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INTUITIONISM AND LOGICAL DETERMINISM

Abstract. In this article, we will consider the application of intuitionistic tense logics to construct logical systems in which the thesis of logical determinism cannot be expressed.

Keywords: logical determinism, tense logic, intuitionism.

Typically, logics of branching time are proposed as nondeterministic systems. The systems, however, have their opponents because the realization of reality occurs according to one branch of the structure, hence there is a reduction of the branching structure to a linear structure. A better proposal for systems in which arguments for logical determinism are rejected seems to be tense logic systems constructed using modal-temporal operators. Considering the formal apparatus employed in the construction of various indeterministic tense logic systems, a pertinent question emerges: is it possible to construct an indeterministic tense logic wherein:

- there are only two truth values,
- no conditions are imposed on the structure of time,
- apart from the tensal operators, no other specific operators appear in the language of this logic?

The answer to this question may lie in the construction of an intuitionistic tense logic. We will show that it is possible to construct an indeterministic tense logic by appropriately combining tense logic with intuitionistic logic.

T_m-intuitionistic tense logic of unchanging time

The system of intuitionistic tense logic of unchanging time was constructed by Kazimierz Trzęsicki[1]. On the one hand, thanks to tense logic, it is possible to properly formalize temporal expressions. On the other hand, thanks to the fact that T_m logic is built on intuitionistic logic, the law of excluded middle is rejected as a means of proof, which means that an argument for determinism based on the law of excluded middle cannot be reconstructed in T_m.

Semantics

Let T be a non-empty set of time moments. Let W be a non-empty set of world states. The elements of set W are indexed by the elements of set T (i.e., by the time moments in which they occurred). W_t is the world state at time t ($\in T$). The binary relation R ($\subseteq W \times W$) is a relation of precedence ("earlier – later") between world states.

World $\mathfrak{M}$ is an ordered triple: $\langle W, R, W_n \rangle$, where W_n is a distinguished element of the set W (intuitively understood as the current world state $\mathfrak{M}$). A temporal precedence relation R ($\subseteq T \times T$) is defined between the elements of the set T in the following way: $t_1 R t_2$ if and only if $W_{t_1} R W_{t_2}$. n is a distinguished element of the set T^2. Time is understood as an ordered triple: $\langle T, R, n \rangle$.

At a time t, the state W_t of the world consists of a non-empty class S_t (sets of possible situations at t). A situation is understood as a semantic correlate of a sentence. Therefore, the set of situations can be understood as a set of sentences. A situation is determined in the world state W_t if it is determined by every possible set of situations that make up W_t. The class of sets of possible situations in a given W_t depends on the time moment (or equivalently, on the world state). The sets of situations possible from the point of view of t_1 need not be the same as the sets of situations possible from the point of view of t_2. Sets of situations are ordered by the set-theoretic relation of inclusion. This relation is reflexive and transitive. Therefore, the world state at time t can be treated as an ordered pair W_t ($= \langle S_t, \leq \rangle$), where S_t is a non-empty class of sets of possible situations from the point of view of the current world state, and $\leq$ ($\subseteq S_t \times S_t$) is a reflexive and transitive relation.

Situations which are semantic correlates of sentences about the past (future) should be linked to certain situations in some past (future) state of the world. Furthermore, situations in a certain past (future) state of the world should be linked to certain situations which are semantic correlates of certain sentences about the future (past) in the current state of the world.

Definition 1

Let f be a function such that for any t, $t_1 \in T$, and $\Gamma_t \in S_t$, $f(\Gamma_t, t_1)$ is a non-empty subset of the set S_{t_1} satisfying the following conditions:

1. For any $\Gamma_{t_1}, \Delta_{t_1} \in f(\Gamma_t, t_1)$, if ($\Gamma_{t_1} \leq \Delta_{t_1}$ or $\Delta_{t_1} \leq \Gamma_{t_1}$), then $\Gamma_{t_1} = \Delta_{t_1}$.
2. If $\Gamma_{t_1} \in f(\Gamma_t, t_1)$, then $\Gamma_t \in f(\Gamma t_1, t)$.
3. For any Γ_t, Δ_t, if $\Gamma_t \leq \Delta_t$, then $f(\Gamma_t, t_1)$ is a non-empty set, and for any $\Gamma_{t_1} \in f(\Gamma_t, t_1)$, there exists $\Delta_{t_1} \in f(\Gamma_t, t_1)$ such that $\Gamma_{t_1} \leq \Delta_{t_1}$.

It is assumed that the function f is defined for any world $\mathfrak{M}$. Thus, world $\mathfrak{M}$ is a quadruple denoted as $\langle W, R, f, n \rangle$.

There exists a relation between situations and sets of situations called the *determination relation* (a situation is determined by a set of situations) and denoted as $\Vdash$.

The definition of the determination relation is as follows:

Definition 2

For any $\varphi, \psi \in \mathfrak{F}$ and $\Gamma \in S_t$:

a) if ($\Gamma \Vdash \varphi$ and $\Gamma \leq \Delta$), then $\Delta \Vdash \varphi$,

b) $\Gamma \Vdash \varphi \wedge \psi \quad \equiv \quad \Gamma \Vdash \varphi$ and $\Gamma \Vdash \psi$,

c) $\Gamma \Vdash \varphi \vee \psi \quad \equiv \quad \Gamma \Vdash \varphi$ or $\Gamma \Vdash \psi$,

d) $\Gamma \Vdash \neg\varphi \quad \equiv \quad$ for any Δ ($\in S_t$) such that $\Gamma \leq \Delta$, it is not the case that $\Delta \Vdash \varphi$,

e) $\Gamma \Vdash \varphi \rightarrow \psi \quad \equiv \quad$ for any Δ ($\in S_t$) such that $\Gamma \leq \Delta$, if $\Delta \Vdash \varphi$, then $\Delta \Vdash \psi$,

f) $\Gamma \Vdash H\varphi \quad \equiv \quad$ for any t_1 ($\in T$) if $t_1 R t$, then for any Δ ($\in (\Gamma, t_1)$), $\Delta \Vdash \varphi$,

g) $\Gamma \Vdash G\varphi \quad \equiv \quad$ for any t_1 ($\in T$) if $t R t_1$, then for any Δ ($\in f(\Gamma, t_1)$), $\Delta \Vdash \varphi$,

h) $\Gamma \Vdash P\varphi \quad \equiv \quad$ there exists t_1 ($\in T_i$) such that $t_1 R t$ and for some Δ ($\in f(\Gamma, t_1)$), $\Delta \Vdash \varphi$,

i) $\Gamma \Vdash F\varphi \quad \equiv \quad$ there exists t_1 ($\in T_i$) such that $t R t_1$ and for some Δ ($\in f(\Gamma, t_1)$), $\Delta \Vdash \varphi$.

