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GENTZEN AND TEMPORAL SEQUENTS

Abstract. The paper presents a comparison of two generalised sequent calculi for temporal logics. In both cases the main technical solution is the multiplication of the sorts of sequents and, additionally, the application of some kind of labelling to formulae. The first approach was proposed by Kaziemierz Trzęsicki at the 1980s. The second, called Multiple Sequent Calculus (MSC), was proposed in the beginning of the present century. Both approaches are examples of the family of multisequent calculi.

Keywords: sequent calculi, temporal logics, multisequent calculi.

Dedicated to Professor Kazimierz Trzęsicki

1. Introduction

Modern proof theory is based on two forms of calculi: natural deduction and sequent calculi, developed independently by Stanisław Jaśkowski [Jaśkowski 34] (natural deduction) and Gerhard Gentzen [Gentzen 34] (both kinds). The groundbreaking work of Gentzen, "Untersuchungen über das Logische Schliessen", was published in *Mathematische Zeitschrift* in 1935 and was translated into Polish by Kazimierz Trzęsicki [Trzęsicki 80][1]. Professor Trzęsicki was not only a translator of one of the most important works in proof theory; a few years later he also applied Gentzen's methodology for the construction of one of the earliest sequent calculi for temporal logics [Trzęsicki 84]. Since the original form of sequent calculus already showed several limitations while applied to non-classical logics, including temporal ones, it was necessary to introduce some modifications into Gentzen's framework. Nowadays, a lot of types of generalised sequent calculi are known, but in the 1980s the system proposed by professor Trzęsicki was one of the first of this kind. In the paper, we recall this formalisation of temporal logics and we try to present it in the wider perspective of several kinds of generalised sequent calculi, but paying attention only to those systems which have been applied to temporal logics. In the next section we briefly characterise Gentzen's calculi, then we describe the most important variants of their generalizations. One of them, not very popular, is based on the idea of introducing several kinds of sequents which have some non-standard properties. We use the name multisequent or multiple sequent calculi for this family. In section 4 we present the original sequent calculus of Trzęsicki which belongs to this class. Next we recall the proposal by Indrzejczak which is also of this kind. We close the paper with some comparative remarks.

2. Gentzen's Sequent Calculus

The abovementioned paper of Gentzen [Gentzen 34] is essentially devoted to the construction of a natural deduction system. However, to obtain some theoretical results, he introduced an auxiliary calculus in which the basic objects are not formulae but sequents of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sequences of formulae. Soon, Gentzen himself started to use this form of calculus as more convenient in theoretical research. Later, many well known logicians, like Kleene and Curry, followed him and developed the basics of proof theory in this framework. It seems that the great utility of sequent calculus is connected with the fact that sequents may be interpreted in several ways. For example:

1. as an implication of the form $\bigwedge \Gamma \rightarrow \bigvee \Delta$ (applied by Gentzen to show the equivalence of sequent calculus with axiomatic system);
2. $\Rightarrow$ understood as a dependence of some chosen formula from Δ on assumptions collected in Γ;
3. $\Rightarrow$ as the symbol of inference relation;
4. interpretation of sequents as conjunctions of the form $\bigwedge \Gamma \wedge \bigwedge \neg\Delta$;
5. interpretation of sequents as disjunctions of the form $\bigvee \neg\Gamma \vee \bigvee \Delta$.

It is worth observing that in the case of interpretations 4 and 5 the rules of sequent calculus read upside down (from the conclusion to premisses) correspond to other procedures of checking the correctness. In the case of interpretation 4 a proof tree provides a formal representation of indirect checking, and in the case of interpretation 5 it becomes a formal representation of the procedure of Post.

Both Γ and Δ in sequents of the form $\Gamma \Rightarrow \Delta$ may be empty, then Γ is interpreted as $\top$ and Δ as $\bot$. As a result:

1. a sequent with empty Γ and Δ is contradictory: $\Rightarrow := \bot$
2. $\Gamma \Rightarrow$ means that elements of Γ are inconsistent;
3. $\Rightarrow \Delta$ means that at least one formula from Δ is provable.

Gentzen's sequent calculus for classical logic LK consists of two types of rules: structural which define admissible operations on sequents, and logical which characterise logical constants. We present only the version for classical propositional logic CPL.

