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LAGUERRE-BESSEL TRANSFORM AND GENERALIZED LIPSCHITZ CLASSES

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ABSTRACT. The aim of this paper is to give necessary and sufficient conditions in terms of the Fourier Laguerre-Bessel transform $\mathcal{W}_{LB}f$ of the function f to ensure that f belongs to the generalized Lipschitz classes $\mathbf{H}_{\alpha}^{k}(\mathbf{X})$ and $\mathbf{h}_{\alpha}^{k}(\mathbf{X})$, where $\mathbf{X} = [0, +\infty) \times [0, +\infty)$.

1. Introduction and Preliminaries

Boas theorems are one of the classical theorems in harmonic analysis and approximation theory, as stated in [8, p. 45], if f is the function defined by $f(x) = \Sigma_{k\in\mathbb{Z}} c_k \exp(ikx)$, where $(c_k)_{k\in\mathbb{Z}}$ is a sequence of complex numbers such that

$$\Sigma_{k\in\mathbb{Z}}|c_k| < \infty.$$

In that case, f is a continuous 2π-periodic function ($f \in \mathcal{C}_{2\pi}$), and $(c_k)_{k\in\mathbb{Z}} \subset \mathbb{R}^+$ (that is, for cosine series with non-negative coefficients). R. P. Boas [8, p. 45] characterized $\mathrm{Lip}(\alpha)$ as the set of functions satisfying the Lipschitz conditions, and showed the sufficient conditions on the Fourier coefficients. Namely we have

Theorem 1.1 ([8]). *Let $\alpha \in (0, 1)$, then the following are equivalents:*

1) $f \in \mathrm{Lip}(\alpha)$.
2) $\sum_{k=n}^{\infty} c_k = O\left(n^{-\alpha}\right)$.

In [16, Theorem 2.5], using a generalized translation operator, the authors give some equivalence relations between behaviour of Fourier-Laguerre transform of a special kind and smoothness of functions.

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Our aim is to generalize these results for the Laguerre-Bessel transform, which is considered a very useful mathematical tool in many fields of signal processing such as signal modeling, system identification, process control, et al.

Given $\alpha \geq 0$. The harmonic analysis on $\mathbf{X} = [0, +\infty) \times [0, +\infty)$ is generated by the following partial differential operators

$$\begin{cases} \mathcal{D}_{1,\alpha} = \frac{\partial^2}{\partial t^2} + \frac{2\alpha}{t}\frac{\partial}{\partial t}, \\ \mathcal{D}_{2,\alpha} = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\mathcal{D}_{1,\alpha}, \end{cases}$$

where $(x, t) \in \mathbf{X}$.

For $(\lambda, m) \in [0, +\infty) \times \mathbb{N}$, the initial value problem

$$\begin{cases} \mathcal{D}_{1,\alpha}u = -\lambda^2 u, \\ \mathcal{D}_{2,\alpha}u = -4\lambda(m + \frac{\alpha+1}{2})u, \\ u(0,0) = 1, \quad \frac{\partial u}{\partial x}(0,0) = \frac{\partial u}{\partial t}(0,0) = 0 \end{cases}$$

has a unique solution $\psi_{\lambda,m}$ given by

$$\psi_{\lambda,m}(x,t) = j_{\alpha-\frac{1}{2}}(\lambda t)\mathfrak{L}_m^\alpha(\lambda x^2), \quad \forall (x,t) \in \mathbf{X}, \tag{1}$$

where $\mathfrak{L}_m^\alpha$ is the Laguerre function defined on $\mathbb{R}_+$ by

$$\mathfrak{L}_m^\alpha(x) = e^{-\frac{x}{2}}\frac{L_m^\alpha(x)}{L_m^\alpha(0)}, \tag{2}$$

L_m^α being the Laguerre polynomial of degree m and order α, given by

$$L_m^\alpha(x) = \sum_{k=0}^{m}(-1)^k\frac{\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)}\frac{1}{k!(m-k)!}x^k, \tag{3}$$

and j_α is the normalized Bessel function given by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{x}{2}\right)^{2k}. \tag{4}$$

Lemma 1.2 ([1]). *For all $(\lambda, m) \in [0, +\infty) \times \mathbb{N}$, the functions $\psi_{\lambda,m}$ are infinitely differentiable on $\mathbb{R}^2$, even with respect to each variable, and we have*

$$sup_{(x,t)\in\mathbf{X}}|\psi_{\lambda,m}(x,t)| = 1. \tag{5}$$

We denote (see [4]):