The expression $\Gamma \Vdash \varphi$ is read as: φ *is (now) determined with respect to* Γ *(where* $\Gamma \in S_t$*)*.

$W_t \models \varphi$ (φ is true in W_t, or φ is determined in W_t) if and only if for any Γ ($\in S_t$):

$$\Gamma \Vdash \varphi.$$

According to the author of the system, since the thesis of predeterminism is symmetric to the thesis of postdeterminism, the function fulfilled by the distinguished element n is not formally significant.

$\langle W, R\rangle \models \varphi$ (φ is true in $\langle W, R\rangle$) if and only if for any $t \in T$, $W_t \models \varphi$.

$\langle W, R\rangle$ is a model of the sentence φ if and only if $\langle W, R\rangle \models \varphi$.

The sentence φ is satisfied in $\langle W, R\rangle$ if and only if there exists W_t ($\in W$) such that $W_t \models \varphi$.

Taking into account the remark regarding the role of the distinguished element n, the time $\mathcal{T}$ is understood as an ordered pair $\langle T, R\rangle$.

The sentence φ is true in time $\mathcal{T}$ if and only if it is true in every model $\langle W, R\rangle$ based on the time $\mathcal{T}$.

φ is a tautology of minimal intuitionistic tense logic of unchanging time if and only if it is true in every time $\mathcal{T}$ $(= \langle T, R\rangle)$.

Minimal tense logic of unchanging time is axiomatizable.

Axiomatization T_m

For any $\varphi, \psi, \gamma \in \mathfrak{F}$:

A0) All tautologies of intuitionistic propositional calculus in the language $\mathcal{L}$.

H1) $H(\varphi \to \psi) \to (H\varphi \to H\psi)$,	G1) $G(\varphi \to \psi) \to (G\varphi \to G\psi)$,
H1') $H(\varphi \to \psi) \to (P\varphi \to P\psi)$,	G1') $G(\varphi \to \psi) \to (F\varphi \to F\psi)$,
H1") $(H\varphi \to P\varphi) \vee H\psi$,	G1") $(G\varphi \to F\varphi) \vee G\psi$,
H2) $\varphi \to HF\varphi$,	G2) $\varphi \to GP\varphi$,
H2') $PG\varphi \to \varphi$,	G2') $FH\varphi \to \varphi$,
HD) $P\varphi \to \neg H \neg\varphi$,	GD) $F\varphi \to \neg G \neg\varphi$,
HD') $\neg P\varphi \to H\neg\varphi$,	GD') $\neg F\varphi \to G\neg\varphi$.

The axioms H1, H2, G1, G2 are also axioms of the system K_t. The remaining listed axioms are theorems of K_t (i.e., $T_m \subseteq K_t$). In the context of intuitionistic tense logic, the operators G, F, H, and P are not mutually definable. The axioms HD and HD' express the relationships between the operators H and P in T_m, while the axioms GD and GD' express the relationships between the operators G and F in T_m. Adopting H1" and G1" as axioms allows semantic considerations to be based on both temporal structures without an initial (or final) moment and temporal structures with such a moment.

As shown later in this work, the formula $H\varphi \to P\varphi$ is valid only in systems where a timeless semantic time is assumed. If a temporal moment is not an initial moment, then $H\varphi \to P\varphi$ is true at that moment. If a temporal moment is an initial moment, then for any sentence ψ, $H\psi$ is true at that moment. Therefore, the sentence $(H\varphi \to P\varphi) \vee H\psi$ is true in any moment, in any temporal structure. The same reasoning applies to the sentence $(G \to F\varphi) \vee G\psi$.

The logic T_m is a minimal logic, which means that for semantic considerations in T_m, a temporal structure with arbitrary properties can be adopted.

The sentence

$$F\varphi \vee F\neg\varphi$$

which would be considered as an expression of the determinism thesis, is not a theorem in the system T_m.

The axiom of T_m is

$$\varphi \to HF\varphi.$$

This sentence has been considered as an expression of the determinism thesis in most presented systems. However, the understanding of tense operators in T_m is such that the sentence $\varphi \to HF\varphi$ does not constitute an expression of the determinism thesis.

2. IT_m – intuitionistic tense logic of changing time

The aim of our considerations is to construct a system of minimal intuitionistic tense logic with changing time. This logic is denoted as IT_m. It is the intuitionistic counterpart of the minimal tense logic system K_t based on classical logic. It seems that the language of intuitionistic logic enriched with tense operators is suitable for describing an indeterminate world. We will show that the constructed intuitionistic tense logic confirms the truth of the proposed hypothesis.

The tense operators will be understood in a standard way, namely:

$G\varphi$ – always in the future, φ will hold,

$F\varphi$ – there will be a moment in the future when φ holds,

$H\varphi$ – always in the past, φ held,

$P\varphi$ – there was a moment in the past when φ held.

2.1. Semantics

Our considerations revolve around the concepts of determination and indetermination of events. These concepts should be considered in a temporal context. Therefore, the semantics of IT_m is a version of Kripke-style semantics for intuitionism. It is a semantics of possible worlds. This semantics must satisfy certain specific conditions.

Since an event can be indeterminate at one time and determinate at another, we must take into account the moments in time where events occur. We assume that there are no *empty moments in time*, i.e., moments where there are no events occurring.[3] Events constitute time. These moments exist presently, not just as possibilities, in which determined events occur. The determination of events is relative and depends on the *location* in the world from which we inquire about determination. Therefore, time, its possibility or actuality, also depends on this world.

We must also consider the *earlier-later* relation between moments in time. Similar to the existence of moments themselves, the occurrence of the *earlier-later* relation is only possible or actual, depending on whether we are dealing with indeterminate or determinate events. The *earlier-later* relation is constituted by the causal relation in the sense that moment t_1 is earlier than moment t_2 if and only if there is an event in t_1 that is the cause of an event in t_2. For this reason, the occurrence of the *earlier-later* relation is relative, depending on the place from which we inquire about its occurrence. The answer to the question of the occurrence of the causal relation depends on this place.

In the proposed semantics, the world is understood as a composition of a set of events, which are semantic correlates of sentences, a set of moments in time, and the relation of temporal succession. A subset of the set of events assigned to a particular moment in time is understood as a set of determined events at that moment. If the world were determinate, everything that would occur would occur out of necessity. In such a world, only what is necessary would be possible. There would be no indeterminate events. Therefore, there would be no other worlds. What could happen would occur in a determinate world. Thus, only one world would exist.

