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A FIXED POINT APPROACH TO THE STABILITY OF A QUADRATIC FUNCTIONAL EQUATION IN MODULAR SPACES WITHOUT Δ_2-CONDITIONS

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ABSTRACT. In this paper, we investigate the Hyers-Ulam-Rassias stability property of a quadratic functional equation. The even and odd cases for the corresponding function are treated separately before combining them into a single stability result. The study is undertaken in a relatively new structure of modular spaces. The theorems are deduced without using the familiar Δ_2-property of that space. This complicated the proofs. In the proofs, a fixed point methodology is used for which a modular space version of Banach contraction mapping principle is utilized. Several corollaries and an illustrative example are provided.

1. Introduction

Functional equations are considered interesting in mathematics because of their theoretical importance as well as their applications to various problems [10, 36, 42]. Here, we consider such an equation to investigate its stability property .

The type of stability investigated for the functional equations considered here is known as Hyers-Ulam-Rassias stability which is of a very general character and is applicable to diverse branches of mathematics [7, 9, 20]. The concept has its

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origin in the mathematical question of Ulam [46] raised in 1940 and its extensions as well as partial answers given in the works of Hyers [18]and Rassias [37]. In the most general terms, following Gruber [16], we can say that a mathematical entity will have the stability of the above kind with respect to a class of mathematical objects if it has an actual approximation from that class (or a subclass of it) when it behaves approximately like the objects of that class. Particularly functional equations of various kinds have been considered for the investigation of Hyers-Ulam-Rassias stability properties [8, 13, 15, 19, 25, 29, 39, 40, 48].

Recently, research on stability properties of equations has focused on different types of equations, such as rational functional equations [35], multiplicative inverse functional equations [43], fractional differential equations [2, 28], integral equations [6, 24], mixed type functional equations [12, 33], etc. Our study framework is modular spaces [30–32, 34, 38, 44, 47]. It is a linear space on which a modular function with specific properties is defined. Such a function introduces another structure into the linear space, thus extending its range. Several studies in different areas of functional analysis have successfully extended this structure. References [26, 27] give technical details of the space mentioned above. In this paper, we establish Hyers-Ulam-Rassias stability result for the quadratic functional equation [3, 4, 11, 14, 41]

$$f(rx + sy) + f(rx - sy) = 2r^2 f(x) + 2s^2 f(y) \tag{1.1}$$

where r, s are some positive integers, in modular spaces without Δ_2-conditions. The notion of Hyers-Ulam-Rassias stability that we consider here is somewhat different from that described by Gruber, as noted above. As an example, in one of our theorems we prove that under suitable conditions, an approximate quadratic function can have an approximation by an additive function. The method we use here is a fixed point method in which an extension of Banach's contraction mapping principle to modular spaces is utilized.

The present problem is to investigate whether the Hyers-Ulam-Rassias stability of a functional equation remains valid in the wider scope of modular spaces. Particularly, it is interesting to derive results without using Δ_2-conditions. The approach we adopt here is a fixed point approach. There are other approaches for the establishment of Hyers-Ulam-Rassias stabilities [5, 21, 22]. The advantage of the present approach is that the version of Banach's contraction mapping principle for modular spaces can be directly applied to the problem. It is already known that fixed point methods have successful applications in problem of Hyers-Ulam-Rassias stabilities in linear spaces. With the advent of new fixed point results on modular spaces, some examples of which are in references [17, 23, 45], it is possible that the methodology of the present paper will be extended by considering these results in stability problems of functional equations on modular spaces.

2. Preliminaries

In this section, we give the definitions that are essential for the discussion. In 1959, Nakano [32] and Musielak and Orlicz [31] defined a modular on a linear space to construct a modular structure on the space.

Definition 2.1 ([46]). Let X, Y be vector spaces over a field $\mathbb{K}$ (here $\mathbb{R}$ or $\mathbb{C}$). A function $f : X \to Y$ is called an even function if $f(-x) = f(x)$ for all $x \in X$ and f is called an odd function if $f(-x) = -f(x)$ for all $x \in X$.

Definition 2.2 ([31]). Let X be a vector space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized functional $m : X \to [0, \infty]$ is called a *modular* if for arbitrary $x, y \in X$,

$(M1)$ $m(x) = 0$ if and only if $x = 0$,

$(M2)$ $m(\alpha x) = m(x)$ for every scalar α with $|\alpha| = 1$,

$(M3)$ $m(z) \leq m(x) + m(y)$ whenever z is a convex combination of x and y.

The corresponding *modular space*, denoted by X_m, is then defined by

$$X_m := \{x \in X : m(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

If $(M3)$ is replaced with $(M3)'$

$$m(\alpha x + \beta y) \leq \alpha m(x) + \beta m(y) \quad \text{if and only if} \quad \alpha + \beta = 1 \quad \text{and} \quad \alpha, \beta \geq 0,$$

then we say that m is a convex modular.

Remark 2.3. In case the modular m is convex, one has $m(x) \leq \delta m((1/\delta)x)$ for all $x \in X_m$, provided that $0 < \delta \leq 1$.

Definition 2.4 ([31]). Let X_m be a modular space and let $\{x_k\}$ be a sequence in X_m. Then,

(i) $\{x_k\}$ is m-convergent to a point $x \in X_m$ and write $x_k \xrightarrow{m} x$ if $m(x_k - x) \to 0$ as $k \to \infty$;

(ii) $\{x_k\}$ is called m-Cauchy if for any $\epsilon > 0$ one has $m(x_k - x_l) < \epsilon$ for sufficiently large $k, l \in \mathbb{N}$;

(iii) A subset $K \subset X_m$ is called m-complete if any m-Cauchy sequence is m-convergent.