- The homogeneous norm on $\mathbf{X}$ defined by

$$|\ x,t\ | = |\ (x,t)\ |_{\mathbf{X}} = \left(x^4 + 4t^2\right)^{\frac{1}{4}}, \quad \text{for all } (x,t) \in \mathbf{X}.$$

- The quasi-norm on $[0,+\infty)\times\mathbb{N}$ defined by

$$|(\lambda,m)| = |\ (\lambda,m)\ |_{[0,+\infty)\times\mathbb{N}} = 4\lambda\left(m+\frac{\alpha+1}{2}\right),$$

for all $(\lambda,m)\in[0,+\infty)\times\mathbb{N}$.

- $\mathbb{B}_r$ the ball centered 0 and of radius r defined by

$$\mathbb{B}_r=\{(\lambda,m)\in[0,+\infty)\times\mathbb{N};|(\lambda,m)|<r\}\quad\text{and}\quad\mathbb{B}_r^c=\big([0,+\infty)\times\mathbb{N}\big)\backslash\mathbb{B}_r.$$

- $L^p_\alpha(\mathbf{X}), p\in[1,+\infty]$, the spaces of measurable functions on $\mathbf{X}$ such that

$$\|f\|_{p,\alpha}=\left[\int\limits_{\mathbf{X}}|f(x,t)|^p dm_\alpha(x,t)\right]^{\frac{1}{p}}<+\infty,\quad\text{if } p\in[1,+\infty),$$

$$\|f\|_{\infty,\alpha}=\operatorname{esssup}_{(x,t)\in\mathbf{X}}|f(x,t)|<+\infty,$$

where dm_α is the positive measure defined on $\mathbf{X}$ by

$$dm_\alpha(x,t)=\frac{x^{2\alpha+1}t^{2\alpha}}{\pi\Gamma(\alpha+1)}dxdt.$$

- $L^p_{\gamma_\alpha}\big([0,+\infty)\times\mathbb{N}\big)$, $p\in[1,+\infty]$, the spaces of measurable functions on $[0,+\infty)\times\mathbb{N}$ such that

$$\|g\|_{\gamma_{\alpha,p}}=\left[\int\limits_{[0,+\infty)\times\mathbb{N}}|g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right]^{\frac{1}{p}}<+\infty,\quad\text{if } p\in[1,+\infty),$$

$$\|g\|_{\gamma_{\alpha,\infty}}=\operatorname{esssup}_{(\lambda,m)\in[0,+\infty)\times\mathbb{N}}|g(\lambda,m)|<+\infty,$$

where γ_α is the positive measure defined on $[0,+\infty)\times\mathbb{N}$ by

$$\int\limits_{[0,+\infty)\times\mathbb{N}}g(\lambda,m)d\gamma_\alpha(\lambda,m)=\frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)}\sum_{m=0}^{\infty}L_m^\alpha(0)\int\limits_0^{+\infty}g(\lambda,m)\lambda^{3\alpha+1}d\lambda.$$

Definition 1.3. [1]

1) The translation operators $T^{(\alpha)}_{(x,t)}$, $(x,t)\in\mathbf{X}$, are defined for a continuous function f on $\mathbf{X}$ by

$$T^{(\alpha)}_{(x,t)}f(y,s)=\begin{cases}\frac{1}{4\pi}\sum_{i,j=0}^{1}\int\limits_0^\pi f\left(\Delta_\theta(x,y),Y+(-1)^it+(-1)^js\right)d\theta & \text{if}\quad\alpha=0,\\ b_\alpha\int\limits_{[0,\pi]^3}f\big(\Delta_\theta(x,y),\Delta_\theta(x,y)\xi\big)\quad d\mu_\alpha(\xi,\psi,\theta) & \text{if}\quad\alpha>0,\end{cases}$$

where

$$\Delta_\theta(x,y)=\sqrt{x^2+y^2+2xy\cos\theta}, b_\alpha=\frac{(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}{\pi^{\frac{3}{4}}\Gamma(\alpha)}, \quad Y=xy\sin\theta$$

and

$$d\mu_\alpha(\xi,\psi,\theta)=(\sin\xi)^{2\alpha-1}(\sin\psi)^{2\alpha-1}(\sin\theta)^{2\alpha}d\xi d\psi d\theta.$$