Our task is to construct a language for describing an indeterminate world. We assume that the world, let's call it m, is indeterminate, meaning that there exist indeterminate events in this world. For example, in world m, the event *There will be a sea battle tomorrow* is indeterminate, as well as the event *There will not be a sea battle tomorrow.* From the perspective of world m, there can exist at least two worlds: one where, in addition to the determinate events in world m, the event *There will be a sea battle tomorrow* is determinate, and another world where, in addition to the determinate events in world m, the event *There will not be a sea battle tomorrow* is determinate. By assuming that world m is not completely determinate, we allow for the possibility of multiple worlds that differ from m. In these worlds, all the events that are determinate in world m are also determinate, and there are additional newly determinate events. These worlds can differ from the original world m not only in terms of the set of newly determinate events and new moments in time but also in terms of new *before-after* relation resulting from the occurrence of new cause-effect relation. These worlds are called worlds accessible from world m. From the perspective of world m, every world in which some indeterminate event in world m is determinate is possible. Furthermore, we assume that from the perspective of world m, world m itself is accessible, which means that the accessibility relation is reflexive. If we did not make this assumption, we would exclude the possibility that world m is completely determinate. However, we do not want the logic to be ontologically engaged. The language of our logic should not decide in favor of determinism or indeterminism but should remain independent in this regard. Resolving the ontological question of whether the world is determinate or not is not the subject of our considerations.

The worlds accessible from world m have no fewer determinate events than there are in world m. Therefore, we say that the worlds accessible from world m are worlds with a level of determinacy no lower than the level of determinacy of world m.

World m_1, accessible from the perspective of world m, can also be indeterminate. From the perspective of world m_1, world m_2 can be accessible, such that in m_2, everything that is determinate in world m_1 is also determinate, and in addition, there are new events that are determinate in m_2 but not in m_1. Since everything that is determinate in world m is also determinate in world m_1, and everything that is determinate in world m_1 is determinate in world m_2, it follows that everything that is determinate in world m is also determinate in world m_2. Therefore, world m_2 is accessible from world m. The accessibility relation is thus transitive.

The accessibility relation is reflexive and transitive, hence it is a partially ordering relation.

In the proposed semantics, possible worlds differ in their level of determinacy. The level of determinacy of a world is determined by its components, which are sets of true events at specific moments in time, as well as time itself, which is a set of moments of time and the temporal succession relation. We say that world m_2 has no lower level of determinacy than world m_1 when the following conditions are satisfied:

1) The set of moments of time in world m_1 is included in the set of moments of time in world m_2. (The existence of moments of time is determined by the occurrence of the specific cause-effect relation. The change in the number of moments in time entails a change in the number of cause-effect relation, i.e., a change in the level of determinacy of the world.)

2) In world m_2, the *before-after* relation that exists between moments of time in world m_1 is preserved. Additionally, in world m_2, there can be a *before-after* relation that did not exist in world m_1.

3) All events that are determinate in world m_1 are also determinate in world m_2. (What is determinate does not cease to be determinate even when new determinate events occur.) Furthermore, at moments in time in world m_2, there can be determinate events that are not determinate at corresponding moments in world m_1. There are relationships between conditions 1), 2), and 3).

The fulfillment of condition 1) implies the fulfillment of condition 2) because in our considerations we exclude situations where "new" moments in time are not related *before-after* to other moments in time. Therefore, a change in the set of moments of time entails a change in the temporal succession relation. However, the reverse is not true. A change in the temporal succession relation does not necessarily entail a change in the set of moments of time. In a world with no lower level of determinacy, a new *before-after* relation can occur between moments in time that exist in worlds with a lower level of determinacy. The fulfillment of condition 2) does not imply the fulfillment of condition 1). Similarly, the fulfillment of condition 3) does not imply the fulfillment of conditions 1) or 2) because new determinate events can occur without the occurrence of new moments in time or a new *before-after* relation.

Each moment in time is associated with a non-empty set of determinate events. If new moments in time exist, then new determinate events also exist. Therefore, the fulfillment of condition 1) implies the fulfillment of condition 3).

The determinacy of events, as considered in our semantics, has certain properties. If an event is determinate in a certain world m, it is also determinate in any world with no lower level of determinacy than world m. On the other hand, if an event is not determinate in world m, it does not mean that the absence of that event is determinate in that world.

Notice that the notion of event determinacy, as considered in our semantics, has properties analogous to the truth properties in intuitionistic logic. Let's consider, for example, a *proof-theoretic* semantics appropriate for intuitionistic logic. In this semantics, intuitionistic truth is understood as having a proof at a given moment in time within a fixed axiomatic system. Intuitionistically true statements are those that have been proven at a given moment. The remaining statements, regardless of whether they will be proven in the future or not, are intuitionistically false. The axiomatic system is understood as dynamic: new statements with proofs gradually emerge. Therefore, some of the statements previously evaluated as intuitionistically false may become intuitionistically true. (The same applies to the semantics for the system IT_m. Events that are indeterminate in world m may be determinate in a world with a higher level of determinacy than world m.) Since no statement that has been proven can lose its proof, a statement once evaluated as intuitionistically true retains this value *forever*. (In the proposed semantics for IT_m, an event that is determinate in world m remains determinate in any world with a level of determinacy not lower than the level of determinacy of world m.) However, these valuations are not isolated but are related to each other in specific ways. Statements to which a valuation assigns a false value had this value in every *earlier* valuation. Statements to which a valuation assigns a true value will retain this value in all *later* valuations.

Taking these considerations into account, we conclude that intuitionistic tense logic will be appropriate for formally expressing our intuitions.

These intuitions will be formalized in the following manner.

2.2. Notation

Let I be a non-empty set. (This is the set of indices of worlds.) Let T_i ($i \in I$) be a non-empty set. (This is the set of moments in time in the world with index i). Let R_i ($\subseteq T_i \times T_i$). This is a binary relation defined on the set of moments in time in the world with index i. The relation R_i is understood as a *before-after* relation on the set of moments in the world with index i. Let $\mathcal{T}_i = \langle T_i, R_i \rangle$. (This is the time in the world with index i). Let $T = \bigcup_{i \in I} T_i$. (This is the set of all time moments existing in any world). Let $R = \bigcup_{i \in I} R_i$. (This is a binary relation on the set of all time moments existing in any world. This relation is understood as a *before-after* relation for time unrelated to any world. Note that $R \subseteq T \times T$). Let $\mathcal{T} = \langle T, R \rangle$. (This is time unrelated to any world). Let $V_i \subseteq T_i \times 2^{\mathfrak{X}}$, where $i \in I$. (V_i is a function assigning subsets $V_i(t)$ of propositional variables to moments t ($\in T_i$)). Let $\mathcal{F} = V_i : i \in I$. Let $m_i = \langle T_i, R_i, V_i \rangle$, where $i \in I$. (m_i is the world with index i). Let $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} = \langle Ti, R_i, V_i \rangle : V_i \in \mathcal{F}, i \in I$, which means $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} = m_i : i \in I$. ($\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$ is a model based on time $\mathcal{T}$ and the class $\mathcal{F}$ of functions).

