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TONK, SYNTAX, SEMANTICS, PRAGMATICS

Abstract. In 1960, A. N. Prior proposed a binary sentential functor defined by inferential rule (introduction) $A \vdash A\,\mathrm{tonk}\,B$, and (elimination) $A\,\mathrm{tonk}\,B \vdash A$. Later, this idea was discussed by several authors, including in the context of the question of whether the above rule defines this functor in a sufficient manner. This paper shows that if we assume the standard matrix (truth-tables) characterization of classical sentential functors, no valuation agrees with rules generating the sense of tonk. Moreover, these inferential prescriptions are at odds with the principle that if premises are true, the conclusion has to be such as well. A new solution is proposed. It consists in considering tonk-rules via so-called rejection consequence operation, that is, a dual with respect to the standard Cn. The general moral stemming from the proposed analysis says that inferentialism, the view that inference rules are purely syntactical, is dubious, because logic has its basis also in semantic presuppositions as well as in pragmatic ones.

Keywords: sentential logic, matrices, valuation, consequence, inferentialism.

How many classical sentential (connectives) functors are there? – "classical" refers to two-valued, functionally complete (that is such that all connectives are definable in it) sentential calculus. The matrix (that is, generated by truth-tables) characterization provides a simple answer. Let A and B denote arbitrary formulas of sentential calculus (in particular, they can be sentential variables). We have four valuations (the expression $v(A)$ means "the value of the formula A is equal to")

(i) $\{1, 1\}; v(A) = 1, v(B) = 1;$
(ii) $\{0, 1\}; v(A) = 0, v(B) = 1;$
(iii) $\{1, 0\}; v(A) = 1, v(B) = 0;$
(iv) $\{0, 0\}; v(A) = 0, v(B) = 0,$

where 1 and 0 refer to logical values, truth and falsity, respectively, although they can be two arbitrary objects such as that for any A, it is not so that $v(A) = 1$ and $v(A) = 0$, and $v(A) = 1$ or $v(A) = 0$ – it implies that 1 and 0 are different. If the chosen metalanguage contains the negation-sign and we accept the convention that 1 can be read as "true sentence", but 0 – "false sentence", we accept equalities $\neg 1 = 0$, $\neg 0 = 1$ (to simplify, negation is denoted by the same symbol in the language and the metalanguage). The above comments on the meaning of 1 and 0 agree with the bivalence principle (or even express it). Although, as the above remarks show, the substantial interpretation of the symbol referring to logical values can be different, I will understand 1 and 0 as semantic concepts, that is, truth and falsity in the intuitive sense.

The valuations (i)–(iv) induce 16 bivalent sentential connectives, that is, expressions having the categorial index p/pp, that is (I limit this to the case with sentential variables) forming a sentence from two sentences. They are defined by the following four-termed sequences of logical values:

(a) $\{0, 0, 0, 0\}$;
(b) $\{1, 0, 0, 0\}$;
(c) $\{0, 1, 0, 0\}$;
(d) $\{0, 0, 1, 0\}$;
(e) $\{0, 0, 0, 1\}$;
(f) $\{1, 1, 0, 0\}$;
(g) $\{1, 0, 1, 0\}$;
(h) $\{1, 0, 0, 1\}$;
(i) $\{0, 1, 1, 0\}$;
(j) $\{0, 0, 1, 1\}$;
(k) $\{0, 1, 0, 1\}$;
(l) $\{1, 1, 1, 0\}$;
(m) $\{1, 1, 0, 1\}$;
(n) $\{1, 0, 1, 1\}$;
(o) $\{0, 1, 1, 1\}$;
(p) $\{1, 1, 1, 1\}$.

We can add 4 unary functors, defined by sequences

(q) $\{0, 0\}$;
(r) $\{0, 1\}$;
(s) $\{1, 0\}$;
(t) $\{1, 1\}$.

It is easily seen that the points (a)–(t) are counterparts of well-known truth-tables describing connectives occurring in classical sentential logic.