Structural rules:

$(AX)\quad \varphi \Rightarrow \varphi$

$$(Cut)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(W{\Rightarrow})\quad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad ({\Rightarrow}W)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(C{\Rightarrow})\quad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad ({\Rightarrow}C)\quad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(P{\Rightarrow})\quad \frac{\Pi, \varphi, \psi, \Gamma \Rightarrow \Delta}{\Pi, \psi, \varphi, \Gamma \Rightarrow \Delta} \qquad\qquad ({\Rightarrow}P)\quad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi, \Pi}{\Gamma \Rightarrow \Delta, \varphi, \psi, \Pi}$$

Logical rules:

$$(\neg{\Rightarrow})\quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad ({\Rightarrow}\neg)\quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\wedge{\Rightarrow})\quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi\wedge\psi, \Gamma \Rightarrow \Delta} \qquad\qquad (\wedge{\Rightarrow})\quad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi\wedge\psi, \Gamma \Rightarrow \Delta}$$

$$({\Rightarrow}\wedge)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi\wedge\psi} \qquad\qquad (\vee{\Rightarrow})\quad \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi\vee\psi, \Gamma \Rightarrow \Delta}$$

$$({\Rightarrow}\vee)\quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi\vee\psi} \qquad\qquad ({\Rightarrow}\vee)\quad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi\vee\psi}$$

$$(\to{\Rightarrow})\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Pi \Rightarrow \Sigma}{\varphi\to\psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad\qquad ({\Rightarrow}\to)\quad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi\to\psi}$$

A proof of the thesis φ is a tree where each leaf is an axiomatic sequent, every new node is generated by means of one of the rules, and the root is a sequent $\Rightarrow \varphi$. Here is a simple example of proof in LK:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p \qquad q \Rightarrow q}{p, p \to q \Rightarrow q} \qquad r \Rightarrow r}{p, p \to q, q \to r \Rightarrow r}}{p \to q, q \to r \Rightarrow p \to r}}{p \to q \Rightarrow (q \to r) \to (p \to r)}}{\Rightarrow (p \to q) \to ((q \to r) \to (p \to r))}$$

The notion of a proof need not be restricted to proofs of theses. Arbitrary sequent $\Gamma \Rightarrow \Delta$ can be the root of a proof tree; a description of the object of such proof depends on the interpretation of a sequent.

The rule (Cut) is of particular interest. Specific instances of its applications represent several important rules, like, e.g.

- Modus Ponens – $\Rightarrow \varphi; \varphi \Rightarrow \psi \ / \ \Rightarrow \psi$
- Hypothetical Syllogism – $\chi \Rightarrow \varphi; \varphi \Rightarrow \psi \ / \ \chi \Rightarrow \psi$

On the general level (Cut) expresses a transitivity of $\Rightarrow$, and, under interpretation 4 also the principle of bivalence, in the sense: if the conjunction corresponding to the conclusion is true, then one of the concjunctions representing premisses must be true.

On the other hand (Cut) is significantly different from other rules, since it does not satisfy one of the most important properties of other rules. In particular, the logical rules satisfy the following properties:

- Subformula property: only formulae present in the conclusion and their subformulae are present in the premisses.
- Invertibility: premisses are deducible from the conclusion (not satisfied e.g. by $(W \Rightarrow)$, $(\to\Rightarrow)$).
- Symmetry: each rule should either introduce a constant to the antecedent or to the succedent.
- Separation: a rule for a constant should not exhibit any other constants in its schema.
- Explicitness: a constant should be present only in the conclusion, moreover only once.
- Independence: correctness of the rule application is not affected by any operation on the parameters (like addition or deletion of parameters in the premisses and conclusion at the same time).

From the standpoint of LK for obtaining a decision procedure for CPL, the subformula property is particularly interesting. But we can easily notice that (Cut) does not satisfy this property. This is why the main theorem of Gentzen is called the cut elimination theorem:

Every proof of $\Gamma \Rightarrow \Delta$ may be transformed into a cut-free proof.

Several consequences of this theorem involve such results as:

– decidability of propositional classical and intuitionistic logic;
– consistency proofs for classical and intuitionistic logic;
– consistency proof for arithmetic without induction.

These results essentially depend on the fact that LK after cut elimination satisfies the subformula property for all rules (not only logical ones); such a system is analytic in one sense of this word.

Let us close this section with the remark that sequent calculus, in addition to natural deduction, provides one of the most important tools for building an antirealist theory of meaning, at least for logical constants. This approach, recently developed under the heading of proof-theoretic semantics, is close to Wittgenstein's idea of defining the meaning of words in terms of their conditions of use. In spite of the troubles with the consistent development of the general programme in the framework of inferentialism, the case of defining logical constants in terms of their use in proofs, is quite satisfactorily dealt with in the setting of sequent calculus.