A particular feature of the space, which one encounters and which often makes analysis difficult, is that the convergence of a sequence $\{x_k\}$ to x does not imply that $\{cx_k\}$ converges to cx, where c is chosen from the corresponding scalar field. For that reason, many mathematicians have established some additional conditions that a modular must satisfy to avoid such difficulties. A condition of the type mentioned above is called Δ_2-condition. A modular m is said to satisfy the Δ_2-condition if there exists $\kappa \geq 2$ such that $m(2x) \leq \kappa m(x)$ for all $x \in X_m$.

Some authors have changed the notion so that only $\kappa > 0$ is required and called it a Δ_2-type condition. In fact, these two notions coincide. There are still a number of equivalent notions related to the Δ_2-conditions. The modular m has the Fatou property if and only if $m(x) \leq \lim_{k\to\infty} m(x_k)$ whenever the sequence $\{x_k\}$ is m-convergent to x. For further details and proofs, we refer the reader to [30]. In [1], Abdou proved a series of fixed point theorems in modular spaces where the modulars do not satisfy Δ_2-conditions. Our results are derived without assuming the Δ_2-conditions.

Definition 2.5 ([26])**.** Given a modular space X_m, a nonempty subset $C \subset X_m$, and a mapping $T : C \to C$. The orbit of T around a point $x \in X_m$ is the set

$$\mathbb{O}(x) := \{x, Tx, T^2x, \ldots\}.$$

The quantity

$$\delta_m(x) := \sup\{m(u - v) : u, v \in \mathbb{O}(x)\}$$

is called the *orbit diameter* of T at x, In particular, if $\delta_m(x) < \infty$, one says that T has a bounded orbit at x.

Definition 2.6 ([26])**.** Let m be a modular defined on a vector space X. Let $C \subset X_m$ be nonempty. A mapping $T : C \to C$ is called m-Lipschitzian if there exists a constant $L \geq 0$ such that

$$m\big(T(x) - T(y)\big) \leq Lm(x - y), \forall x, y \in C.$$

If $L < 1$, then T is called m-contraction.

The following result is a modular version of Banach contraction mapping principle.

Theorem 2.7 ([1])**.** *Assume X_m is m-complete. Let C be a nonempty m-closed subset of X_m. Let $T : C \to C$ be an m-contraction mapping. Then, T has a fixed point z if and only if T has an m-bounded orbit. Moreover, if $m(x - z) < \infty$, then $\{T^k(x)\}$ m-converges to z, for any $x \in C$.*

The above theorem allows us to conclude that if z_1 and z_2 are two fixed points of T such that $m(z_1 - z_2) < \infty$, then we have $z_1 = z_2$. In particular, if C is m-bounded, then T has a unique fixed point in C.

3. The generalized Hyers-Ulam stability for (1.1) in modular spaces

Throughout this paper, we assume that the convex modular m has the Fatou property.

Theorem 3.1. *Let V be a vector space and X_m be an m-complete convex modular space. Suppose that a mapping $f : V \to X_m$, which is even, satisfies $f(0)=0$ and the functional inequality*

$$m\big(4f(rx+sy)+4f(rx-sy)-8r^2f(x)-8s^2f(y)\big) \le \phi(x,y) \tag{3.1}$$

for all $x, y \in V$ and some $r, s \in \mathbb{N}$, where $\phi : V^2 \to [0,\infty)$ is a mapping satisfying

$$\phi(2x,2y) \le 4L\phi(x,y) \tag{3.2}$$

for all $x, y \in V$ and some L with $0 < L < 1$.
Then, there exists a unique quadratic mapping $Q : V \to X_m$ such that

$$m\big(2Q(x)-f(x)\big) \le \frac{1}{4(1-L)}\psi(x,x) \tag{3.3}$$

where

$$\psi(x,y) = \frac{1}{4}\left[\phi\left(\frac{x}{r},\frac{y}{s}\right)+\phi\left(\frac{x}{r},0\right)+\phi\left(0,\frac{y}{s}\right)\right]$$

for all $x, y \in V$.

Proof. Consider the set

$$M = \{g : V \to X_m : g(0)=0\}$$

and define a mapping $\tilde{m}$ on M by

$$\tilde{m}(g) = \inf\{c>0 : m\big(g(x)\big) \le c\psi(x,x)\}, \quad g \in M.$$

Now, we prove that $\tilde{m}$ is a convex modular on M. It is sufficient to show that $\tilde{m}$ satisfies the following condition

$$\tilde{m}(\alpha g+\beta h) \le \alpha\tilde{m}(g)+\beta\tilde{m}(h),$$

if

$$\alpha+\beta = 1 \quad \text{and} \quad \alpha,\beta \ge 0.$$

Let $\epsilon > 0$ be given. Then, there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \le \tilde{m}(g)+\epsilon, \qquad c_2 \le \tilde{m}(h)+\epsilon$$

and

$$m\big(g(x)\big) \le c_1\psi(x,x), \qquad m\big(h(x)\big) \le c_2\psi(x,x)$$

for all $x \in V$. If $\alpha+\beta = 1$ and $\alpha,\beta \ge 0$, then

$$\begin{aligned} m\big(\alpha g(x)+\beta h(x)\big) &\le \alpha m\big(g(x)\big)+\beta m\big(h(x)\big) \\ &\le (c_1\alpha+c_2\beta)\psi(x,x) \quad \text{for all } x \in V. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{m}(\alpha g+\beta h) &\le c_1\alpha+c_2\beta \\ &\le \big(\alpha\tilde{m}(g)+\beta\tilde{m}(h)\big)+(\alpha+\beta)\epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\tilde{m}(\alpha g + \beta h) \leq \alpha \tilde{m}(g) + \beta \tilde{m}(h)$. Thus, we have a convex modular space $M_{\tilde{m}}$. □

Let $\{g_k\}$ be a $\tilde{m}$-Cauchy sequence in $M_{\tilde{m}}$. Let $\epsilon > 0$ be given. Then, there exists $p \in N$ such that $\tilde{m}(g_k - g_l) \leq \epsilon$ for all $k, l \geq p$. Then,

$$m\big(g_k(x) - g_l(x)\big) \leq \epsilon\psi(x, x) \quad \text{for all } x \in V \quad \text{and} \quad k, l \geq p. \tag{3.4}$$