These combinatorial settings together with the fact that every n-ary connective ($n > 2$) is reducible to a binary one, show that there are exactly 20 different two-argument functors of sentential calculus. As is known (I list three cases only, just those which are relevant for further analysis), (b) expresses conjunction ($A \wedge B$), (l) – disjunction ($A \vee B$) and (r) – negation ($\neg A$);

introducing zeroary connectives **1** (the constant truth) and **0** (the constant falsity) does not increase the global number of functors because the former can be defined as $A \vee \neg A$ ($v(A \vee \neg A) = 1$, for any valuation) but the latter as $A \wedge \neg A$ ($v(A \wedge \neg A) = 0$, for any valuation). Thus, if one would like to introduce a new connective to the language of sentential calculus, it has to be reducible to one of (a)–(t). This issue is relevant for modal logic. Jan Łukasiewicz proved (see [8], [9]) the possibility functor, satisfying the condition $\Diamond A \Leftrightarrow \Diamond \neg A$ (A is possible if and only if $\neg A$ is possible) cannot be defined in two-valued sentential calculus – this observation was the starting point for his attempts to build a three-valued logic. David Lewis (see [7]) demonstrated that conditional probability $P(A/B)$ (B is probable with respect to conditions A) cannot be reduced to the implication $A \Rightarrow B$ – a similar situation occurs in the case of temporal operators (see [15]). Consequently, modal systems in the wide understanding, that is, covering alethic, epistemic, deontic, probabilistic, temporal, etc. modalities, based on classical sentential logic, must be constructed by adding new axioms, for example the formula $\Box A \Rightarrow \Diamond A$ (if A is necessary, then A is possible). There is still one additional metalogical reason for the combinatorial completeness of the list (a)–(t). Assume that we admit a new functor, non-reducible to the cases (a)–(t). Denote this connective by the symbol #. It means that there is a theorem of sentential logic, that is, a formula A such that it contains #, possibly other connectives and sentential variables. Yet it is at odds with the property of Post-completeness – sentential calculus has this property, that is, adding to it a formula which is not a tautology (its theorem) leads to a contradiction. Thus, #-formulas are not theorems of sentential formulas or become tautulogical after a suitable extension as in the case of modal logic.

The above settings are semantic, that is, employ logical values. The appearance of systems formalized via natural deduction (Stanisław Jaśkowski, Gerhard Gentzen) led to syntactic ways of defining functors. Omitting various details, particularly related to Gentzen's sequents calculi, it is possible to provide elimination (**EL**) and introduction (**IN**) schemes (rules) for disjunction and conjunction (I recall that other connectives are not taken into account):

(**EL**$\vee$) $A \vee B, \neg A \vdash B$; (**IN**$\vee$) $A \vdash (A \vee B)$;

(**EL**$\wedge$) $A \wedge B \vdash A$; (**IN**$\wedge$) $A, B \vdash (A \wedge B)$.

One of differences in the above rule consists in the fact that occurs in (**EL**$\vee$) but not in the schemes for conjunction. However, if we remember about the entire sentential calculus, we can state the rule $\neg(A \wedge B) \vdash \neg A$, where the symbol $\vdash$ denotes the deducibility relation, which is syntactic in its nature. This rule can also be shaped by writing $B \in Cn\{A \vee B, \neg A\}$ – in words, B belongs to the set of logical consequences of the set $\{A \vee B, \neg A\}$; the operations Cn can be introduced axiomatically, and its basic sense (similarly as the symbol $\vdash$) is associated with the following condition of soundness

(*) if A is deducible (logically follows, is logically entailed by) from the set of sentences X ($X \vdash A$; $A \in CnX$), then A is true, provided that X is true (= all sentences belonging to X are true).

Applying (*) to (**EL**) and (**IN**) for $\wedge$ and $\vee$, we immediately get semantic characterizations of both connectives. Assume that $A \Leftrightarrow v(A) = 1$ (A if and only if the value A is equal to 1; A if and only if A is true). Clearly, the conjunction satisfies the condition (b), and the disjunction – the condition (l).