3. Sequent Calculus and Nonclassical Logics

Gentzen's sequent calculus provided a convenient formalisation not only for classical logic and some elementary theories, but also for intuitionistic logic. However, the first attempts to cover other nonclassical logics, like modal or many-valued ones, revealed its significant limitations. For many logics, the rules characterising additional logical constants missed several important properties formulated above. Moreover, very often it was impossible to prove the elimination of cut. In response to these problems, several generalised versions of sequent calculus were proposed. The overview of these generalisations may be found in [Wansing 99], [Wansing 02] and [Poggiolesi 11]. Among the systems which were applied to temporal logics we have:

– labelled systems, which use some kind of indexing of the formulae, for example to make the relativisation to time points explicit (e.g. [Fitting 84], [Indrzejczak 03]);

– display calculus, which introduces additional structural constants (e.g. [Belnap 82], [Wansing 99]);
– hypersequent calculus, where rules operate on the collections of sequents (e.g. [Avron 96], [Indrzejczak 19]);
– nested sequent calculus where sequents may appear as members of other sequents (e.g. [Kashima 94], [Poggiolesi 11]);
– n-argument sequent calculus, where the number of arguments of sequents is enriched (e.g. [Nishimura 80]);
– multiple sequent calculi, which apply different kinds of sequents (e.g. [Curry 63], [Zeman 73]).

It is worth noting that one of the first proposal of generalised sequent calculus for temporal logics was presented by Trzęsicki [Trzęsicki 84]. It belongs to the last group of generalised calculi, i.e. to multiple sequent calculi. This family was introduced earlier but its application to temporal logics and systematic development was proposed first by Trzęsicki, and then in the papers [Indrzejczak 96], [Indrzejczak 97], [Indrzejczak 00]. Before we present these systems, we briefly recall the basic facts concerning temporal logics in the sense of Prior, which were dealt with by means of these multiple sequent calculi.

4. Temporal Logics

One can find an excellent and detailed presentation of temporal logics in Trzęsicki [Trzęsicki 08]; here we recall briefly only the most basic information and establish the notation. We focus only on propositional logics, called often tense logics or Priorean logics. We add four unary temporal constants G, F, H, P to the propositional language of CPL. Their informal reading is the following:

G for "in all future times"
F for "it will happen"
H for "in all past times"
P for "it happened"

Semantically this language is interpreted in relational models $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ where $\mathcal{W} \neq \emptyset$ is the set of time points, $\mathcal{R}$ is the earlier-later relation, and V is the valuation function for propositional variables, i.e. $V : VAR \longrightarrow \mathcal{P}(\mathcal{W})$.

Satisfaction of formulae φ in the point w of a model $\mathfrak{M}$ ($\mathfrak{M}, w \models \varphi$) is defined in the following way:

$\mathfrak{M}, w \models \varphi$	iff	$w \in V(\varphi)$ for any $\varphi \in VAR$
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$
$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi \vee \psi$	iff	$\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi \to \psi$	iff	$\mathfrak{M}, w \not\models \varphi$ or $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models G\varphi$	iff	$\mathfrak{M}, w' \models \varphi$ for any w' such that $\mathcal{R}(ww')$
$\mathfrak{M}, w \models F\varphi$	iff	$\mathfrak{M}, w' \models \varphi$ for some w' such that $\mathcal{R}(ww')$
$\mathfrak{M}, w \models H\varphi$	iff	$\mathfrak{M}, w' \models \varphi$ for any w' such that $\mathcal{R}(w'w)$
$\mathfrak{M}, w \models P\varphi$	iff	$\mathfrak{M}, w' \models \varphi$ for some w' such that $\mathcal{R}(w'w)$

The set of all models characterises temporal logic Kt in the sense that the set of its tautologies coincides with the set of formulae satisfied in all points of all models. Kt is the weakest temporal logic and has rather technical importance since no interesting features of time are expressed here. Reasonable temporal logics, which represent several theories of time require the restriction of the classes of models to those which satisfy some properties of $\mathcal{R}$, like transitivity or discreteness. Restricting the class of models, we obtain richer sets of tautologies, which are extensions of Kt. In the table we list some properties of $\mathcal{R}$ which are attributed to logics considered in the remaining parts of the paper:

Table 1

Name	Condition	Axiom
F-seriality	$\forall x \exists y \mathcal{R}xy$	(D^+)
P-seriality	$\forall y \exists x \mathcal{R}xy$	(D^-)
transitivity	$\forall xyz(\mathcal{R}xy \wedge \mathcal{R}yz \to \mathcal{R}xz)$	(4F), (4P)
density	$\forall xy(\mathcal{R}xy \to \exists z(\mathcal{R}xz \wedge \mathcal{R}zy))$	(4C)
strong F-connectedenss	$\forall xyz(\mathcal{R}xy \wedge \mathcal{R}xz \to \mathcal{R}yz \vee \mathcal{R}zy)$	(3F)
weak F-connectedness	$\forall xyz(\mathcal{R}xy \wedge \mathcal{R}xz \to \mathcal{R}yz \vee \mathcal{R}zy \vee y = z)$	(LF)

In the right column we have listed the names of axioms corresponding to suitable conditions, since the logics under consideration have adequate axiomatic characterisations demanding the addition of suitable axioms and two rules to arbitrary axiomatization of CPL (with Modus Ponens):

(RG) and (RH) If $\vdash \varphi$, then $\vdash G\varphi$ and $\vdash H\varphi$.

Kt may be axiomatised by means of the following schemata:

$$G(\varphi \to \psi) \to (G\varphi \to G\psi)$$
$$H(\varphi \to \psi) \to (H\varphi \to H\psi)$$

$$\varphi \to GP\varphi$$
$$\varphi \to HF\varphi$$
$$G\varphi \leftrightarrow \neg F \neg \varphi$$
$$H\varphi \leftrightarrow \neg P \neg \varphi$$

In the case of extensions of Kt we should remember that for each property from the table we have a suitable axiom; in the case of transitivity even two:

(4F) $\quad G\varphi \to GG\varphi$ and (4P) $\quad H\varphi \to HH\varphi$

To obtain Kt4 it is sufficient to add to Kt only one of them since the other is provable. The remaining axioms:

(D^+) $\quad G\varphi \to F\varphi$
(D^-) $\quad H\varphi \to P\varphi$
$(4C)$ $\quad GG\varphi \to G\varphi$ [or $HH\varphi \to H\varphi$]
$(3F)$ $\quad F\varphi \wedge F\psi \to F(F\varphi \wedge \psi) \vee F(\varphi \wedge F\psi)$
(LF) $\quad F\varphi \wedge F\psi \to F(F\varphi \wedge \psi) \vee F(\varphi \wedge F\psi) \vee F(\varphi \wedge \psi)$

By their addition to Kt we can obtain adequate axiomatisations of stronger logics in a modular way. For example, adding $(D^+), (4F), (LF)$ to Kt yields an axiomatisation of the logic of transitive, nonbranching and future-infinite time.

5. The Calculus of Trzęsicki

As we already mentioned, the system of Trzęsicki may be seen as belonging to a certain group of generalised sequent calculi called here multisequent calculi. The characteristic feature of this kind of generalised sequent calculi is the application of several kinds of sequents in one calculus. Such an approach started with Curry [Curry 63] in the formalisation of modal logics S4 and S5, in which, in addition to the standard sequent, a notion of modal sequent was introduced. A similar solution was applied by Zeman [Zeman 73] to other modal logics. Both Curry's and Zeman's proposals are not fully satisfactory. For example, in the case of Curry's system a proof of cut elimination for S4 does not depend on the fact that a special modal sequent is used. On the other hand, in the case of S5, the addition of the modal sequent still does not allow us to prove cut elimination. This problem was resolved much later in [Indrzejczak 96]. The application of specific kinds of sequents for the characterisation of temporal logics seems to be specific for Polish attempts, since one may find only two such solutions, in [Trzęsicki 84] and [Indrzejczak 00]. We describe both approaches which belong to the same family but develop totally different calculi.

Since the system of Trzęsicki belongs to generalised sequent calculi with the proliferation of sequents, we should ask what kinds of sequents we find there. In the case of formalisation of the logic Kt we have the following sorts of sequents:

$$\Gamma \Rightarrow \Delta, \quad \Gamma \Rightarrow^{G} \Delta, \quad \Gamma \Rightarrow^{H} \Delta.$$