This shows that $\{g_k(x)\}$ is a m-Cauchy sequence in X_m for each fixed $x \in V$. Since X_m is m-complete, $\{g_k(x)\}$ is m-convergent in X_m for each fixed $x \in V$. Hence, we can define a function $g : V \to X_m$ by

$$g(x) = \lim_{k\to\infty} g_k(x), x \in V.$$

Clearly, $g \in M_{\tilde{m}}$. Since m has the Fatou property, taking limit as $l \to \infty$ in (3.4), we get

$$m\big(g_k(x) - g(x)\big) \leq \epsilon\psi(x, x) \quad \text{for all } x \in V \quad \text{and} \quad k \geq p.$$

Thus, $\tilde{m}(g_k - g) \leq \epsilon$ for all $k \geq p$ and so $\{g_k\}$ is $\tilde{m}$-convergent sequence in $M_{\tilde{m}}$. Hence, $M_{\tilde{m}}$ is complete.

Define

$$J : M_{\tilde{m}} \to M_{\tilde{m}} \text{ by } Jg(x) = \frac{1}{4}g(2x) \text{ for all } g \in M_{\tilde{m}} \text{ and } x \in V.$$

Let $g, h \in M_{\tilde{m}}$ and $\epsilon > 0$. Then,

$$m\big(g(x) - h(x)\big) \leq (\tilde{m}(g - h) + \epsilon)\psi(x, x) \quad \text{for all } x \in V.$$

Now,

$$\begin{aligned} m\big(Jg(x) - Jh(x)\big) &= m\left(\frac{1}{4}g(2x) - \frac{1}{4}h(2x)\right) \\ &\leq \frac{1}{4}m\big(g(2x) - h(2x)\big) \\ &\leq \frac{1}{4}\big(\tilde{m}(g - h) + \epsilon\big)\psi(2x, 2x) \\ &\leq L\big(\tilde{m}(g - h) + \epsilon\big)\psi(x, x) \quad \text{for all } x \in V. \end{aligned}$$

Therefore,

$$\tilde{m}(Jg - Jh) \leq L\big(\tilde{m}(g - h) + \epsilon\big).$$

Taking $\epsilon \to 0^+$, we get

$$\tilde{m}(Jg - Jh) \leq L\tilde{m}(g - h) \quad \text{for all } g, h \in M_{\tilde{m}}.$$

That is, J is an $\tilde{m}$-contraction. Putting $y = 0$ in (3.1), we get

$$m\big(8f(rx) - 8r^2 f(x)\big) \leq \quad \phi(x, 0) \quad \text{for all } x \in V. \tag{3.5}$$

Again, putting $x = 0$ in (3.1), we get

$$m\big(8f(sy) - 8s^2 f(y)\big) \leq \quad \phi(0, y) \quad \text{for all } y \in V. \tag{3.6}$$

Now,

$$
\begin{aligned}
&m\big(f(rx+sy)+f(rx-sy)-2f(rx)-2f(sy)\big)\\
&\le \frac{1}{4}m\big(4f(rx+sy)+4f(rx-sy)-8r^2f(x)-8s^2f(y)\big)\\
&\quad+\frac{1}{4}m\big(8f(rx)-8r^2f(x)\big)+\frac{1}{4}m\big(8f(sy)-8s^2f(y)\big)\\
&\le \frac{1}{4}[\phi(x,y)+\phi(x,0)+\phi(0,y)] \quad \text{for all } x,y\in V \ \ [\text{by } (3.1),(3.5),(3.6)].
\end{aligned}
$$

Replacing x with $\frac{x}{r}$ and y with $\frac{y}{s}$, we obtain

$$
\begin{aligned}
&m\big(f(x+y)+f(x-y)-2f(x)-2f(y)\big)\\
&\le \frac{1}{4}\left[\phi\left(\frac{x}{r},\frac{y}{s}\right)+\phi\left(\frac{x}{r},0\right)+\phi\left(0,\frac{y}{s}\right)\right]\\
&=\psi(x,y) \quad \text{for all } x,y\in V.
\end{aligned}
\tag{3.7}
$$

Putting $y=x$ in (3.7), we get

$$
m\big(f(2x)-4f(x)\big)\le\psi(x,x) \qquad \text{for all } x\in V. \tag{3.8}
$$

Replacing x with $2x$ in (3.8), we get

$$
m\big(f(2^2x)-4f(2x)\big)\le\psi(2x,2x) \quad \text{for all } x\in V. \tag{3.9}
$$

Now,

$$
\begin{aligned}
&m\left(\frac{f(2^2x)}{4^2}-f(x)\right)\\
&\le \frac{1}{4^2}m\big(f(2^2x)-4f(2x)\big)+\frac{1}{4}m\big(f(2x)-4f(x)\big)\\
&\le \frac{1}{4^2}\psi(2x,2x)+\frac{1}{4}\psi(x,x) \quad [\text{by } (3.8),\ (3.9)]\\
&=\frac{1}{4}\sum_{i=0}^{1}\frac{1}{4^i}\psi(2^ix,2^ix) \text{ for all } x\in V.
\end{aligned}
\tag{3.10}
$$

By Principle of mathematical induction, we can easily see that

$$
m\left(\frac{f(2^kx)}{4^k}-f(x)\right)\le\frac{1}{4}\sum_{i=0}^{k-1}\frac{1}{4^i}\psi(2^ix,2^ix) \qquad \text{for all } x\in V \ \text{ and } \ k\in\mathbb{N}.
$$

Thus,

$$m\left(\frac{f(2^k x)}{4^k} - f(x)\right) \leq \frac{\psi(x,x)}{4}\sum_{i=0}^{k-1} L^i$$
$$\leq \frac{1}{4(1-L)}\psi(x,x) \quad \text{for all } \ x \in V \ \text{ and } \ k \in \mathbb{N}. \tag{3.11}$$