2) The convolution product of two continuous functions f, g on $\mathbf{X}$ with compact support is defined by

$$(f*g)(x,t) = \int_{\mathbf{X}} T^{(\alpha)}_{(x,t)}f(y,s)g(y,s)dm_\alpha(y,s), \quad (x,t)\in\mathbf{X}.$$

The Fourier Laguerre-Bessel transform of a function in $L^1_\alpha(\mathbf{X})$ is given by

$$\mathcal{W}_{LB}f(\lambda,m) = \int_{\mathbf{X}} f(x,t)\psi_{\lambda,m}(x,t)dm_\alpha(x,t), \quad (\lambda,m)\in[0,+\infty)\times\mathbb{N}.$$

From [1], it is well-known that the Fourier Laguerre-Bessel transform can be inverted to

$$\mathcal{W}^{-1}_{LB}f(x,t) = \int_{[0,+\infty)\times\mathbb{N}} f(\lambda,m)\psi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \quad (x,t)\in\mathbf{X}.$$

It is well-known (see [1], [3], [4]) that the Fourier Laguerre-Bessel transform $\mathcal{W}_{LB}$ satisfies the following properties.

- We have the following Plancherel formula:

$$\|f\|_{2,\alpha}=\|\mathcal{W}_{LB}f\|_{\gamma_\alpha,2}, \quad \text{for } f\in L^1_\alpha(\mathbf{X})\cap L^2_\alpha(\mathbf{X}).$$

- Let f be in $L^1_\alpha(\mathbf{X})$ such that $\mathcal{W}_{LB}(f)\in L^1_{\gamma_\alpha}\big([0,+\infty)\times\mathbb{N}\big)$, then we have the inversion formula

$$f(x,t) = \int_{[0,+\infty)\times\mathbb{N}} \mathcal{W}_{LB}f(\lambda,m)\psi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \quad \text{a.e. } \mathbf{X}.$$

- For $f\in L^1_\alpha(\mathbf{X})$, we have

$$\mathcal{W}_{LB}(T^{(\alpha)}_{(x,t)}f)(\lambda,m) = \psi_{\lambda,m}(x,t)\mathcal{W}_{LB}(f)(\lambda,m), \tag{6}$$

  where $(x,t)\in\mathbf{X}, (\lambda,m)\in[0,+\infty)\times\mathbb{N}$.

- For $f\in L^p_\alpha(\mathbf{X}), p\in[1,+\infty]$, we have $T^{(\alpha)}_{(x,t)}f\in L^p_\alpha(\mathbf{X})$ and

$$\|T^{(\alpha)}_{(x,t)}f\|_{p,\alpha}\le\|f\|_{p,\alpha}.$$

We now define finite differences of order $k \in \mathbb{N}$ and step $(x,t) \in \mathbf{X}$ by

$$\Delta^k_{(x,t)} f(y,s) = \left(T^{(\alpha)}_{(x,t)} - I\right)^k f(y,s),$$

where I denotes the unit operator.

Remark 1. For all $k \in \mathbb{N}$, we have

$$\Delta^k_{(x,t)} f(y,s) = \sum_{0 \le i \le k} (-1)^{k-i} \binom{k}{i} \left(T^{(\alpha)}_{(x,t)}\right)^i f(y,s).$$

Lemma 1.4. *For a fixed $(x,t) \in \mathbf{X}$, we have*

$$\mathcal{W}_{LB}(\Delta^k_{(x,t)} f)(\lambda,m) = \big(\psi_{\lambda,m}(x,t) - 1\big)^k \mathcal{W}_{LB}(f)(\lambda,m). \tag{7}$$

P r o o f. Based on (6), we have

$$\mathcal{W}_{LB}\left(T^{(\alpha)}_{(x,t)} f\right)(\lambda,m) = \psi_{\lambda,m}(x,t) \mathcal{W}_{LB}(f)(\lambda,m).$$

Then, by recurrence on i, we get

$$\mathcal{W}_{LB}\left(\left(T^{(\alpha)}_{(x,t)}\right)^i f\right)(\lambda,m) = \big(\psi_{\lambda,m}(x,t)\big)^i \mathcal{W}_{LB}(f)(\lambda,m).$$

Hence

$$\begin{aligned}\mathcal{W}_{LB}\big(\Delta^k_{(x,t)} f\big)(\lambda,m) &= \sum_{0 \le i \le k} (-1)^{k-i} \binom{k}{i} \big(\psi_{\lambda,m}(x,t)\big)^i \mathcal{W}_{LB}(f)(\lambda,m), \\ &= \left(\sum_{0 \le i \le k} (-1)^{k-i} \binom{k}{i} \big(\psi_{\lambda,m}(x,t)\big)^i\right) \mathcal{W}_{LB}(f)(\lambda,m).\end{aligned}$$

Using Newton's formula, we obtain (7). □

Throughout this paper, C denotes a positive constant which can differ from one line to another.