The possibility of introducing connectives via rules is the starting point of inferentialism, which says that meanings of logical constants are partially (the weak version) or entirely (the strong version) established by inferential rules, like (**EL**) and (**IN**) for $\vee$ and $\wedge$, which leads to the construction of logic by natural deduction in its various forms (see [3], [5], passim, [10], chapter 10, and [11] for general characterization of inferentialism. Arthur N. Prior's proposal (see [12]) of defining the functor tonk, was an episode in the history of this approach. The connective tonk is to satisfy two rules

(**IN**tonk) $A \vdash A \,\mathrm{tonk}\, B$;

(**EL**tonk) $A \,\mathrm{tonk}\, B \vdash A$.

The status of tonk raised a discussion (see [14], [1], [13], [4], [5], [10], [11]). Since (**IN**tonk) is similar to (**IN**$\vee$), but (**EL**tonk) to (**EL**$\wedge$), tonk was called (by Jaroslav Peregrin) – a hybrid functor. Yet the problem arose of whether this hybrid expresses a natural (in the sense of (a)–(p)) connective. It was observed, for example, that properties of the relation $\vdash$ are not strictly defined with respect to tonk and that there is no related matrix for tonk. Roy Cook showed that we can define this functor via many-valued logic and associated consequence operations. For Lloyd Humberstone, who followed Nuel Belnap, tonk does not behave in an elegant way according to classical logic. In particular, adding tonk to classical sentential logic does not lead to a conservative extension of the system with functors defined by the conditions (a)–(t). I will argue that tonk is partially conjunction and partially disjunction, but it requires a change of metalogical perspective.

Prior and other authors assumed that the relation $\vdash$ used in tonk-rules satisfies the condition (*), that is, it does not produce a false conclusion from true premises. Following this assumption, one should check (**EL**tonk) and (**IN**tonk) from this point of view, that is, the property of soundness. As far as the issue concerns the former, we must take into account valuations (i) and (iii), because $v(A) = 1$ in this case. The survey of (a)–(p) shows that the cases (g), (l), (n) and (p) are relevant only, because the formulas A and $A \,\mathrm{tonk}\, B$ are simultaneously true just in these cases. In particular, (l) represents a tautology (a possible reading "A and B joined by tonk form a tautology"), that is, a sentence deducible from any sentence. The rule (**IN**tonk) is more difficult for its analysis, because it is not known in advance which valuations from the set {(i), (ii), (iii), (iv)} transform $A \,\mathrm{tonk}\, B$ in such a way that $v(A \,\mathrm{tonk}\, B) = 1$. Yet we can investigate what happens if $A \,\mathrm{tonk}\, B$ is true in the cases, that is, when $A \vdash A \,\mathrm{tonk}\, B$. The possibility (g) holds if $v(A) = 1$, $v(B) = 1$ and $v(A) = 1$, $v(B) = 0$. A simple calculation demonstrates that $(A \,\mathrm{tonk}\, A) \Leftrightarrow A$ in such cases – as a consequence, the rule (**IN**tonk) appears as reduced to $A \vdash A$. However, we cannot conclude that the dependence $(A \,\mathrm{tonk}\, A) \vdash A$ defines the constant true sentence, that is, **1**, because it is not excluded that $v(A \,\mathrm{tonk}\, A) = 1$ przy $v(A) = 0$ – similarly for $A \vdash (A \,\mathrm{tonk}\, A)$ for the possibility of $v(A) = 1$ and $v(A \,\mathrm{tonk}\, A) = 0$. Furthermore, (l) identifies tonk with $\vee$, (n) – with the so-called converse implication, (p) – with tautology-forming functor, ($A \,\mathrm{tonk}\, B$ is a tautology), but (**EL**tonk) would not be valid in any of these cases because A can be false and $A \,\mathrm{tonk}\, B$ true. Since it is not excluded that $v(A) = 0$ and $v(A \,\mathrm{tonk}\, A) = 1$, the dependence $v(A \,\mathrm{tonk}\, A) \vdash A$ does not hold. Consequently, there is no such situation that (**EL**tonk) and (**IN**tonk) simultaneously hold for a standard connective of sentential calculus. And it is a reason for the conclusion that these rules do not define any of the twenty functors of sentential calculus, that is unary, if we consider $A \,\mathrm{tonk}\, A$ or binary, if $A \,\mathrm{tonk}\, B$ and $A \neq B$. It is not strange that embedding tonk into classical logic leads to difficulties.