Now, for any $k, l \in N$, we get

$$m\left(\frac{f(2^k x)}{2.4^k} - \frac{f(2^l x)}{2.4^l}\right)$$
$$\leq \frac{1}{2}m\left(\frac{f(2^k x)}{4^k} - f(x)\right) + \frac{1}{2}m\left(\frac{f(2^l x)}{4^l} - f(x)\right)$$
$$\leq \frac{1}{4(1-L)}\psi(x,x) \quad \text{for all } \ x \in V \ \ [\text{by (3.11)}].$$

Therefore,

$$\tilde{m}\left(J^k\left(\frac{1}{2}f\right) - J^l\left(\frac{1}{2}f\right)\right) \leq \frac{1}{4(1-L)} \ < \ \infty \quad \text{for all } k, l \in N.$$

This shows that J has a bounded orbit at $\frac{1}{2}f$. Then, by Theorem 2.7

(i) the sequence $\left\{J^k\left(\frac{1}{2}f\right)\right\}$ is $\tilde{m}$-convergent to some $Q \in M_{\tilde{m}}$, i.e.,

$$\lim_{k\to\infty} \tilde{m}\left(J^k\left(\frac{1}{2}f\right) - Q\right) = 0,$$

(ii) Q is a fixed point of J and so, $J(Q) = Q$, i.e.,

$$\frac{1}{4}Q(2x) = Q(x) \quad \text{for all } \ x \in V.$$

This implies $\lim_{k\to\infty} m\big(J^k \frac{1}{2} f(x) - Q(x)\big) = 0$, i.e.,

$$\lim_{k\to\infty} m\left(\frac{1}{2.4^k}f(2^k x) - Q(x)\right) = 0 \quad \text{for all} \quad x \in V.$$

Replacing x with $2^k x$ and y with $2^k y$ in (3.7), we get

$$m\Big(f\big(2^k(x+y)\big) + f\big(2^k(x-y)\big) - 2f(2^k x) - 2f(2^k y)\Big) \leq \psi(2^k x, 2^k y)$$

for all $x, y \in V$ and $k \in \mathbb{N}$.

Now, for all $x, y \in V$ and $k \in \mathbb{N}$,

$$\begin{aligned} & m\left(\frac{1}{2.4^k}\left[f\big(2^k(x+y)\big)+f\big(2^k(x-y)\big)-2f(2^kx)-2f(2^ky)\right]\right) \\ & \le \frac{1}{2.4^k}m\Big(f\big(2^k(x+y)\big)+f\big(2^k(x-y)\big)-2f(2^kx)-2f(2^ky)\Big) \\ & \le \frac{1}{2.4^k}\psi(2^kx, 2^ky) \\ & \le \frac{1}{2}L^k\psi(x,y). \end{aligned}$$

Since ρ has the Fatou property and $0 < L < 1$, taking limit as $k \to \infty$, we get

$$m\big(Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\big) = 0 \quad \text{for all } \ x, y \in V.$$

Thus,

$$Q(x+y)+Q(x-y) = 2Q(x)+2Q(y) \quad \text{for all } x, y \in V,$$

that is, Q is quadratic.

Again, since m has the Fatou property, by (3.11) we get

$$m\big(2Q(x)-f(x)\big) \le \frac{1}{4(1-L)}\psi(x,x) \quad \text{for all } \ x \in V. \tag{3.12}$$

To prove the uniqueness of Q, let Q' be another fixed point of J satisfying (3.12). Now,

$$\begin{aligned} m\big(Q(x)-Q'(x)\big) &= m\Big(Q(x)-\frac{1}{2}f(x)+\frac{1}{2}f(x)-Q'(x)\Big) \\ &\le \frac{1}{2}m\big(2Q(x)-f(x)\big)+\frac{1}{2}m\big(2Q'(x)-f(x)\big) \\ &\le \frac{1}{4(1-L)}\psi(x,x) \quad \text{for all } \ x \in V \ \ [\text{by } \ (3.12)]. \end{aligned}$$

It implies that $\tilde{m}(Q-Q') \le \frac{1}{4(1-L)} < \infty$. Since J is an $\tilde{m}-$contraction, we get

$$\tilde{m}(Q-Q') = \tilde{m}\big(J(Q)-J(Q')\big) \le L\tilde{m}(Q-Q')$$

which implies that

$$\tilde{m}(Q-Q') = 0, \quad \text{i.e., } \ Q = Q'.$$

This completes the theorem.

Corollary 3.2. *Let V be a linear space and X_m be an m-complete convex modular space. Suppose that a mapping $f : V \to X_m$, which is even, satisfies $f(0) = 0$ and the functional inequality*

$$m\big(4f(rx+sy)+4f(rx-sy)-8r^2f(x)-8s^2f(y)\big) \le \epsilon$$

for all $x, y \in V$ and some $r, s \in \mathbb{N}, \epsilon \ge 0$. Then, there exists a unique quadratic mapping $Q : V \to X_m$ such that

$$m\big(2Q(x)-f(x)\big) \le \frac{\epsilon}{4} \quad \textit{for all } x \in V.$$

P r o o f. Define $\phi(x,y) = \epsilon$ for all $x, y \in V$ and take $L = \frac{1}{4}$. Then, it is clear that (3.2) is satisfied. Thus, applying Theorem 3.1, we have the above result.