2. Auxiliary results

We start with an auxiliary result interesting in itself. Let g be a non-negative, measurable function defined on $[0,+\infty) \times \mathbb{N}$.

Lemma 2.1. *Let $r > 0$.*

1) *Furthermore, the behaviour of the kernel $\psi_{\lambda,m}$ in 0 could be expressed as follows:*

$$\psi_{\lambda,m}(x,t) = 1 - \frac{(\lambda t)^2}{4(\alpha + \frac{1}{2})} - \frac{|(\lambda,m)|x^2}{4(\alpha+1)} + \kappa_{\alpha,m} \lambda^2 x^4 + o\big(\lambda^2 \mid x,t \mid^4\big), \tag{8}$$

where

$$\kappa_{\alpha,m} = \frac{m^2}{2(\alpha+1)(\alpha+2)} + \frac{m}{2(\alpha+2)} + \frac{1}{8}.$$

2) *There exists* $C > 0$ *such that for all* $(\lambda, m) \in \mathbb{B}_r$ *and* $(x, t) \in \mathbf{X}$,

$$| \psi_{\lambda,m}(x,t) - 1 | \leq C|(\lambda, m)| \, | x, t | . \tag{9}$$

P r o o f.

1) [6, 7] From relations (2) and (3), we have

$$\mathfrak{L}_m^{\alpha}(x) = 1 - \frac{m + \frac{\alpha+1}{2}}{\alpha+1} x + \left(\frac{m^2}{2(\alpha+1)(\alpha+2)} + \frac{m}{2(\alpha+2)} + \frac{1}{8} \right) x^2 + o\big(x^2\big). \tag{10}$$

Then, (8) could be easily deduced using relations (1), (10) and the behaviour of the normalized Bessel function in 0 which states

$$j_\alpha(u) = 1 - \frac{1}{4(\alpha+1)} u^2 + o(u^2).$$

2) Denote $r = \frac{\eta}{|x,t|^2}$. Using relation (8), there exist $C > 0$ and $\eta > 0$ such that for all $(x, t) \in \mathbf{X}$,

$$|(\lambda, m)||x, t|^2 < \eta \Rightarrow |\psi_{\lambda,m}(x,t) - 1|^2 \leq C|(\lambda, m)|^2 |x, t|^2. \tag{11}$$

Then, for all $(\lambda, m) \in \mathbb{B}_{\frac{\eta}{|x,t|^2}}$, we have

$$|\psi_{\lambda,m}(x,t) - 1| \leq \sqrt{C}|(\lambda, m)||x, t|. \qquad \square$$

Lemma 2.2. *Let* $k \in \mathbb{N}$. *Then*

1) *If* $|(\lambda, m)|^k g(\lambda, m) \in L^1_{\gamma_\alpha}\big([0, +\infty) \times \mathbb{N}\big) \cap \mathrm{Loc}\big([0, +\infty) \times \mathbb{N}\big)$, $0 < \alpha \leq k$ *and*

$$\int\limits_{\mathbb{B}_r} |(\lambda, m)|^k g(\lambda, m) d\gamma_\alpha(\lambda, m) = O\big(r^{k-\alpha}\big) \quad \text{for all } r > 0, \tag{12}$$

then

$$g \cdot \chi_{\mathbb{B}_r^c} \in L^1_{\gamma_\alpha}\big([0, +\infty) \times \mathbb{N}\big),$$

where $\chi_{\mathbb{B}_r^c}(\cdot)$ *is the characteristic function of* $\mathbb{B}_r^c$, *and*

$$\int\limits_{\mathbb{B}_r^c} g(\lambda, m) d\gamma_\alpha(\lambda, m) = O\big(r^{-\alpha}\big) \quad \text{for all } r > 0. \tag{13}$$

2) *Inversely, if* $0 \leq \alpha < k$ *and* (13) *holds, then* (12) *also holds.*

P r o o f.