In the set $\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$, we define the relation $\leq$ ($\subseteq \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \times \mathfrak{M}_{(\mathcal{T},\mathcal{F})}$):

Definition 3

For any $i, j \in I$:

$$m_i \leq m_j \equiv (T_i \subseteq T_j, \text{ and } R_i \subseteq R_j \text{ and } \forall t \in T_i \; V_i(t) \subseteq V_j(t)).$$

We understand that for worlds m_i and m_j, the relation $\leq$ (i.e., $m_i \leq m_j$) means that world m_j has an equal or higher level of determination (an equal or greater degree of determination) than world m_i.

The relation $\leq$ is determined by the inclusions: set of time moments, temporal succession relation, and sets of determined events at individual time moments. Thus, it possesses all properties of inclusion. In particular, it is reflexive and transitive.

Notation

By m_i^* (where $i \in I$), we mean any $m_j \in \mathfrak{M}_{(\mathcal{T},\mathcal{F})}$ such that

$$m_i \leq m_j.$$

The concept of sentence satisfaction in the model will now be introduced. This concept will be defined inductively.

Definition 4

For the model $\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$, world m_i $(= \langle T_i, R_i, V_i \rangle)$, element t $(\in T_i)$, tense logic sentence φ, it holds that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_i]$, i.e., the sentence φ is satisfied in the model $\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$, in world m_i, at time t, if and only if:

a) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_i] \equiv \varphi \in V_i(t)$, if $\varphi \in \mathfrak{X}$,

b) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \neg\varphi[t, m_i] \equiv \forall\, m_i{}^* \in \mathfrak{M}_{(\mathcal{T},\mathcal{F})}\ \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \varphi[t, m_i^*]$,

c) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models (\varphi \vee \psi)[t, m_i] \equiv \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_i]$ or $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_i]$,

d) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models (\varphi \wedge \psi)[t, m_i] \equiv \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_i]$ and $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_i]$,

e) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models (\varphi \to \psi)[t, m_i] \equiv \forall\, m_i{}^* \in \mathfrak{M}_{(\mathcal{T},\mathcal{F})}$
$(\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \varphi[t, m_i^*]$ or $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_i^*])$,

f) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models F\varphi[t, m_i] \equiv \exists\, t_1 \in T_i$
$(tR_it_1$ such that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t_1, m_i])$,

g) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models G\varphi[t, m_i] \equiv \forall\, m_i{}^* \in \mathfrak{M}_{(\mathcal{T},\mathcal{F})}\ \forall\, t_1 \in T_i^*$
(if $tR_i^*t_1$ then $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t_1, m_i^*])$,

h) $\mathfrak{M}_{(\mathcal{T},\mathcal{F}} \models P\varphi[t, m_i] \equiv \exists\, t_1 \in T_i$
$(t_1R_it$ such that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t_1, m_i])$,

i) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models H\varphi[t, m_i] \equiv \forall\, m_i{}^* \in \mathfrak{M}_{(\mathcal{T},\mathcal{F})}\ \forall\, t_1 \in T_i^*$
(if $t_1R_i^*t$, then $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t_1, m_i^*])$.

Let us note that a necessary condition for the sentence $F\varphi$ to be true in world m_i at time t $(\in T_i)$ is the existence, in the time structure of world m_i, of a moment t_1 $(\in T_i)$ that is later than t (tR_it_1) in which the sentence φ is true. If such a moment exists in the time structure of world m_i, then from the definition of the relation $\leq$ and the set-theoretic properties of inclusion, it follows that such a moment also exists in the time structure of every world no less determined than world m_i. Therefore, the verification of the truth of the sentence $F\varphi$ in world m_i can be limited to world m_i. Notice that if the sentence $F\varphi$ is not true at time t, it does not mean that the sentence $F\neg\varphi$ is true at t. This will be demonstrated in the further part of the work.

In the case of the operator G, the situation is different. According to the given way of understanding this operator, the sentence $G\varphi$ is read as *always in the future, it will be the case that* φ. For the sentence $G\varphi$ to be true in world m_i at time t ($\in T_i$), it is necessary for the sentence φ to be true in any world m_i^* (not less determined than world m_i) and at any moment t_1 ($\in T_i^*$) later than t ($tR_i^*t_1$). Therefore, the truth of the sentence $G\varphi$ cannot be considered only within the temporal boundaries of world at a given level of determination. The fact that the sentence φ is always true in the future means that φ is true at any moment in the future. Since world m_i is assigned only a certain fragment of the time structure, when defining the truth of a sentence constructed using the operator G, all worlds of no lower level of determination than world m_i must be taken into account.

If the truth of the sentence $G\varphi$ were defined as it is, for example, in the system T_m (intuitionistic tense logic of immutable time[4]), that is:

$$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models G\varphi[t, m_i] \;\equiv\; \forall t_1 \in T_i \quad (\text{if} \;\; tR_it_1 \;\; \text{then} \;\; \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t_1, m_i])^5$$

it could lead to a contradiction. It would be possible for a situation to arise where in a certain world m_i at time t the following conditions hold:

1) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models G\varphi[t, m_i]$.

and in another world m_j no less determined than world m_i ($m_i \leq m_j$), there exists a moment t_1 ($\in T_j$) such that: $t_1 \notin T_i$, tR_jt_1, and $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \varphi[t_1, m_j]$. Thus, according to the definition of truth for the operator G, we have

2) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models G\varphi[t, m_j]$

What is determined does not cease to be determined as the level of determination of world increases. Since world m_j is no less determined than world m_i, it follows that if $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models G\varphi[t, m_i]$ (based on 1), then $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models G\varphi[t, m_j]$, which contradicts 2).

The introduced understanding of the truth of the sentence $G\varphi$ in world m_i at time t excludes the situation described above.

The truth of a sentence, truth of a sentence in time, and truth of a sentence in a model are defined in the standard way.

In some sciences (e.g., physics), it happens that statements initially considered true are found to be false as the world evolves. In our terminology, we would express this fact as follows: a statement considered true in a certain world m_i may be false in some world no less determined than world m_i.

In the system IT_m, such a situation is impossible. What is determined in world m_i is also determined in any world no less determined than world m_i.

