}{(|D_iu_1|+|D_iu_2|)^{\frac{p_i(x)(2-p_i(x))}{2}}} \right\|_{L^{\frac{2}{p_i(\cdot)}}(\Omega_i^2)}\\
&\quad\times \left\| (|D_iu_1|+|D_iu_2|)^{\frac{p_i(x)(2-p_i(x))}{2}} \right\|_{L^{\frac{2}{2-p_i(\cdot)}}(\Omega_i^2)}\\
&\le 2\max\left\{ \left(\int_{\Omega_i^2} \frac{|D_i(u_1-u_2)|^2}{(|D_iu_1|+|D_iu_2|)^{2-p_i(x)}}\,\mathrm{d}x \right)^{\frac{p_i^-}{2}},\right.\\
&\qquad\left. \int_{\Omega} \left(\int_{\Omega_i^2} \frac{|D_i(u_1-u_2)|^2}{(|D_iu_1|+|D_iu_2|)^{2-p_i(x)}}\,\mathrm{d}x \right)^{\frac{p_i^+}{2}} \right\}\\
&\quad\times\max\left\{ \left(\int_{\Omega} (|D_iu_1|+|D_iu_2|)^{p_i(x)}\,\mathrm{d}x \right)^{\frac{2-p_i^+}{2}},\right.\\
&\qquad\left. \left(\int_{\Omega} (|D_iu_1|+|D_iu_2|)^{p_i(x)}\,\mathrm{d}x \right)^{\frac{2-p_i^-}{2}} \right\}\\
&\le 2c\max\left\{ (I_i)^{\frac{p_i^-}{2}}, (I_i)^{\frac{p_i^+}{2}} \right\} \big(1+\rho_{p_i}(|D_iu_1|+|D_iu_2|)\big)^{\frac{2-p_-^-}{2}}\\
&\le c'\max\left\{ (I_i)^{\frac{p_i^-}{2}}, (I_i)^{\frac{p_i^+}{2}} \right\}. \qquad (32)
\end{aligned}
$$

By combining (30), (32), and (29), we obtain

$$
\int_{\Omega} |D_i(u_1-u_2)|^{p_i(x)}\,\mathrm{d}x = 0, \quad i=1,\dots,N. \qquad (33)
$$

Then, from (33) and (8) we conclude that

$$
\|D_i(u_1-u_2)\|_{p_i(\cdot)} = 0, \quad i=1,\dots,N. \qquad (34)
$$

By using (10) and (34) we get

$$
\|u_1-u_2\|_{\overrightarrow{p}(\cdot)} = 0, \quad i=1,\dots,N. \qquad (35)
$$

Then, (35) implies that $u_1 = u_2$ and so the solution of (22) is unique. Moreover, for all $(v,\sigma) \in X \times [0,1]$

$$\|u\|_X \leq \rho, \tag{36}$$

where, $\rho > 0$ dependent on $\|v\|_X$ and n.

We will now prove the continuity of Ψ:
Let us fix $n \in \mathbb{N}^*$, and let (v_m, σ_m) be a sequence of $X \times [0,1]$ converging to (v,σ) in this space. Then, we get

$$v_m \longrightarrow v, \quad \text{strongly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{37}$$

$$\sigma_m \longrightarrow \sigma, \quad \text{in } \mathbb{R}. \tag{38}$$

After considering the sequence (u_m) defined by $u_m = \Psi(v_m, \sigma_m)$, $m \in \mathbb{N}^*$, we obtain for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\sum_{i=1}^{N} \int_\Omega \theta_n(x)|D_i u_m|^{p_i(x)-2} D_i u_m D_i\varphi \, \mathrm{d}x =$$

$$\sigma_m \left(\int_\Omega f_n \varphi \, dx + \sum_{i=1}^{N} \int_\Omega b_n(x)|D_i v_m|^{p_i(x)} \varphi \, \mathrm{d}x - \int_\Omega a_n(x) v_m \sum_{i=1}^{N} |v_m|^{p_i(x)-2} \varphi \, \mathrm{d}x \right). \tag{39}$$

For v, σ defined in (37), (38), we putting $u = \Psi(v,\sigma)$, then we have for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\sum_{i=1}^{N} \int_\Omega \theta_n(x)|D_i u|^{p_i(x)-2} D_i u D_i\varphi \, \mathrm{d}x =$$

$$\sigma \left(\int_\Omega f_n \varphi \, dx + \sum_{i=1}^{N} \int_\Omega b_n(x)|D_i v|^{p_i(x)} \varphi \, \mathrm{d}x - \int_\Omega a_n(x) v \sum_{i=1}^{N} |v|^{p_i(x)-2} \varphi \, \mathrm{d}x \right). \tag{40}$$