The problem can be considered by applying the dual consequence operation dCn ([17] and [16] – the latter for the relation of dCn to the concept of being false), defined in the following way

(1) $A \in dCnX$ if and only if there exists a set Y such that $Y \subseteq X$ and $\cap(Cn\{B\} : B \in Y) \subseteq Cn\{A\}$.

Let us define duality of formulas (I touch on variables, negation conjunction and disjunction)

(2) (a) $dp = p$, for every sentential variable;
(b) $d(\neg A) = \neg dA$;
(c) $d(A \wedge B) = dA \wedge dB$;
(d) $d(A \vee B) = dA \vee dB$.

Having (1) and (2), we define

(3) $A \in dCnX$ if and only if $dA \in CndX$, provided that $A \in dX$ if and only if $dA \in X$.

The intended interpretation dCn consists in understanding it as related to rejecting sentences on the basis of rejecting others. Respectively, the context $A \in dCnX$ can be read "A is rejected on the basis of the set X of rejected sentences"; I will use the notation $X \dashv A$ in my further considerations. We note

(4) (a) if $\dashv A$, then $\dashv (A \wedge B)$;
(b) if $\dashv (A \wedge B)$, then $\dashv A$ i $\dashv B$.

(4a) defines the introduction of conjunction via rejection, that is, introduction of rejected conjunction on the basis of rejection of one of its components, but (4b) shows how to eliminate disjunction via rejecting both its disjuncts. We also have

(5) (a) if $\dashv (A \vee B)$, then $\dashv A$ i $\dashv B$;
(b) if $\dashv A$ i $\dashv B$, then $\dashv (A \vee B)$.

These formulas almost reproduce tonk-rules, "almost" because (4b) has $\dashv B$ in its consequent, but (5) has this formula in both points – there are no obstacles to write "if $\dashv (A \vee B)$, then $\dashv B$" and "if $\dashv B$, then $\dashv (A \wedge B)$". If we admit that tonk can be understood as conjunction (Prior himself applied the term "contonction"; see [13]) or disjunction, omitting the sign $\dashv$ in consequents of (4) and (5) results in incomplete characterization of both functors. Otherwise speaking, combining $\vdash$ and $\dashv$ (that is, Cn and dCn) strengthens inferential characterization to its equivalence with the matrix approach.

The above remarks suggest that a "mystery" of tonk is partially caused by the fact that the rules (**EL**tonk) and (**IN**tonk) do not provide a univocal characterization of this functor (see also [1]), relatively to possibilities determined by truth-tables, particularly, and do not determine its relation to negation. On the other hand, (4) and (5) fill the gap in question because the formula $\dashv A$ can be interpreted as expressing that $v(A) = 0$ (A is rejected as false). This claim seems to provide an additional justification of inferentialism because it correlates conditions of assertion and rejection with those of truth and falsity. However, the triumph of the idea of defining the meaning of logical constants (in this case sentential connectives) should be considered as premature, because the entire argumentation proceeds in the metalanguage in which we assume a counterpart between assertion and rejection, and truth and falsity, respectively. Now, the condition (*) can be easily modified as

(**) if $X \vdash A$ ($A \in CnX$), then one should assert A, provided that sentences belonging to X are asserted as well.

Yet this rephrasing replaces the semantic explanation with a pragmatic one, because asserting is a notion belonging to linguistic pragmatics. Thereby, inferentialism cannot be maintained without an additional justification that defines the meaning of logical constants by syntax exclusively.

Further related considerations require an appeal to a cognitive theory of reasoning, for instance, the idea of mental models (see [6]) or natural logic (see [2]. It seems that strong inferentialism in the syntactic version cannot be accepted because every characterization of the logical consequence operations must appeal either to semantics (the condition (*) or analogical for falsity – dual consequences of falsities cannot be true) or to pragmatics (logical consequences of asserted dCn consequences of rejections are rejected). These observations confirm one of the basic intuitions of the founding-fathers of semiotics (Charles Morris, Rudolf Carnap) that semantics is an abstraction from pragmatics, but syntax from semantics. As a consequence, we have that syntactic characterization of logical constants in the object-language requires something non-syntactic in the metalanguage.

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