Corollary 3.3. *Let V be a normed linear space and X_ρ an m-complete convex modular space. Suppose that a mapping $f : V \to X_m$, which is even, satisfies $f(0) = 0$ and the functional inequality*

$$m\big(4f(rx+sy) + 4f(rx-sy) - 8r^2 f(x) - 8s^2 f(y)\big) \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$ and some $r, s \in \mathbb{N}, \epsilon \geq 0, 0 \leq p < 2$. Then, there exists a unique quadratic mapping $Q : V \to X_m$ such that

$$m\big(2Q(x) - f(x)\big) \leq \frac{\epsilon}{2(4-2^p)} \left[\frac{1}{r^p} + \frac{1}{s^p}\right] \|x\|^p \quad \text{for all} \quad x \in V. \qquad \square$$

P r o o f. Define $\phi(x,y) = \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in V$ and take $L = 2^{p-2}$. Then, it is clear that (3.2) is satisfied. Thus, applying Theorem 3.1, we have the above result. $\square$

The next theorem is for odd functions.

Theorem 3.4. *Let V be a linear space and X_m be an m-complete convex modular space. Suppose that a mapping $f : V \to X_m$, which is odd, satisfies the functional inequality*

$$m\big(4f(rx+sy) + 4f(rx-sy) - 8r^2 f(x) - 8s^2 f(y)\big) \leq \phi(x,y)$$

for all $x, y \in V$ and some $r, s \in \mathbb{N}$, where $\phi : V^2 \to [0, \infty)$ is a mapping satisfying

$$\phi(2x, 2y) \leq 2L\phi(x,y) \quad \text{for all } x, y \in V \text{ and some } L \text{ with } 0 < L < 1. \tag{3.13}$$

Then, there exists a unique additive mapping $A : V \to X_m$ such that

$$m\big(2A(x) - f(x)\big) \leq \frac{1}{2(1-L)} \psi(x,x), \tag{3.14}$$

where

$$\psi(x,y) = \frac{1}{4}\left[\phi\left(\frac{x}{r}, \frac{y}{s}\right) + \phi\left(\frac{x}{r}, 0\right) + \phi\left(0, \frac{y}{s}\right)\right] \quad \text{for all } x, y \in V.$$

P r o o f. Proceeding analogously as in Theorem 3.1, the result can be proved. $\square$

Corollary 3.5. *Let V be a linear space and X_m be an m-complete convex modular space. Suppose that a mapping $f : V \to X_m$, which is odd, satisfies the functional inequality*

$$m\big(4f(rx+sy) + 4f(rx-sy) - 8r^2 f(x) - 8s^2 f(y)\big) \leq \epsilon$$

for all $x, y \in V$ and some $r, s \in \mathbb{N}, \epsilon \geq 0$. Then, there exists a unique additive mapping $A : V \to X_m$ such that

$$m\big(2A(x) - f(x)\big) \leq \frac{3\epsilon}{4} \quad \text{for all } x \in V.$$

P r o o f. Define $\phi(x, y) = \epsilon$ for all $x, y \in V$ and take $L = \frac{1}{2}$. Then, it is clear that (3.13) is satisfied. Thus, applying Theorem 3.4, we have the above result. $\square$

Corollary 3.6. *Let V be a normed linear space and X_m be an m-complete convex modular space. Suppose that a mapping $f : V \to X_m$, which is odd, satisfies the functional inequality*

$$m\big(4f(rx+sy) + 4f(rx-sy) - 8r^2 f(x) - 8s^2 f(y)\big) \le \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$ and some $r, s \in \mathbb{N}, \epsilon \ge 0, 0 \le p < 1$. Then, there exists a unique additive mapping $A : V \to X_m$ such that

$$m\big(2A(x) - f(x)\big) \le \frac{\epsilon}{2(2-2^p)}\left[\frac{1}{r^p} + \frac{1}{s^p}\right]\|x\|^p \quad \textit{for all } x \in V.$$

P r o o f. Define $\phi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in V$ and take $L = 2^{p-1}$. Then, clearly, (3.13) is satisfied. Thus, applying Theorem 3.4, we have the above result. $\square$

Theorem 3.7. *Let V be a linear space and X_m be an m-complete convex modular space. Suppose that a mapping $f : V \to X_m$ which satisfies $f(0) = 0$ and the functional inequality*

$$m\big(4f(rx+sy) + 4f(rx-sy) - 8r^2 f(x) - 8s^2 f(y)\big) \le \phi(x, y)$$

for all $x, y \in V$ and for some $r, s \in \mathbb{N}$, where $\phi : V^2 \to [0, \infty)$ is a mapping satisfying

$$\phi(2x, 2y) \le 2L\phi(x, y)$$

for all $x, y \in V$ and some L with $0 < L < 1$. Then, there exist a unique quadratic mapping $Q : V \to X_m$ and a unique additive mapping $A : V \to X_m$ such that

$$m\left(A(x) + Q(x) - \frac{f(x)}{2}\right) \le \frac{3}{8(1-L)}\psi(x, x)$$

where

$$\psi(x, y) = \frac{1}{8}\Big[\phi\left(\frac{x}{r}, \frac{y}{s}\right) + \phi\left(-\frac{x}{r}, -\frac{y}{s}\right) + \phi\left(\frac{x}{r}, 0\right) + \phi\left(-\frac{x}{r}, 0\right) + \phi\left(0, \frac{y}{s}\right) + \phi\left(0, -\frac{y}{s}\right)\Big] \quad \textit{for all } x, y \in V.$$

P r o o f. Let

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2} \quad \text{for all } x \in V.$$

Then,

$$f(x) = f_e(x) + f_o(x) \quad \text{for all } x \in V.$$

Clearly, f_e is even and f_o is odd and $f_e(0) = f_o(0) = 0$.