1) By (12), there exists a constant $C > 0$ such that

$$\int\limits_{\mathbb{B}_{2^{i+1}}\setminus\mathbb{B}_{2^i}} |(\lambda,m)|^k g(\lambda,m)d\gamma_\alpha(\lambda,m) \le \int\limits_{\mathbb{B}_{2^{i+1}}} |(\lambda,m)|^k g(\lambda,m)d\gamma_\alpha(\lambda,m),$$
$$\le C2^{(k-\alpha)(i+1)} \quad \text{for all } i \in \mathbb{Z}.$$

We have

$$2^{ki}\int\limits_{\mathbb{B}_{2^{i+1}}\setminus\mathbb{B}_{2^i}} g(\lambda,m)d\gamma_\alpha(\lambda,m) \le \int\limits_{\mathbb{B}_{2^{i+1}}\setminus\mathbb{B}_{2^i}} |(\lambda,m)|^k g(\lambda,m)d\gamma_\alpha(\lambda,m)$$
$$\le C2^{(k-\alpha)(i+1)},$$

whence it follows that

$$\int\limits_{\mathbb{B}_{2^{i+1}}\setminus\mathbb{B}_{2^i}} g(\lambda,m)d\gamma_\alpha(\lambda,m) \le C2^{(k-\alpha)(-i\alpha)},$$

then

$$\int\limits_{\mathbb{B}^c_{2^i}} g(\lambda,m)d\gamma_\alpha(\lambda,m) \le C2^{k-\alpha}(2^{-\alpha i} + 2^{-\alpha(i+1)} + \cdots)$$
$$= O(2^{-\alpha i}),$$

that is equivalent to (13) in the case when $r = 2^i, i \in \mathbb{Z}$. When $0 < r < +\infty$, let $i \in \mathbb{Z}$ such that $2^i \le r < 2^{i+1}$ for some $i \in \mathbb{Z}$. It follows that

$$\int\limits_{\mathbb{B}^c_r} g(\lambda,m)d\gamma_\alpha(\lambda,m) \le \int\limits_{\mathbb{B}^c_{2^i}} g(\lambda,m)d\gamma_\alpha(\lambda,m)$$
$$\le C2^{-\alpha i}$$
$$= 2^\alpha C2^{-\alpha(i+1)} \le 2^\alpha Cr^{-\alpha}.$$

This proves (13) in the general case.

2) By (13), there exists a constant C such that

$$\int\limits_{\mathbb{B}_{2^i}\setminus\mathbb{B}_{2^{i-1}}} g(\lambda,m)d\gamma_\alpha(\lambda,m) \le \int\limits_{\mathbb{B}^c_{2^{i-1}}} g(\lambda,m)d\gamma_\alpha(\lambda,m)$$
$$\le C2^{-\alpha(i-1)} \qquad \text{for all } i \in \mathbb{Z}.$$

Then, we have

$$\begin{aligned}\int\limits_{\mathbb{B}_{2^i}\setminus\mathbb{B}_{2^{i-1}}} |(\lambda,m)|^k g(\lambda,m)d\gamma_\alpha(\lambda,m) &\le 2^{ik}\int\limits_{\mathbb{B}^c_{2^{i-1}}} g(\lambda,m)d\gamma_\alpha(\lambda,m),\\ &\le C2^{ki}2^{-\alpha(i-1)},\\ &\le C2^{\alpha}2^{(k-\alpha)i},\end{aligned}$$

whence it follows that

$$\begin{aligned}\int\limits_{\mathbb{B}_{2^i}} |(\lambda,m)|^k g(\lambda,m)d\gamma_\alpha(\lambda,m) &\le C2^{\alpha}(2^{(k-\alpha)i}+2^{(k-\alpha)(i-1)}+\cdots),\\ &= O(2^{i(k-\alpha)}),\end{aligned} \tag{14}$$

which is equivalent to (12) in the case $r=2^i, i\in\mathbb{Z}$.

When $0<r<+\infty$, let $i\in\mathbb{Z}$ such that $2^i\le r<2^{i+1}$. It follows

$$\begin{aligned}\int\limits_{\mathbb{B}_r} |(\lambda,m)|^k g(\lambda,m)d\gamma_\alpha(\lambda,m) &\le \int\limits_{\mathbb{B}_{2^{i+1}}} |(\lambda,m)|^k g(\lambda,m)d\gamma_\alpha(\lambda,m)\\ &\le C2^{(i+1)(k-\alpha)} \le C2^{k-\alpha}2^{i(k-\alpha)}\\ &\le C2^{k-\alpha}r^{k-\alpha}.\end{aligned}$$