By (36) and the boundedness of (v_m) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ (due (37)):

$$\left\|u_m\right\|_{\overrightarrow{p}(\cdot)} = \left\|\Psi(v_m,\sigma_m)\right\|_{\overrightarrow{p}(\cdot)} \leq \rho, \tag{41}$$

with $\rho > 0$ independent of m.
From (41) we conclude that the sequence (u_m) is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$. So, there exists a function $w \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by (u_m)) such that

$$u_m \rightharpoonup w \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{42}$$

by (42), (10), and (9), we obtain that

$$u_m \longrightarrow w \quad \text{strongly in } L^{p_+(\cdot)}(\Omega). \tag{43}$$

Then, by passing to the limit in (39) as $m \longrightarrow +\infty$, we get for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$,

$$\sum_{i=1}^{N} \int_{\Omega} \theta_n(x)|D_i w|^{p_i(x)-2} D_i w D_i \varphi \, \mathrm{d}x =$$

$$\sigma\left(\int_{\Omega} f_n \varphi \, dx + \sum_{i=1}^{N} \int_{\Omega} b_n(x)|D_i v|^{p_i(x)} \varphi \, \mathrm{d}x - \int_{\Omega} a_n(x) v \sum_{i=1}^{N} |v|^{p_i(x)-2} \varphi \, \mathrm{d}x \right), \quad (44)$$

and this implies that $w = \Psi(v, \sigma)$.

The uniqueness of the weak solution of problem (22) implies that

$$w = u = \Psi(v, \sigma).$$

So,

$$\Psi(v_m, \sigma_m) = u_m \longrightarrow u = \Psi(v, \sigma) \quad \text{strongly in } X.$$

Which shows the continuity of Ψ.

Compactness of Ψ: Let $\tilde{B}$ be a bounded of $X \times [0,1]$. Thus $\tilde{B}$ is contained in a product of the type $B \times [0,1]$ with B a bounded of X, which can be assumed to be a ball of center O and of radius $r > 0$. For $u \in \Psi(\tilde{B})$, we have, thanks to (36):

$$\|u\|_{\overrightarrow{p}(\cdot)} \leq \rho.$$

For

$$u = \Psi(v, \sigma) \quad \text{with} \quad (v, \sigma) \in B \times [0,1]; \quad \|v\|_{\overrightarrow{p}(\cdot)} \leq r.$$

This proves that Ψ applies $\tilde{B}$ in the closed ball of center O and radius ρ ((ρ depends on n and r)) in X. Let u_n be a sequence of elements of $\Psi(\tilde{B})$. Therefore, $u_n = \Psi(v_n, \sigma_n)$ with $(v_n, \delta_n) \in \tilde{B}$. Since u_n remains in a bounded of X, then we can extract a sub-sequence $u_{n_k} = \Psi(v_{n_k}, \delta_{n_k})$ which converges weakly to $u = \Psi(v, \sigma)$ in X. Through (10) and (9) we deduce that u_{n_k} converges strongly to an element w of $L^{p_+(\cdot)}(\Omega)$. Since the existence and uniqueness of the weak solution to the problem (22) and the continuity of Ψ, we have that $w = u$. This proves that

$$\overline{\Psi(\tilde{B})}^{X}$$

is compact. So, Ψ is compact.

It is clear that $\Psi(v, 0) = 0$ for all $v \in X$, because $u = 0 \in X$ the only weak solition of the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D_i \left(\theta_n(x)|D_i u|^{p_i(x)-2} D_i u \right) & = & 0 \quad \text{in } \Omega, \\ u & = & 0 \quad \text{on } \partial\Omega. \end{cases}$$

Now, let us prove that

$$\exists M > 0,\ \forall (v, \sigma) \in X \times [0,1] : v = \Psi(v, \sigma) \Rightarrow \|v\|_X \le M.$$

For that, we give the estimate of elements of X such that

$$v = \Psi(v, \sigma),$$

then we have for all

$$\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega),$$

$$\sum_{i=1}^{N} \int_{\Omega} \theta_n(x)|D_i v|^{p_i(x)-2} D_i v D_i \varphi \,\mathrm{d}x = \sigma \int_{\Omega} \left(b_n(x) \sum_{i=1}^{N} |D_i v|^{p_i(x)} + f_n - a_n(x) v \sum_{i=1}^{N} |v|^{p_i(x)-2} \right) \varphi \,\mathrm{d}x. \quad (45)$$