There are many differences between systems of temporal logics built upon classical logic and those built upon intuitionistic logic. One of them is that in the latter, the falsity of φ does not imply the truth of $\neg\varphi$. Let's consider the following situation: the statement φ is undetermined in world m_i at time t ($\in T_i$), but it is determined at that moment in world m_j that is no less determined than m_i. If we were to conclude based on the fact that φ is undetermined at time t in world m_i that in that world, at time t, the statement $\neg\varphi$ is determined, then, according to the adopted definition of the truth of the statement $\neg\varphi$, the statement φ could not be determined at time t in any world no less determined than world m_i. In particular, the statement φ could not be determined at time t in world m_j. This leads to a contradiction, as we would obtain that the statement φ is both determined and undetermined at time t in world m_j.

When the statement φ is determined at a certain moment in world m_i, it is not undetermined in any world no less determined at that moment. However, when at a certain moment the statement $\neg\varphi$ is undetermined, it does not imply that at that moment in every world no less determined the statement φ is determined. It simply means that the statement φ is undetermined at that moment in not every world no less determined.

We will now prove a theorem that expresses the monotonicity of determination. What is determined in world m_i is also determined in every world whose level of determination is not lower than the level of determination of world m_i.

Theorem 1

For any sentence φ and any worlds m_i, m_j ($\in \mathfrak{M}_{(\mathcal{T},\mathcal{F})}$):

$$\text{if } (m_i \leq m_j \text{ and } \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_i]), \text{ then } \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_j].$$

Proof.

We will prove the theorem by induction on the length of the sentence φ.

Assume that $m_i \leq m_j$.

I) First, consider the case when φ is a propositional letter ($\varphi \in \mathfrak{X}$). By definition, if $m_i \leq m_j$, then for any $t \in T_i$, we have **1)** $V_i(t) \subseteq V_j(t)$. If $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_i]$, then by def. 4.a, we have **2)** $\varphi \in V_i(t)$. From 1), 2), and the definition of inclusion, we obtain **3)** $\varphi \in V_j(t)$. Since φ is a propositional letter, from 3) and def. 4.a, we have **4)** $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_j]$.

II) Inductive Assumption

Let φ, ψ be such that:

a) If $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_i]$, then $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_j]$,
b) If $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_i]$, then $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_j]$.

III) Now we will consider compound sentences constructed from sentences φ, ψ using propositional connectives and tensor operators.

$(\neg\varphi)$

Let's assume that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \neg\varphi[t, m_i]$. From the definition of negation, we obtain: **1)** For any m_k such that $m_i \leq m_k$, we have $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \varphi[t, m_k]$. Let's consider any world m_l that is no less determined than world m_j, i.e., **2)** $m_j \leq m_l$. From 2), the assumption that $m_i \leq m_j$, and the transitivity of the $\leq$ relation, we conclude that $m_i \leq m_l$, which means that world m_l is no less determined than world m_i. Therefore, based on 1), we have $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \varphi[t, m_l]$. Since m_l is an arbitrary world that is no less determined than world m_j, we obtain: **3)** For any m_l such that $m_j \leq m_l$, we have $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \varphi[t, m_l]$. From 3) and the definition of truth for negation, we conclude $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \neg\varphi[t, m_j]$.

$(\varphi \wedge \psi)$

Let's assume that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models (\varphi \wedge \psi)[t, m_i]$. Thus, from the definition of truth for conjunction, we have: **1)** $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_i]$, and **2)** $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_i]$. From 1) and assumption II.a, we obtain: **3)** $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t, m_j]$. Similarly, from 2) and assumption II.b, we obtain: **4)** $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_j]$. From 3), 4), and definition 2.15.d, we have $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models (\varphi \wedge \psi)[t, m_j]$.

$(\varphi \vee \psi)$

The reasoning is analogous to the case of conjunction.

$(\varphi \to \psi)$

Let us assume that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models (\varphi \to \psi)[t, m_i]$. According to the definition of the truth of implication **1)**, for any m_k such that $m_i \leq m_k$, at least one of the following facts holds: $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \varphi[t, m_k]$ or $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_k]$. Let's consider an arbitrary world m_l, which is no less deterministic than world m_j, i.e., **2)** $m_j \leq m_l$. Based on 2), the assumption that $m_i \leq m_j$, and the transitivity of the relation $\leq$, we have $m_i \leq m_l$, which means that world m_l is no less deterministic than world m_i. Therefore, based on 1), we have $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \varphi[t, m_l]$ or $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_l]$. Since m_l is any world that is no less deterministic than world m_j, we obtain: **3)** for any m_l such that $m_j \leq m_l$, either $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \varphi[t, m_l]$ or $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t, m_l]$. Using 3) and the definition of the truth of implication, we get $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models (\varphi \to \psi)[t, m_j]$.

$(G\varphi)$

Let's assume that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models G\varphi[t, m_i]$. From the definition of truth for the G operator, for any m_k such that $m_i \leq m_k$ and for any $t_1 \in T_k$ such that tR_kt_1, we have $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t1, m_k]$. Let's consider an arbitrary world m_l

that is no less determined than world m_j, which means **2)** $m_j \leq m_l$. From 2), the assumption $m_i \leq m_j$, and the transitivity of the $\leq$ relation, we conclude that $m_i \leq m_l$, which means that world m_l is no less determined than world m_i. Based on 1), we have $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t1, m_l]$. Since m_l is an arbitrary world that is no less determined than m_j, we obtain: **3)** for any m_l such that $m_j \leq m_l$, we have $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t_1, m_l]$. From 3) and the definition of truth for the G operator, we have $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models G\varphi[t, m_j]$.

$(H\varphi)$

Analogous reasoning as in the case of the G operator.

$(F\varphi)$

Let's assume that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models F\varphi[t, m_i]$. According to the definition of the truth of the operator F, there exists a moment t_1 $(\in T_i)$ such that tR_it_1 and the following holds: **1)** $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t_1, m_i]$.

Based on 1) and assumption II.a, we obtain: **2)** $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \varphi[t_1, m_j]$. By the assumption that $m_i \leq m_j$, the definition of a world being no less deterministic, and the definition of inclusion, we get: **3)** $t \in T_j$, **4)** $t_1 \in T_j$, and **5)** tR_jt_1. Therefore, using 2), 3), 4), 5), and the definition of the truth of the operator F, we obtain $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models F\varphi[t, m_j]$.

$(P\varphi)$

The reasoning is analogous to the case of the operator F. □

The system IT_m is axiomatizable.

Axioms

For any $\varphi, \psi, \gamma \in \mathfrak{F}$:

A0) All tautologies of intuitionistic propositional logic in the language $\mathcal{L}$.