This proves (12) in the general case. □

Lemma 2.3. *Let $k\in\mathbb{N}$. Then*

1) *If*

$$|(\lambda,m)|^k g(\lambda,m)\in L^1_{\gamma_\alpha}\big([0,+\infty)\times\mathbb{N}\big)\cap \mathrm{Loc}\big([0,+\infty)\times\mathbb{N}\big),\quad 0<\alpha<k$$

and

$$\int\limits_{\mathbb{B}_r} |(\lambda,m)|^k g(\lambda,m)d\gamma_\alpha(\lambda,m)=o\big(r^{k-\alpha}\big)\quad \textit{whenever } r\to+\infty, \tag{15}$$

then $g\cdot\chi_{\mathbb{B}^c_r}\in L^1_{\gamma_\alpha}\big([0,+\infty)\times\mathbb{N}\big)$ for large r, where $\chi_{\mathbb{B}^c_r}(\cdot)$ is the characteristic function of $\mathbb{B}^c_r$, and

$$\int\limits_{\mathbb{B}^c_r} g(\lambda,m)d\gamma_\alpha(\lambda,m)=o(r^{-\alpha})\quad \textit{as } r\to+\infty. \tag{16}$$

2) *Inversely, if $0<\alpha\le k$ and both conditions* (13) *and* (16) *hold, then* (15) *also holds.*

P r o o f.

(1) By (15), for every $\varepsilon > 0$, there exists $i_0 \in \mathbb{Z}$ such that

$$\int\limits_{\mathbb{B}_{2^{i+1}} \setminus \mathbb{B}_{2^i}} |(\lambda, m)|^k g(\lambda, m) d\gamma_\alpha(\lambda, m) \le \int\limits_{\mathbb{B}_{2^{i+1}}} |(\lambda, m)|^k g(\lambda, m) d\gamma_\alpha(\lambda, m)$$
$$\le \varepsilon 2^{(k-\alpha)(i+1)} \quad \text{whenever} \quad i \ge i_0.$$

Let $i \ge i_0$. Clearly, we have

$$2^{ki} \int\limits_{\mathbb{B}_{2^{i+1}} \setminus \mathbb{B}_{2^i}} g(\lambda, m) d\gamma_\alpha(\lambda, m) \le \int\limits_{\mathbb{B}_{2^{i+1}} \setminus \mathbb{B}_{2^i}} |(\lambda, m)|^k g(\lambda, m) d\gamma_\alpha(\lambda, m),$$
$$\le \varepsilon 2^{(k-\alpha)(i+1)} \quad \text{whenever} \quad i \ge i_0.$$

whence it follows that

$$\int\limits_{\mathbb{B}_{2^{i+1}} \setminus \mathbb{B}_{2^i}} g(\lambda, m) d\gamma_\alpha(\lambda, m) \le \varepsilon 2^{(k-\alpha)} 2^{-i\alpha},$$

then

$$\int\limits_{\mathbb{B}^c_{2^i}} g(\lambda, m) d\gamma_\alpha(\lambda, m) \le \varepsilon 2^{k-\alpha} \left(2^{-\alpha i} + 2^{-\alpha(i+1)} + \cdots\right)$$
$$= c_9 \varepsilon 2^{-\alpha i}.$$

Since $\varepsilon > 0$ is arbitrary, then

$$\int\limits_{\mathbb{B}^c_{2^i}} g(\lambda, m) d\gamma_\alpha(\lambda, m) = o\big((2^i)^{-\alpha}\big).$$

(2) By (16), for every $\varepsilon > 0$, there exists $i_0 \in \mathbb{Z}$ such that

$$\int\limits_{\mathbb{B}_{2^{i+1}} \setminus \mathbb{B}_{2^i}} g(\lambda, m) d\gamma_\alpha(\lambda, m) \le \int\limits_{\mathbb{B}^c_{2^i}} g(\lambda, m) d\gamma_\alpha(\lambda, m)$$
$$\le \varepsilon 2^{-i\alpha} \quad \text{whenever} \quad i \ge i_0.$$

Analogously to (14), we have

$$\int\limits_{\mathbb{B}_{2^{i+1}} \setminus \mathbb{B}_{2^i}} |(\lambda, m)|^k g(\lambda, m) d\gamma_\alpha(\lambda, m) \le \varepsilon 2^{(k-\alpha)i},$$

and

$$\int\limits_{\mathbb{B}_{2^i} \setminus \mathbb{B}_{2^{i_0}}} |(\lambda, m)|^k g(\lambda, m) d\gamma_\alpha(\lambda, m) \le C\varepsilon \frac{2^\alpha}{1 - 2^{-\alpha}} 2^{(k-\alpha)i}, \quad \text{for every } i > i_0.$$