Replacing x with $-x$ and y with $-y$ in (3.1), we get

$$\begin{aligned}&m\big(4f(-rx-sy)+4f(-rx+sy)-8r^2f(-x)-8s^2f(-y)\big)\\&\le \phi(-x,-y)\quad\text{for all }\ x,y\in V.\end{aligned}\tag{3.15}$$

Now,

$$\begin{aligned}&m\big(4f_e(rx+sy)+4f_e(rx-sy)-8r^2f_e(x)-8s^2f_e(y)\big)\\&\le \frac{1}{2}m\big(4f(rx+sy)+4f(rx-sy)-8r^2f(x)-8s^2f(y)\big)\\&\quad+\frac{1}{2}m\big(4f(-rx-sy)+4f(-rx+sy)-8r^2f(-x)-8s^2f(-y)\big)\\&\le \frac{1}{2}[\phi(x,y)+\phi(-x,-y)]\quad[\text{by (3.1) and (3.15)}]\\&=\theta(x,y)\quad\text{say, for all }\ x,y\in V.\end{aligned}$$

Now,

$$\begin{aligned}\theta(2x,2y)&=\frac{1}{2}[\phi(2x,2y)+\phi(-2x,-2y)]\\&\le\frac{1}{2}[2L\phi(x,y)+2L\phi(-x,-y)]\quad[\text{by (3.13)}],\end{aligned}$$

i.e.,

$$\theta(2x,2y)\le 2L\theta(x,y)\le 4L\theta(x,y)\quad\text{for all }\ x,y\in V.$$

By Theorem 3.1, there exists a unique quadratic mapping $Q:V\to X_m$ such that

$$m\big(2Q(x)-f_e(x)\big)\le\frac{1}{4(1-L)}\psi(x,x),\tag{3.16}$$

where

$$\begin{aligned}\psi(x,x)&=\frac{1}{4}\left[\theta\left(\frac{x}{r},\frac{x}{s}\right)+\theta\left(\frac{x}{r},0\right)+\theta\left(0,\frac{x}{s}\right)\right]\\&=\frac{1}{8}\left[\phi\left(\frac{x}{r},\frac{x}{s}\right)+\phi\left(-\frac{x}{r},-\frac{x}{s}\right)\right.\\&\quad+\phi\left(\frac{x}{r},0\right)+\phi\left(-\frac{x}{r},0\right)\\&\quad\left.+\phi\left(0,\frac{x}{s}\right)+\phi\left(0,-\frac{x}{s}\right)\right]\quad\text{for all }\ x\in V.\end{aligned}$$

Again,

$$
\begin{aligned}
& m\big(4f_o(rx+sy)+4f_o(rx-sy)-8r^2f_o(x)-8s^2f_o(y)\big) \\
& \le \frac{1}{2}m\big(4f(rx+sy)+4f(rx-sy)-8r^2f(x)-8s^2f(y)\big) \\
& \quad + \frac{1}{2}m\big(4f(-rx-sy)+4f(-rx+sy)-8r^2f(-x)-8s^2f(-y)\big) \\
& \le \frac{1}{2}[\phi(x,y)+\phi(-x,-y)] \quad [\text{by}(3.1) \text{ and } (3.15)] \\
& = \theta(x,y) \quad \text{for all } \ x,y \in V.
\end{aligned}
$$

So, by Theorem 3.4, there exists a unique additive mapping $A : V \to X_m$ such that

$$
m\big(2A(x)-f_o(x)\big) \le \frac{1}{2(1-L)}\psi(x,x) \quad \text{for all } \ x \in V. \tag{3.17}
$$

Now,

$$
\begin{aligned}
& m\left(A(x)+Q(x)-\frac{f(x)}{2}\right) \\
& \le \frac{1}{2}m\big(2A(x)-f_o(x)\big)+\frac{1}{2}m\big(2Q(x)-f_e(x)\big) \\
& \le \frac{1}{2}\left[\frac{1}{2(1-L)}+\frac{1}{4(1-L)}\right]\psi(x,x) \ \ [\text{by } \ (3.16), \ (3.17)] \\
& = \frac{3}{8(1-L)}\psi(x,x) \quad \text{for all } x \in V.
\end{aligned}
$$

This completes the proof. □

Corollary 3.8. *Let V be a linear space and X_m be an m-complete convex modular space. Suppose that a mapping $f : V \to X_m$ satisfies the functional inequality*

$$
m\big(4f(rx+sy)+4f(rx-sy)-8r^2f(x)-8s^2f(y)\big) \le \epsilon
$$

for all $x,y \in V$ and some $r,s \in \mathbb{N}, \epsilon \ge 0$. Then, there exist a unique quadratic mapping $Q : V \to X_m$ and a unique additive mapping $A : V \to X_m$ such that

$$
m\left(A(x)+Q(x)-\frac{f(x)}{2}\right) \le \frac{9\epsilon}{16} \quad \textit{for all } \ x \in V.
$$

Proof. Define

$$m(x,y)=\epsilon \quad \text{for all } x,y\in V \quad \text{and take } L=\frac{1}{2}.$$

Then, it is clear that (3.13) is satisfied. Thus, applying Theorem 3.7, we have the above result. □

Corollary 3.9. *Let V be a normed linear space and X_m be an m-complete convex modular space. Suppose that a mapping $f : V \to X_m$ satisfies the functional inequality*

$$m\big(4f(rx+sy)+4f(rx-sy)-8r^2f(x)-8s^2f(y)\big)\le \epsilon(\|x\|^p+\|y\|^p)$$

for all $x,y\in V$ and some $r,s\in\mathbb{N},\epsilon\ge 0, 0\le p<1$. Then, there exists a unique quadratic mapping $Q : V\to X_m$ and a unique additive mapping $A : V\to X_m$ such that

$$m\left(A(x)+Q(x)-\frac{f(x)}{2}\right)\le \frac{3\epsilon}{8(2-2^p)}\left[\frac{1}{r^p}+\frac{1}{s^p}\right]\|x\|^p \quad \textit{for all } x\in V.$$

Proof. Define

$$\phi(x,y)=\epsilon(\|x\|^p+\|y\|^p) \text{ for all } x,y\in V \text{ and take } L=2^{p-1}.$$

Then, clearly (3.13) is satisfied. Thus, applying Theorem 3.7, we have the above result. □