We cosider the following functions defined as for all $t, s \in \mathbb{R}$,

$$\Phi_\lambda(t) = [e^{\lambda|t|} - 1]\mathrm{sgn}(t), \quad \lambda > 0,$$

$$G_k(s) = \begin{cases} 0 & \text{if} \quad |s| \le k, \\ s - k & \text{if} \quad s > k, \quad k > 0, \\ s + k & \text{if} \quad s < -k. \end{cases}$$

For fixed $\lambda > \gamma$, using

$$(\Phi_\lambda \circ G_\beta)(v) = \left[e^{\lambda |G_\beta(v)|} - 1 \right] \mathrm{sgn}\big(G_\beta(v)\big),$$

as a test function in (45), and after noting that

$$v_n \Phi_\lambda(v) = |v| \left| e^{\lambda |G_\beta(v)|} - 1 \right|,$$

we can obtain that

$$\lambda \sum_{i=1}^{N} \int_{\Omega} \theta_n(x) |D_i G_\beta(v)|^{p_i(x)} e^{\lambda |G_\beta(v)|} \,\mathrm{d}x + \sigma \int_{\Omega} \sum_{i=1}^{N} |a_n(x)||v|^{p_i(x)-1} |e^{\lambda |G_\beta(v)|} - 1| \,\mathrm{d}x$$
$$\le \sigma \left(\int_{\Omega} |f_n| |e^{\lambda |G_\beta(v)|} - 1| \,\mathrm{d}x + \int_{\Omega} \sum_{i=1}^{N} |b_n(x)||D_i v|^{p_i(x)} \left[e^{\lambda |G_\beta(v)|} - 1 \right] \mathrm{d}x \right). \quad (46)$$

By using (14), (15), (16), we get

$$\frac{\alpha}{1+\alpha}(\lambda-\gamma)\sum_{i=1}^{N}\int_{\Omega}|D_i v|^{p_i(x)}e^{\lambda|G_\beta(v)|}\,\mathrm{d}x+ \\ \sigma\int_{\Omega}|a_n(x)|\left(\sum_{i=1}^{N}|v|^{p_i(x)-1}-\beta\right)\left[e^{\lambda|G_\beta(v)|}-1\right]\mathrm{d}x\le 0, \tag{47}$$

and (47) gives us

$$\sum_{i=1}^{N}|v|^{p_i(x)-1}\le\beta. \tag{48}$$

By the fact that $1+|v|^{p_i(x)-1}\ge|v|^{p_-^- -1}$ and (48), we get

$$|v|\le\left(1+\frac{\beta}{N}\right)^{\frac{1}{p_-^- -1}}. \tag{49}$$

Now, choosing the increasing function $\Phi_\gamma(v)$ as test function in (45), we have

$$\gamma\sum_{i=1}^{N}\int_{\Omega}\theta_n(x)|D_i v|^{p_i(x)}e^{\gamma|v|}\,\mathrm{d}x+\sigma\sum_{i=1}^{N}a_n(x)|v|^{p_i(x)-2}v\Phi_\gamma(v)\,\mathrm{d}x= \\ \sigma\int_{\Omega}\left(\sum_{i=1}^{N}b_n(x)|D_i v|^{p_i(x)}+f_n\right)\Phi_\gamma(v)\,\mathrm{d}x. \tag{50}$$

Then, we have

$$\gamma\sum_{i=1}^{N}\int_{\Omega}\theta_n(x)|D_i v|^{p_i(x)}e^{\gamma|v|}\,\mathrm{d}x+\sigma\sum_{i=1}^{N}a_n(x)|v|^{p_i(x)-2}v\Phi_\gamma(v)\,\mathrm{d}x\le \\ \int_{\Omega}|b_n(x)|\sum_{i=1}^{N}|D_i v|^{p_i(x)}|\Phi_\gamma(v)|\,dx+\int_{\Omega}|f_n||\Phi_\gamma(v)|\,\mathrm{d}x. \tag{51}$$