On the other hand, by condition (13), we have

$$\int\limits_{\mathbb{B}_{2^{i_0}}} |(\lambda,m)|^k g(\lambda,m) d\gamma_\alpha(\lambda,m) \leq C 2^{(k-\alpha)i_0},$$

then

$$\begin{aligned}\int\limits_{\mathbb{B}_{2^{i}}} |(\lambda,m)|^k g(\lambda,m) d\gamma_\alpha(\lambda,m) &\leq C 2^{(k-\alpha)i_0} + C\varepsilon \frac{2^\alpha}{1-2^{-\alpha}} 2^{(k-\alpha)i}, \\ &\leq C 2^{(k-\alpha)i_0} + C\varepsilon \frac{1}{2^\alpha - 1} 2^{(k-\alpha)i}.\end{aligned}$$

Thus,

$$\int\limits_{\mathbb{B}_{2^{i}}} |(\lambda,m)|^k g(\lambda,m) d\gamma_\alpha(\lambda,m) \leq \varepsilon(1 + \frac{C}{2^\alpha - 1}) 2^{(k-\alpha)i}, \quad \text{for large } i.$$

Since ε is arbitrary, then (15) holds.
The proof is completed. □

3. Main results

In this section, we present the main results of this paper. First, we need to define the generalized Lipschitz classes $\mathbf{H}^k_\alpha(\mathbf{X})$ and $\mathbf{h}^k_\alpha(\mathbf{X})$.

Definition 3.1. A function $f : \mathbf{X} \to \mathbb{C}$ is said to belong to the generalized Lipschitz classes $\mathbf{H}^k_\alpha(\mathbf{X})$ for $\alpha > 0$ if

$$\left|\Delta^k_{d_h(x,t)} f(y,s)\right| = O(h^\alpha), \tag{17}$$

and is said to belong to the generalized Lipschitz classes $\mathbf{h}^k_\alpha(\mathbf{X})$ for $\alpha > 0$ if

$$\left|\Delta^k_{d_h(x,t)} f(y,s)\right| = o(h^\alpha) \quad \text{as } h \to 0, \tag{18}$$

where $d_s(x,t)$ is the dilation of $(x,t) \in \mathbf{X}$ given by $d_s(x,t) = (sx, s^2 t)$.

For $\alpha > 0$.

- The spaces $\mathbf{H}^1_\alpha(\mathbf{X})$ and $\mathbf{h}^1_\alpha(\mathbf{X})$, are called the Lipschitz class $Lip(\alpha)$ and little Lipschitz class $lip(\alpha)$, respectively.

- The spaces $\mathbf{H}^2_\alpha(\mathbf{X})$ and $\mathbf{h}^2_\alpha(\mathbf{X})$, are called the Zygmund class $Zyg(\alpha)$ and little Zygmund class $zyg(\alpha)$, respectively.

Theorem 3.2. *Let $f : \mathbf{X} \to \mathbb{C}, k \in \mathbb{N}$ and $0 < \alpha \leq k$. Suppose that $f \in L^1_\alpha(\mathbf{X})$. If*

$$\int_{\mathbb{B}_r} |(\lambda, m)|^k \mid \mathcal{W}_{LB} f(\lambda, m) \mid d\gamma_\alpha(\lambda, m) = O\left(r^{k-\alpha}\right) \quad \text{for all } r > 0, \tag{19}$$

then

$$\mathcal{W}_{LB}(f) \in L^1_{\gamma_\alpha}\big([0, +\infty) \times \mathbb{N}\big) \quad \text{and} \quad f \in \mathbf{H}^k_\alpha(\mathbf{X}).$$

Remark 2. From Lemma 2.2, statement (19) refers to

$$\int_{\mathbb{B}^c_r} \mid \mathcal{W}_{LB} f(\lambda, m) \mid d\gamma_\alpha(\lambda, m) = O\left(r^{-\alpha}\right).$$

Then,

$$\mathcal{W}_{LB}(f) \in L^1_{\gamma_\alpha}\big([0, +\infty) \times \mathbb{N}\big).$$

P r o o f. Let $(x, t), (y, s) \in \mathbf{X}$ and $r = \frac{1}{h|x,t|} > 0$.