H1)	$H(\varphi \to \psi) \to (H\varphi \to H\psi)$,	G1)	$G(\varphi \to \psi) \to (G\varphi \to G\psi)$,
H2)	$H(\varphi \to \psi) \to (P\varphi \to P\psi)$,	G2)	$G(\varphi \to \psi) \to (F\varphi \to F\psi)$,
H3)	$\varphi \to HF\varphi$,	G3)	$\varphi \to GP\varphi$,
H4)	$PG\varphi \to \varphi$,	G4)	$FH\varphi \to \varphi$,
H5)	$P(\varphi \vee \psi) \to (P\varphi \vee P\psi)$,	G5)	$F(\varphi \vee \psi) \to (F\varphi \vee F\psi)$,
H6)	$(P\varphi \to H\psi) \to H(\varphi \to \psi)$,	G6)	$(F\varphi \to G\psi) \to G(\varphi \to \psi)$,
H7)	$P\varphi \to \neg H \neg \varphi$,	G7)	$F\varphi \to \neg G \neg \varphi$.

Rules:

MP: Modus Ponens $\qquad RH$: $\dfrac{\vdash_{IT_m} \varphi}{\vdash_{IT_m} H\varphi}$, $\qquad RG$: $\dfrac{\vdash_{IT_m} \varphi}{\vdash_{IT_m} G\varphi}$.

In the system K_t (the minimal tense logic system based on classical logic), the specific axioms are only H1, G1, H3, and G3. The remaining specific axioms of IT_m are theorems of K_t. Since the proofs of these theorems in K_t utilize classical logic statements that are not theorems of intuitionalistic logic (which will be shown later in this work), and these statements are true in any model, in order to ensure the completeness of IT_m under the proposed semantics, the specific axioms H2, H4, H5, H6, H7, G2, G4, G5, G6, and G7 are included as additional specific axioms alongside H1, G1, H3, and G3.

Theorem 2 (Surowik 2001)
The specific axioms H2, H4, H5, H6, H7, G2, G4, G5, G6, and G7[6] are theorems of the K_t system.

From the above theorem, the following conclusion follows:

Corollary 1

$$IT_m \subseteq K_t.$$

The natural questions arise about the relationship between the proposed semantics and the adopted axioms for this logic. Basic theorems hold for this logic.

Theorem 3 (Surowik 2001)
The IT_m system is consistent.

The axioms of IT_m are true in any model. In the proposed semantics for IT_m, no conditions have been imposed on the relation of temporal consequence. Strong completeness theorems hold for the considered logic.

Theorem 4 (Surowik 2001)

$$\Sigma \vdash_{IT_m} \varphi \text{ if and only if } \Sigma \models \varphi.$$

Now we will show one of the basic differences between the minimal tense logic system built upon classical logic and the minimal tense logic system built upon intuitionistic logic.

Theorem 5
The intuitionistic tense operators F, G, P, and H are not mutually definable in IT_m.

Proof.
First, we will show that the operator F is not definable using intuitionistic propositional connectives and other tense operators. To do this, we

will demonstrate that the formula Fp is not equivalent to any tense formula without the operator F.

Let $T_1 = t_1, t_2$, $T_2 = t_1, t_2, t_3$, $R_1 = (t_1, t_2)$, $R_2 = (t_1, t_2), (t_1, t_3)$, and $T = T_1 \cup T_2$. Let $V_1 : T_1 \to 2^{\mathfrak{X}}$ be a function such that $p \notin V_1(t_2)$. Let $V_2 : T_2 \to 2^{\mathfrak{X}}$ be a function defined as follows: $V_2(t_1) = V_1(t_1)$, $V_2(t_2) = V_1(t_2) \cup p$, $V_2(t_3) = V_1(t_2)$. Let $\mathcal{F} = V_1, V_2$. Let $m_1 = \langle T_1, R_1, V_1 \rangle$, $m_2 = \langle T_2, R_2, V_2 \rangle$. Let $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} = \{m_1, m_2\}$. By structural induction, it can be shown that for any formula ψ that does not contain the operator F, $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t_1, m_1]$ if and only if $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t_1, m_2]$. However, $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models Fp[t_1, m_2]$ and $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models Fp[t_1, m_2]$. Therefore, the operator F is not definable in IT_m.

Next, we will show that the operator G is not definable using intuitionistic propositional connectives and other tense operators. We will demonstrate that the formula Gp is not equivalent to any tense formula without the operator G. Let $T_1 = t_1, t_2$, $T_2 = t_1, t_2, t_3$, $R_1 = (t_1, t_2)$, $R_2 = (t_1, t_2), (t_1, t_3)$, and $T = T_1 \cup T_2$. Let $V_1 : T_1 \to 2^{\mathfrak{X}}$ be a function such that $p \notin V_1(t_2)$. Let $V_2 : T_2 \to 2^{\mathfrak{X}}$ be a function defined as follows: $V_2(t_1) = V_1(t_1)$, $V_2(t_2) = V_1(t_2) \cup p$, $V_2(t_3) = V_1(t_3)$, and $p \in V_1(t_3)$. Let $\mathcal{F} = V_1, V_2$. Let $m_1 = \langle T_1, R_1, V_1 \rangle$, $m_2 = \langle T_2, R_2, V_2 \rangle$. Let $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} = m_1, m_2$. By structural induction, it can be shown that for any formula ψ that does not contain the operator G, the following holds:

$$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t_1, m_1] \text{ if and only if } \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \psi[t_1, m_2].$$

However, $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models Gp[t_1, m_2]$ and $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models Gp[t_1, m_2]$. Therefore, the operator G is not definable in IT_m. Similarly, the undefinability of the operators P and H can be shown on the grounds of IT_m using analogous arguments. $\square$

3. The Relationship between Intuitionistic Tense Logic and the Thesis of Logical Determinism

Our considerations will focus on the question of the relationship between intuitionistic tense logic IT_m and the thesis of logical determinism.