After dropping the nonnegative term in (51) due to the fact that

$$v\Phi_\gamma(v)=|v||\Phi_\gamma(v)|,$$

and using the fact that (due (49))

$$\Phi_\gamma(v)\le\Phi_\gamma\left(\left(1+\frac{\beta}{N}\right)^{\frac{1}{p_-^- -1}}\right),$$

and the fact that $e^{\gamma|v|} - |\Phi_\gamma(v)| = 1$, and (15), we obtain

$$\gamma \sum_{i=1}^{N} \int_\Omega \theta_n(x)|D_i v|^{p_i(x)}\, \mathrm{d}x \le \|f_n\|_{L^1(\Omega)} \Phi_\gamma\left(\left(1+\frac{\beta}{N}\right)^{\frac{1}{p_-^- - 1}}\right). \tag{52}$$

Then, by (16) we have

$$\frac{\gamma\alpha}{1+\alpha} \sum_{i=1}^{N} \int_\Omega |D_i v|^{p_i(x)}\, \mathrm{d}x \le \|f\|_{L^1(\Omega)} \Phi_\gamma\left(\left(1+\frac{\beta}{N}\right)^{\frac{1}{p_-^- - 1}}\right). \tag{53}$$

On the other hand, (like [14], or [17–19]), we have

$$\sum_{i=1}^{N} \int_\Omega |D_i v|^{p_i(x)}\, \mathrm{d}x \ge \left(\frac{1}{N}\|v\|_{\overrightarrow{p}(\cdot)}\right)^{p_-^-} - N. \tag{54}$$

From (54), and (53) we get the existence of $C > 0$ independent of n such that

$$\|v\|_{\overrightarrow{p}(\cdot)} \le C. \tag{55}$$

It then follows from the Leray-Schauder Theorem that the operator

$$\Psi_1 : X \longrightarrow X \quad \text{defined by } \ \Psi_1(u) = \Psi(u,1)$$

has a fixed point, which shows the existence of a solution of the approximated problems (19) in the sense of (20).

In order to prove (21), we can use the function

$$(\Phi_\lambda \circ G_\beta)(u_n) = [e^{\lambda|G_\beta(u_n)|} - 1]\,\mathrm{sgn}(G_\beta(u_n)),$$

as a test function in (20) for fixed $\lambda > \gamma$, and in the same ways as proof (48) we can simply get (21). □

Remark 2. By the fact that $1 + |u_n|^{p_i(x)-1} \ge |u_n|^{p_-^- - 1}$ and (21), we get

$$|u_n| \le \left(1+\frac{\beta}{N}\right)^{\frac{1}{p_-^- - 1}},$$

and this implies that

$$(u_n) \text{ is bounded in } \ L^\infty(\Omega). \tag{56}$$

Remark 3. By going back to (20) and the same way to prove (52) (of course, by replacing v_n by u_n and taking $\sigma = 1$) we can simply get

$$\int_\Omega \theta_n(x)|D_i u_n|^{p_i(x)}\, \mathrm{d}x \le C, \quad i = 1, \dots, N,$$

through this and (15), we obtain:

$$\int_\Omega |b_n(x)||D_i u_n|^{p_i(x)}\,\mathrm{d}x \le C, \quad i = 1, \dots, N,$$

and this implies that, for all $i = 1, \dots, N$

$$\big(b_n(x)|D_i u_n|^{p_i(x)}\big) \quad \text{is bounded in } \ L^1(\Omega). \tag{57}$$

3.1.1. A priori estimates

Lemma 3.4. *Let f, a, b, θ and p_i, $i = 1, \dots, N$ be restricted as in Theorem 3.2. Then*

$$u_n \quad \text{is bounded in } \ \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{58}$$

where u_n the weak solution to the problem (19) .

P r o o f. By choosing the increasing function $\Phi_\gamma(u_n)$ as test function in (20) and the same way as proof (55) (of course, with replacing v_n by u_n and putting $\sigma = 1$) we can get (58). □

Lemma 3.5. *There exists a subsequence (still denoted (u_n)) such that, for all $i = 1, \dots, N$*

$$D_i u_n \longrightarrow D_i u \quad \text{a.e. in } \ \Omega, \tag{59}$$

where u is the weak limit of the sequence (u_n) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

P r o o f. From (58) the sequence (u_n) is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$. So, there exists a function $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by (u_n)) such that

$$u_n \rightharpoonup u \quad \text{weakly in } \ \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \quad \text{and} \quad \text{a.e. in } \ \Omega. \tag{60}$$