Example 3.10. Let $(X,\|.\|)$ be a commutative Banach algebra and X_m be an m-complete convex modular space, where $m(x)=\|x\|$. Define

$$f : X\to X_m \quad \text{by } f(x)=ax^2+A\|x\|x_0 \quad \text{for all } x\in X,$$

where $a\in\mathbb{R}, A\in\mathbb{R}^+$ and x_0 is a unit vector in X. Then,

$$\begin{aligned}&m\big(4f(rx+sy)+4f(rx-sy)-8r^2f(x)-8s^2f(y)\big)\\&\le 8A[r(r-1)\|x\|-s(s-1)\|y\|]\end{aligned}$$

for all $x,y\in X$ and some $r,s\in\mathbb{N}$. Let

$$\phi(x,y)=8A[r(r-1)\|x\|-s(s-1)\|y\|] \quad \text{for all } x,y\in X \quad \text{and take } L=\frac{1}{2}.$$

Clearly, (3.2) is satisfied. Thus, all the conditions of Theorem 3.1 are satisfied. Then, there exists a unique quadratic mapping $Q : X\to X_m$ such that

$$m\big(2Q(x)-f(x)\big)\le 4A(r+s-2)\|x\| \quad \text{for all } x\in X.$$

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REFERENCES

[1] ABDOU, A. A. N.—KHAMSI, M. A.: *Fixed point theorems in modular vector spaces*, J. Nonlinear Sci. Appl. **10** (2017), no. 8, 4046–4057.

[2] M. A. ALMALAHI—K. A. ALDWOAH—K. SHAH—T. ABDELJAWAD: *Stability and numerical analysis of a coupled system of piecewise atangana–baleanu fractional differential equations with delays*, Qualitative Theory of Dynamical Systems, **23** 2024, no. 3, 1–27.

[3] BANTAOJAI, T.—SUANOOM, C.: *Stability of a generalization of cauchy's and the quadratic functional equations in quasi-banach spaces*, Thai J. Math. **18** (2020), no. 3, 963–975.

[4] BETTENCOURT, G. H.—MENDES, S.: *On the stability of a quadratic functional equation over non-archimedean spaces*, **35** (2021), no. 8, 2693–2704.

[5] BODAGHI, A.—MOSHTAGH, H.—MOUSIVAND, A. ET AL: *Characterization and stability of multi-Euler-Lagrange quadratic functional equations*, J. Function Spaces **2022**, paper no. 3021457.

[6] BOYKOV, I. V.—ROUDNEV, V. A.—BOYKOVA, A. I.: *Stability of solutions of systems of Volterra integral equations*, Appl. Math. Comput. **475** (2024), paper no. 128728.

[7] CADARIU L.—RADU, V.: *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4** (2003), no. (1), article 4, 7 pp.

[8] CZERWIK, S.: *On the stability of the quadratic mapping in normed spaces*, In: Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, Vol. 62, Springer--Veralg, Berlin, 1992, pp. 59–64.

[9] DONG, Y.: *On approximate isometries and application to stability of a functional equation*, J. Math. Anal. Appl. **426** (2015), no. 1, 125–137.

[10] DUTTA, H.—SENTHIL KUMAR, B. V.—SABARINATHAN, S.: *Fuzzy stabilities of a new hexic functional equation in various spaces*, Analele ştiinţifice ale Universităţii "Ovidius" Constanţa. Seria Matematică **30** (2022), no. 3, 143–171.

[11] EL-FASSI, IZ-IDDINE: *Approximate solution of a generalized multi-quadratic type functional equation in Lipschitz spaces*, J. Math. Anal. Appl. **519** (2023), no. 2, paper no. 126840, 16 pp.

[12] ELUMALAI, P.—SANGEETHA, S.—SELVAN, A.P.: *Fixed point approach to the stability of a cubic and quartic mixed type functional equation in non-Archimedean spaces*, J. Math. Comput. Sci, **33** (2024), no. 2, 124–136.

[13] GAVRUTA, P.: *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), no. 3, 431–436.

[14] GOVINDAN, V.—PARK, C.—PINELAS, S.—RASSIAS T. M.: *Hyers-Ulam stability of an additive-quadratic functional equation*, Cubo (Temuco), **22** (2020), no. 2, 233–255.

[15] GRABIEC, A.: *The generalized Hyers-Ulam stability of a class of functional equations*, Publ. Math. Debrecen **48** (1996), no. 3–4, 217–235.

[16] GRUBER, P. M.: *Stability of isometries*, Trans. Amer. Math. Soc. **245** (1978), 263–277.

[17] GÜNDOĞDU, H,—ÖZTÜRK, M. —GÖZÜKIZIL, Ö. F : *Fixed point theorems of new generalized c-conditions for (psi; gamma)-mappings in modular metric spaces and its applications*, Nonlinear Anal. Model. Control **28** (2023), 1–22.

[18] HYERS, D.-H.: *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** 1941, 222–224.

[19] JUNG, S.: *Quadratic functional equations of pexider type*, Int. J. Math. Math. Sci. **24** (2000), no. 5, 351–359.

[20] JUNG, S.: *Hyers-Ulam stability of linear differential equations of first order, II*, Appl. Math. Lett. **19** (2006), no. 9, 854–858.

[21] CHUGH, J. R.—JAISWAL, S.—DUBEY, R.: *Stability of various functional equations in non-Archimedean (n,β)-normed spaces*, The Journal of Analysis, **30** (2022), no. 4, 1653–1669.

[22] JAYARAMAN, U.—KALAICHELVAN, R.: *Generalized Hyers-Ulam-Rassias stability of an Euler-Lagrange type cubic functional equation in non-Archimedean quasi-Banach spaces*, Mathematical Modelling of Engineering Problems **11** (2024), no. 4, 1021–1028.

[23] KARAPINAR, E.—AKSOY, Ü—FULGA, A.—ERHAN, I. M.: *Fixed point theorems for mappings with a contractive iterate at a point in modular metric spaces*, Fixed Point Theory **23** (2022), no. 2, 519–531.