We have identified an expression that could be considered as a formulation of the thesis of predeterminism in the language of intuitionistic tense logic:

$$F\varphi \vee F\neg\varphi.$$

This expression states that for any formula φ, there exists a future moment in time where either φ holds or its negation $\neg\varphi$ holds. The semantic counterparts of these statements are events, so the expression $F\varphi \vee F\neg\varphi$ can be understood as follows: for an event Z (which is the semantic counterpart of the formula φ), it is determined that there exists a future moment in time where event Z occurs or it is determined that there exists a future moment in time where event Z does not occur. Since φ in $F\varphi \vee F\neg\varphi$ can be any formula, it follows that for any event it is determined that it will occur at some future moment in time or it is determined that it will not occur at some future moment in time. In other words, all future events are determined. Therefore, if the axiom $F\varphi \vee F\neg\varphi$ is valid in a system, the language of that system can only describe completely determined worlds. This raises the question of whether the possibility of constructing a logical system whose language can only describe completely determined worlds implies a resolution to the problem of determinism or indeterminism in the physical world. Does it mean that the actual world is completely determined? The answer to this question is negative. Whether the actual world is determined or indetermined is independent of the language or logic used. We may agree with the views of Zbigniew Jordan when he writes:

> The contemporary determinist is liberated from the fear that his views make him a hostage of destiny. He also rejects the conditional period: if every sentence is either true or false, then everything that happens in the world has been causally determined for ages. Because from the semantic principles of identity, non-contradiction, and excluded middle, we can only conclude that the future is logically determined, i.e., it will be what it will be, and this truism does not imply that the future is strictly and unambiguously causally determined. If a state of affairs A is causally determined, then state of affairs A is also logically determined. But the successor of this conditional does not imply its antecedent. The state of affairs stated in a logically determined sentence may be the result of preceding events, but whether it is actually the case, we cannot determine through reasoning. Sentences belonging to logic and mathematics do not say anything about the world[7]. Tautologies are useful tools in understanding reality because they determine the range of possible states of affairs. However, we cannot infer how things actually are from tautologies, and this applies to the thesis of determinism and strict determinism. From sentences that do not speak about reality, we cannot deduce sentences about reality that require confirmation by observational sentences for their validity[8].

The views expressed by Zbigniew Jordan in the quoted passage are debatable. It raises the question of how one can speak about an indeterminate world using a language that is suited only for describing a determined world. If the thesis of determinism is a consistent principle of a logical system, then the language of such a system is limited in its ability to describe the world. With such a formal system, one can only investigate properties of worlds that belong to the class of determined worlds. In this case, logic is ontologically involved in the issue of determinism.

One can avoid the need to accept the thesis of determinism, as Jordan does, by asserting that even if the thesis of determinism is a principle of a given theory, the actual world is not determined because the sentences of that theory say nothing about reality. However, this situation is rather uninteresting. A much more intriguing situation arises when one acknowledges that the sentences of a theory do describe reality, and the thesis of determinism is not a principle of that theory. In such a case, it is possible to provide a formal description of the actual world without making a determination about its determinism or indeterminism. This can be achieved through the language of intuitionistic tense logic. Classical tense logic based on classical logic does not offer this opportunity.

We will demonstrate that the thesis of predeterminism, formulated as $F\varphi \vee F\neg\varphi$, is a principle of logic that arises by adding an expression characterizing the class of endless times to the system K_t.

Theorem 6

$$K_t \cup \{G\varphi \to F\varphi\} \vdash F\varphi \vee F\neg\varphi.$$

Proof.

1) $\varphi \vee \neg\varphi$	A0,
2) $G(\varphi \vee \neg\varphi)$	1, RG,
3) $G(\varphi \vee \neg\varphi) \to F(\varphi \vee \neg\varphi)$	thesis attached to the K_t,
4) $F(\varphi \vee \neg\varphi)$	2, 3, MP,
5) $F(\varphi \vee \neg\varphi) \to (F\varphi \vee F\neg\varphi)$	G5,
6) $F\varphi \vee F\neg\varphi$	4, 5, MP.

□

In our considerations, we consider as deterministic logics those logics in which the thesis of determinism is a principle. Based on the above theorem, tense logic of endless time built on classical logic is a deterministic logic.

The law of excluded middle is not a principle of intuitionistic logic, so in the system IT_m and its intuitionistic extensions, the presented proof of the above theorem would be unjustified.

In the case of assuming that the physical world is indeterminate, what matters is not whether the thesis of logical determinism is not a principle of minimal intuitionistic tense logic, but whether the thesis of logical determinism is a principle of intuitionistic tense logic in which semantic time assumes properties typically attributed to physical time. Therefore, what is important is whether the thesis of logical determinism is a principle of intuitionistic linear, dense, endless tense logic.

4. The Thesis of Determinism and Properties of Physical Time

We will show that the thesis of logical predeterminism expressed in the language of IT_m – that is, $F\varphi \vee F\neg\varphi$ – is not a principle of intuitionistic linear, dense, endless tense logic.

Theorem 7

$Fp \vee F\neg p$ is not a tautology of linear, dense, endless time.

Proof.

To prove this theorem, we will show that the sentence $Fp \vee F\neg p$ has a countermodel in which time is linear, dense, and there is no last moment. Let $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} = m_i : i \in I$. Let in any world m_i $(= \langle T_i, R_i, V_i\rangle)$, the relation R_i satisfies the following conditions:

1) $\forall t \in T_i, \forall t_1 \in T_i, \forall t_2 \in T_i$ if $(t_1R_it$ and $t_2R_it)$, then $(t_1 = t_2$ or $t_1R_it_2$ or $t_2R_it_1)$ – linear past,

2) $\forall t \in T_i, \forall t_1 \in T_i, \forall t_2 \in T_i$ if $(tR_it_1$ and $tR_it_2)$, then $(t_1 = t_2$ or $t_1R_it_2$ or $t_2R_it_1)$ – linear future,

3) $\forall t \in T_i, \forall t_1 \in T_i, \forall t_2 \in T_i$ there exists $(tR_it_2$ and $t_2R_it_1)$ – density,

4) $\forall t \in T_i, \exists t_1 \in T_i$ such that tR_it_1 – no last moment.

Let m_1 $(= \langle T_1, R_1, V_1\rangle)$ and m_2 $(= \langle T_2, R_2, V_2\rangle) \in \mathfrak{M}_{(\mathcal{T},\mathcal{F})}$ be such that $t_0 \in T_1$ and the following conditions are satisfied: **5)** $\forall t \in T_1$, if t_0R_1t, then $p \notin V_1(t)$, **6)** $\forall t \in T_2$, if t_0R_2t, then $p \in V_2(t)$, **7)** $m_1 \leq m_2$.

In world m_1, there is no moment t later than t_0 (t_0R_1t) in which the sentence p is satisfied.

By the definition of the truth of the operator F, we have that **8)** $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models Fp[t_0, m_1]$.

In world m_1, the sentence $\neg p$ is also not satisfied at any moment later than t_0. In order for the sentence $\neg p$ to be satisfied at some moment t in world m_1, it is necessary for the sentence p not to be satisfied at moment t in every world no less deterministic than world m_1. However, in world m_2 – which is no less deterministic than world m_1 – at any moment t later than t_0 (t_0R_1t), the sentence p is satisfied.