$$\begin{aligned}
&\left|\Delta^k_{d_h(x,t)} f(y, s)\right| \\
&= \left| \int_{[0,+\infty)\times\mathbb{N}} \Big(\psi_{\lambda,m}\big(d_h(x, t)\big) - 1\Big)^k \mathcal{W}_{LB}(f)(\lambda, m)\psi_{\lambda,m}(y, s) d\gamma_\alpha(\lambda, m) \right| \\
&\leq I_1 + I_2, \qquad (20)
\end{aligned}$$

where

$$I_1 = \int_{\mathbb{B}_r} \left| \Big(\psi_{\lambda,m}\big(d_h(x, t)\big) - 1\Big)^k \mathcal{W}_{LB}(f)(\lambda, m)\psi_{\lambda,m}(y, s) \right| d\gamma_\alpha(\lambda, m),$$

$$I_2 = \int_{\mathbb{B}^c_r} \left| \Big(\psi_{\lambda,m}\big(d_h(x, t)\big) - 1\Big)^k \mathcal{W}_{LB}(f)(\lambda, m)\psi_{\lambda,m}(y, s) \right| d\gamma_\alpha(\lambda, m),$$

By Lemma 2.1, we get

$$\begin{aligned}
I_1 &\leq Ch^k \mid x, t \mid^k \int_{\mathbb{B}_r} |(\lambda, m)|^k \mid \mathcal{W}_{LB}(f)(\lambda, m) \mid d\gamma_\alpha(\lambda, m), \\
&= h^k O\left(\left(\frac{1}{h \mid x, t \mid}\right)^{k-\alpha}\right), \\
&= O\left(h^\alpha\right). \qquad (21)
\end{aligned}$$

On the other hand, by Lemma 1.2, we have

$$I_2 \leq C \int\limits_{\mathbb{B}_r^c} \mid \mathcal{W}_{LB}(f)(\lambda, m) \mid d\gamma_\alpha(\lambda, m).$$

By (19) and Lemma 2.2, we get

$$\begin{aligned} I_2 &= O\left(\left(\frac{1}{h \mid x, t \mid}\right)^{-\alpha}\right), \\ &= O(h^\alpha). \end{aligned} \tag{22}$$

Combining (21) and (22), we get $f \in \mathbf{H}_\alpha^k(\mathbf{X})$. □

Theorem 3.3. *Both statements in Theorem 3.2 remain valid if the right-hand side in* (19) *is replaced by*

$$o\left(r^{k-\alpha}\right) \quad as \ \ r \to +\infty, \tag{23}$$

and $f \in \mathbf{H}_\alpha^k(\mathbf{X})$ *is replaced by* $f \in \mathbf{h}_\alpha^k(\mathbf{X})$*, provided that* $0 < \alpha < k$.

Remark 3. From Lemma 2.3, statement (23) refers to

$$\int\limits_{\mathbb{B}_r^c} \mid \mathcal{W}_{LB} f(\lambda, m) \mid d\gamma_\alpha(\lambda, m) = o(r^{-\alpha}) \quad \text{as} \ \ r \to +\infty.$$

Then,

$$\mathcal{W}_{LB}(f) \in L^1_{\gamma_\alpha}\big([0, +\infty) \times \mathbb{N}\big).$$

P r o o f. By (23), for every $\varepsilon > 0$ there exists $r_0 > 0$ such that

$$\int\limits_{\mathbb{B}_r} |(\lambda, m)|^k \mid \mathcal{W}_{LB} f(\lambda, m) \mid d\gamma_\alpha(\lambda, m) \leq \varepsilon \left(r^{k-\alpha}\right) \quad \text{whenever} \ \ r > r_0. \tag{24}$$

Again, we start with the estimate (20). It follows from Lemma 2.1 and (24) with $r = \frac{1}{h}$ that

$$I_1 \leq h^k \varepsilon \left(\frac{1}{h}\right)^{k-\alpha} = \varepsilon h^\alpha \quad \text{whenever} \ \ 0 < h < \frac{1}{r_0}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$I_1 = o(h^\alpha) \quad \text{as} \ \ h \to 0. \tag{25}$$

On the other hand, by (23), Lemma 2.3 and Lemma 2.1, we obtain

$$I_2 \leq \varepsilon \left(\frac{1}{h}\right)^{-\alpha} = \varepsilon h^\alpha \quad \text{whenever} \ \ 0 < h < \frac{1}{r_0}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$I_2 = o(h^\alpha) \quad \text{as} \ \ h \to 0. \tag{26}$$

Combining (25) and (26), we get $f \in \mathbf{h}_\alpha^k(\mathbf{X})$. □

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