[24] KAWANO, A.—MATSUNAGA, H.: *Exponential stability and asymptotic periodic solutions of linear integral equations with two delays*, J. Dynam. Differential Equations **35** (2023), no. 2, 1309–1335.

[25] KAYAL, N. C.—MONDAL, P.—SAMANTA, T. K.: *On the stability of a pexiderized functional equation in intuitionistic fuzzy Banach spaces*, App. Appl. Math. **10** (2015), no. (2), 783–794.

[26] KHAMSI, M. A.—KOZLOWSKI, W. M.: *Fixed Point Theory in Modular Function Spaces*. [With a foreword by W. A. Kirk.], Birkhäuser, Springer, Cham, 2015.

[27] KOZLOWSKI, W. M.: *Modular Function Spaces*. Marcel Dekker, Inc., New York, 1988.

[28] MATAR, M. M.—SAMEI, M. E.—ETEMAD, S.— AMARA, A.— REZAPOUR, S.–ALZABUT, J.: *Stability analysis and existence criteria with numerical illustrations to fractional Jerk differential system involving generalized caputo derivative*, Qualitative Theory of Dynamical Systems, **23** (2024), no. 3, 1–36,

[29] MIRMOSTAFAEE, A. K.—MOSLEHIAN, M. S.: *Stability of additive mappings in non-Archimedean fuzzy normed spaces*, Fuzzy Sets and Systems, **160** (2009), no. 11, 1643–1652.

[30] MUSIELA, J.: *Orlicz spaces and modular spaces*, Lecture notes in Math. **1034** (1983), pp. 216.

[31] MUSIELAK, J.—ORLICZ, W.: *On modular spaces*, Studia Mathematica **18** (1950), no. 1, 49–65.

[32] NAKANO, H.: *Modular Semi-ordered Spaces*. Tokyo Mathematical Book Series, Maruzen Co. Ltd, Tokyo, Japan, 1950.

[33] NARASIMMAN, P.—BODAGHI, A.: *Solution and stability of a mixed type functional equation*, Filomat, **31** (2017), no. 5, 1229–1239.

[34] PACHAIYAPPAN, D.—RAMDOSS, M.—LEE, J. R.—MIN, S. W.: *Multifarious functional equations in connection with three geometrical means*, J. Comput. Anal. Appl. **32** (2024), no. 1.

[35] RASSIAS, J. M.—RAVI, K.—SENTHIL KUMAR, B. V.: *Stabilities and instabilities of rational functional equations and Euler-Lagrange-Jensen (a, b)-sextic functional equations*, Math. Anal. Appl. Selected Topics (2018), Wiley Online Library, 341–400.

[36] RASSIAS, J. M.—THANDAPANI, E.—RAVI, K.—SENTHIL KUMAR, B. V.: *Functional equations and inequalities: solutions and stability results.* Series on Concrete and Applicable Mathematics, Vol. 21. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

[37] RASSIAS, T. M.: *On the stability of the linear mapping in banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. (2), 297–300.

[38] SAHA, P.—MONDAL, P.—CHQUDHURY, B. S. : *A fixed point approach to the hyers-ulam-rassias stability problem of pexiderized functional equation in modular spaces*, Tatra Mt. Math. Publ. **78** (2021), no. 1, 59–72.

[39] SAHA, P.—SAMANTA, T. K.—MONDAL, P.—CHOUDHURY, B. S.: *Stability of a two-variable pexiderized additive functional equation in intuitionistic fuzzy banach spaces: A fixed point approach*, Tamsui Oxford Journal of Information & Mathematical Sciences (TOJIMS) **33** (2019), no. 1.

[40] P. SAHA, T. K. SAMANTA, P. MONDAL, AND B. S. CHOUDHURY: *Stability of two variable pexiderized quadratic functional equation in intuitionistic fuzzy Banach spaces*, Proyecciones (Antofagasta), **38** (2019), no. 3, 447–468.

[41] SAYYARI, Y.—DEHGHANIAN, M.—C. PARK, C.: *System of bi-additive and bi-quadratic functional equations*, The Journal of Analysi, (2024), 1–12.

[42] SENTHIL KUMAR, B. V.—AL-SHAQSI, K.—DUTTA, H.: *Classical stabilities of multiplicative inverse difference and adjoint functional equations*, Adv. Difference Equ. **2020** (2020), no. 1, Article no. 215.

[43] SENTHIL KUMAR, B. V.—AL-SHAQSI, K.—SABARINATHAN, S.: *Dislocated quasi-metric stability of a multiplicative inverse functional equation*, J. Math. Comput. Sci. **24** (2022), 140–146.

[44] TAMILVANAN, K.—ALKHALDI, A. H.—JAKHAR, J.—CHUGH, R.— JAKHAR, J.—RASSIAS, J. M.: *Ulam stability results of functional equations in modular spaces and 2-Banach spaces*, Mathematics, **11** (2023), no. 2, Article no. 371.

[45] TAMRAKAR, E.—PATHAK, H. K.: *Fixed point results for multivalued contraction mappings in modular and non-Archimedean modular metric spaces*, J. Anal. **32** (2024), no. 1, 103–123.

[46] ULAM, S. M.: *Problems in Modern Mathematics*, Courier Corporation, Dover Publications, Inc. Mineola New York, 2004.

[47] WONGKUM, K.—CHAIPUNYA, P.—KUMAM, P. ET AL: *On the generalized Ulam-Hyers-Rassias stability of quadratic mappings in modular spaces without-conditions*, J. Funct. Spaces Appl. **2015** (2015), no. 1, paper no. 461719.

[48] XU, T. Z.—RASSIAS, M. J.: *A fixed point approach to the intuitionistic fuzzy stability of quintic and sextic functional equations*, Iran. J. Fuzzy Syst. **9** (2012), no. 5, 21–40.

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