Therefore, there is no moment t in world m_1, later than t_0 (t_0R_1t), in which the sentence $\neg p$ is satisfied. By the definition of the truth of the operator F, we have **9)** $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models F\neg p[t_0, m_1]$. From 8), 9), and the definition 4.c, we obtain $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models (Fp \vee F\neg p)[t_0, m_1]$. □

From the proven theorem, two conclusions follow.

Corollary 2

$F\varphi \vee F\neg\varphi$ is not a theorem of intuitionistic linear temporal logic without an end.

The theses of IT_m are true in any model, regardless of the semantic properties of time on which the model is based.

Corollary 3

The formula $F\varphi \vee F\neg\varphi$ is not a theorem of IT_m.

Let us note that the formula $Fp \vee F\neg p$ is not a tautology in tense logic, whose semantics is based on time with a last moment[9]. If t is the last moment, there is no moment t' later than t. Therefore, there is no moment t' later than t in which either φ or $\neg\varphi$ is satisfied. Hence, $Fp \vee F\neg p$ is not satisfied at the last moment.

Based on Theorem 7, we can conclude that intuitionistic tense logic is suitable for describing an indeterministic world, even when the semantic time is linear, dense, and without a last moment. Sufficient evidence to support the thesis of indeterminism is to indicate at least one indeterministic event. The fact that the formula $Fp \vee F\neg p$ is not a tautology in intuitionistic tense logic, even when the time is linear, dense, justifies the statement that in the language of IT_m, it is possible to express that an event is indeterministic, although it may be the case that there are no such events in the real world. However, intuitionistic tense logic does not make a definitive judgment on this matter. With the language of this logic, it is possible to describe both indeterministic and deterministic events.

Now let us consider the relationship between the expressed thesis of determinism, i.e., $F\varphi \vee F\neg\varphi$, and the Aristotelian example of a deterministic thesis, namely, "Tomorrow there will be a sea battle or there will not be a sea battle." The Aristotelian example cannot be encoded in the language of IT_m because in this language it is possible to express "there will be an event Z in the future," but it is not possible to express "tomorrow there will be an event Z." However, we can imagine a language in which, in addition to the operator F, there exists an operator F^T[10] that is read as "tomorrow φ." If p encodes the expression "sea battle," then in the language with the operator F^T, the Aristotelian statement "tomorrow there will be a sea battle or there will not be a sea battle" would be written as $F^T p \vee F^T \neg p$.

The truth of the statement $F^T p \vee F^T \neg p$ implies the truth of the statement $Fp \vee F\neg p$, but not vice versa. The fact that $Fp \vee F\neg p$ is true does not imply the truth of $F^T p \vee F^T \neg p$. The truth of $F^T p \vee F^T \neg p$ is a sufficient condition for the truth of $Fp \vee F\neg p$. On the other hand, the falsehood of $Fp \vee F\neg p$ implies the falsehood of $F^T p \vee F^T \neg p$. Therefore, the truth of $Fp \vee F\neg p$ is a necessary condition for the truth of $F^T p \vee F^T \neg p$.

Since $F^T p \vee F^T \neg p$ is not an expression in the language of IT_m, within the IT_m system itself it is not possible to examine the syntactic relationships between the expressions $Fp \vee F\neg p$ and $F^T p \vee F^T \neg p$. However, intuitively, $Fp \vee F\neg p$ semantically follows from $F^T p \vee F^T \neg p$. The fact that tomorrow there will be either p or $\neg p$ is sufficient to conclude that there is a moment in the future in which either p or $\neg p$ is satisfied. However, the reverse is not true. The fact that there is a moment in the future in which either p or $\neg p$ is satisfied is not sufficient to conclude that either p or $\neg p$ will be satisfied exactly tomorrow.

Taking into account the semantic implication between the statements $Fp \vee F\neg p$ and $F^T p \vee F^T \neg p$, we observe that the rejection of $Fp \vee F\neg p$ entails the rejection of $F^T p \vee F^T \neg p$. Since in IT_m the statement $Fp \vee F\neg p$ is not a tautology, we conclude that the Aristotelian argument for rejecting determinism can be reconstructed within IT_m.

Indeterminists do not postulate that all events must be indeterministic. If all events in the real world were indeterministic, everything that happened would be a result of chance. Chaos would prevail. Therefore, the existence of deterministic events or moderately deterministic events is permissible. Moderately deterministic events are those whose occurrence is determined but not from the beginning of the world. Such events follow specific laws and create causal chains. However, the existence of indeterministic events is not excluded. This is in line with the views of Zbigniew Jordan, who writes:

> *...determinism is a specific interpretation of the principle of causality. We do not contradict ourselves if, after rejecting determinism, we still acknowledge the limited value of the principle of causality. It is possible to agree with the idea that nothing happens without prior causes and at the same time not accept the assumption that everything is the result of eternal causes.*

NOTES

[1] Cf. K. Trzęsicki, *Intuitionism and Indeterminism...*, 1994, pp. 271–296.

[2] Intuitively, n is understood as the present moment in time.

[3] The view that there exists an event for every moment in time, such that moments of time are eventfully non-empty and cannot exist without events occurring in them, has strong theoretical and experimental grounds in both special and general relativity theories. (See Z. Augustynek, *Własności czasu*, Państwowe Wydawnictwo Naukowe, Warszawa, 1972, p. 23).

This view is opposed to Newton's position, according to which, as Z. Augustynek writes, *...objects exist independently (of anything) in contrast to their properties, which exist dependent on objects. Therefore, moments of time, being objects, exist independently of events. Thus, moments can exist without events occurring in them, i.e., there can be eventless moments, empty ones. Consequently, time as a specific object, being a set of moments, exists independently of the material world. Therefore, time can also exist without the material world, i.e., it can be empty, and probably some of its elements, i.e., moments, are eventless.* See Z. Augustynek, *Własności czasu*, p. 21.

[4] By **unchanging time** (in the adopted terminology) is meant such a time that for any $i, j \in I$, it holds that $(T_i = T_j$ and $R_i = R_j)$.

[5] Cf. e.g., K. Trzęsicki, *Intuitionism and indeterminism...*, 1994.

[6] The remaining specific axioms of IT_m (H1, G1, H3, G3) are specific axioms of K_t.

[7] In the literature, the problem of why mathematics is so effective in describing the real world is discussed (cf. P. J. Davis, *The World of Mathematics*). The basis for posing such a problem is the acceptance of the view that mathematical sentences "say something" about reality. This is a position opposed to that presented by Zbigniew Jordan in the quoted passage.

[8] Cf. Z. Jordan, *On Logical Determinism*, Studia Logica, XIV, 1963, p. 65.

[9] This holds true for both intuitionistic and classical tense logic without a last moment of time.

[10] This is an example of a metric tense operator.

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