Choosing $\frac{1}{\theta_n}\varphi$ as test function in (20), then $\forall \varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega |D_i u_n|^{p_i(x)-2} D_i u_n D_i \phi \ \mathrm{d}x \\
&= \sum_{i=1}^N \int_\Omega \frac{1}{\theta_n(x)} b_n(x)|D_i u_n|^{p_i(x)} \phi \ \mathrm{d}x \\
&\quad + \sum_{i=1}^N \int_\Omega \frac{1}{\theta_n(x)} |D_i u_n|^{p_i(x)-2} D_i u_n (D_i \theta_n) \phi \,\mathrm{d}x \\
&\quad - \sum_{i=1}^N \int_\Omega \frac{1}{\theta_n(x)} a_n(x)|u_n|^{p_i(x)-2} u_n \phi \ \mathrm{d}x + \int_\Omega \frac{1}{\theta_n(x)} f_n \phi \,\mathrm{d}x.
\end{aligned} \tag{61}$$

Then, we can write

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_n|^{p_i(x)-2}D_iu_nD_i\phi\;\mathrm{d}x\;=\int_{\Omega}F_n(x)\phi\;\mathrm{d}x, \tag{62}$$

where,

$$F_n(x)=\frac{1}{\theta_n(x)}\sum_{i=1}^{N}\big(b_n(x)|D_iu_n|^{p_i(x)}+|D_iu_n|^{p_i(x)-2}D_iu_n(D_i\theta_n)-\\ a_n(x)|u_n|^{p_i(x)-2}u_n+f_n\big).$$

$*$ By (21), we have

$$a_n(x)\sum_{i=1}^{N}|u_n|^{p_i(x)-1}\leq \alpha a_n(x). \tag{63}$$

So, from (63), and since $a_n\in L^1(\Omega)$ (due Remark 1), we get

$$\left(a_n(x)u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}\right)\quad\text{is bounded in }\;L^1(\Omega),\;i=1,\dots,N. \tag{64}$$

$*$ By using Young's inequality, and since $u_n\in X$ we obtain for all $\varepsilon>0$,

$$\int_{\Omega}|D_iu_n|^{p_i(x)-1}\;\mathrm{d}x\leq\int_{\Omega}\Big(C(\varepsilon)+\varepsilon|D_iu_n|^{p_i(x)}\Big)\;\mathrm{d}x\leq c,$$

where $c>0$ dependent of ε. This implies that

$$|D_iu_n|^{p_i(x)-2}D_iu_n\in L^1(\Omega). \tag{65}$$

$*$ From (16) we have for all $x\in\overline{\Omega}$

$$\frac{1}{n}\leq\frac{1}{\theta_n(x)}\leq 1+\frac{1}{\alpha}. \tag{66}$$

So, from (64), (65), (57), and (66), we conclude that

$$(F_n)\quad\text{is bounded in }\;L^1(\Omega).$$

So, we can now apply to the sequence (u_n) the results of [2] in order to get (59).
□

3.2. Proof of the Theorem 3.2:

By (59) and (58) we have, for all $i=1,\dots,N$

$$|D_iu_n|^{p_i(x)-2}D_iu_n\rightharpoonup|D_iu|^{p_i(x)-2}D_iu\quad\text{weakly in }\;L^{p_i'(\cdot)}(\Omega), \tag{67}$$

such that,

$$p_i'(\cdot)=\frac{p_i(\cdot)}{p_i(\cdot)-1}.$$

From (18) we conclude that, for all $i=1,\dots,N$

$$\theta_n(\cdot)\longrightarrow\theta(\cdot)\quad\text{strongly in }\;L^{p_i(\cdot)}(\Omega). \tag{68}$$

Furthermore, through (56), (64), and $a_n(\cdot) \in L^1(\Omega)$ due (13), we have

$$a_n(x)u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2} \longrightarrow a(x)u \sum_{i=1}^{N} |u|^{p_i(x)-2} \quad \text{strongly in } L^1(\Omega). \tag{69}$$

Also, from (59), (57), and $b_n(\cdot) \in L^1(\Omega)$ due (13), we have

$$b_n(x) \sum_{i=1}^{N} |D_i u_n|^{p_i(x)} \longrightarrow b(x) \sum_{i=1}^{N} |D_i u|^{p_i(x)} \quad \text{strongly in } L^1(\Omega). \tag{70}$$

Then, through (69), (70), and using (67), (68) to pass to the limit in the weak formulation of (20). This proves Theorem 3.2.

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