The additional technical device is based on the labelling of formulae as $\varphi^{\sharp}$, which is connected with the application of temporal rules. It should be noted that temporal sequents and labelled formulae may occur only in the proof trees but are not allowed as their roots. The system consists of the following rules:

$$(G1) \quad \frac{\Gamma \Rightarrow \varphi}{G\Gamma \Rightarrow G\varphi} \qquad (H1) \quad \frac{\Gamma \Rightarrow \varphi}{H\Gamma \Rightarrow H\varphi}$$

$$(G2) \quad \frac{\Gamma, \Sigma \Rightarrow \Delta}{\Gamma^{\sharp}, G\Sigma \Rightarrow^{H} \Delta^{\sharp}} \qquad (H2) \quad \frac{\Gamma, \Sigma \Rightarrow \Delta}{\Gamma^{\sharp}, H\Sigma \Rightarrow^{G} \Delta^{\sharp}}$$

$$(G3) \quad \frac{\Gamma^{\sharp}, \Sigma \Rightarrow^{G} \Delta, \Theta^{\sharp}}{\Gamma, G\Sigma \Rightarrow G\Delta, \Theta} \qquad (H3) \quad \frac{\Gamma^{\sharp}, \Sigma \Rightarrow^{H} \Delta, \Theta^{\sharp}}{\Gamma, H\Sigma \Rightarrow H\Delta, \Theta}$$

$$(G4) \quad \frac{\Gamma^{\sharp}, \Sigma, \Lambda \Rightarrow^{G} \Delta, \Theta^{\sharp}}{\Gamma^{\sharp}, G\Sigma^{\sharp}, \Lambda \Rightarrow^{G} \Delta, \Theta^{\sharp}} \qquad (H4) \quad \frac{\Gamma^{\sharp}, \Sigma, \Lambda \Rightarrow^{H} \Delta, \Theta^{\sharp}}{\Gamma^{\sharp}, H\Sigma^{\sharp}, \Lambda \Rightarrow^{H} \Delta, \Theta^{\sharp}}$$

where in $(G3)$, Δ contains at least one element or there is a formula φ such that $\neg G\varphi$ is an element of Γ or $G\varphi$ is an element of Θ, and in $(H3)$ Δ contains at least one element or there is a formula φ such that $\neg H\varphi$ is an element of Γ or $H\varphi$ is an element of Θ.

Structural rules (except (Cut)) and logical rules for exstensional constants (except $(\rightarrow\Rightarrow)$) work on temporal sequents of the type $\Rightarrow^{G}, \Rightarrow^{H}$ in the same way as in the case of standard sequents. In the case of cut, in addition to its standard form we have:

$$(Cut-a) \quad \frac{\Gamma^{\sharp}, \Theta \Rightarrow^{G} \Delta^{\sharp}, \Xi, \varphi \qquad \varphi, \Pi^{\sharp}, \Lambda \Rightarrow^{G} \Sigma^{\sharp}, \Upsilon}{\Gamma^{\sharp}, \Theta, \Pi^{\sharp}, \Lambda \Rightarrow^{G} \Delta^{\sharp}, \Xi, \Sigma^{\sharp}, \Upsilon}$$

$$(Cut-b) \quad \frac{\Gamma^{\sharp} \Rightarrow \varphi \qquad \varphi, \Pi^{\sharp}, \Lambda \Rightarrow^{G} \Sigma^{\sharp}, \Upsilon}{G\Gamma^{\sharp}, \Pi^{\sharp}, \Lambda \Rightarrow^{G} \Sigma^{\sharp}, \Upsilon}$$

$$(Cut-c)\quad \frac{\Gamma^\sharp, \Theta \Rightarrow^G \Delta^\sharp, \Xi, \varphi \qquad \varphi, \Pi \Rightarrow \psi}{\Gamma^\sharp, \Theta, G\Pi^\sharp \Rightarrow^G \Delta^\sharp, \Xi, \psi}$$

$$(Cut-d)\quad \frac{\Gamma^\sharp, \Theta \Rightarrow^G \Delta^\sharp, \Xi, \varphi \qquad \varphi, \Pi \Rightarrow}{\Gamma^\sharp, \Theta, G\Pi^\sharp \Rightarrow^G \Delta^\sharp, \Xi}$$

and four dual rules with H instead of G in all schemata.

In the case of the rule for the introduction of $\rightarrow$ to the antecedent, in addition to the standard rule working only with standard sequents, we have:

$$(\rightarrow\Rightarrow a)\quad \frac{\Gamma^\sharp, \Theta \Rightarrow^G \Delta^\sharp, \Xi, \varphi \qquad \psi, \Pi^\sharp, \Lambda \Rightarrow^G \Sigma^\sharp, \Upsilon}{\varphi \rightarrow \psi, \Gamma^\sharp, \Theta, \Pi^\sharp, \Lambda \Rightarrow^G \Delta^\sharp, \Xi, \Sigma^\sharp, \Upsilon}$$

$$(\rightarrow\Rightarrow b)\quad \frac{\Gamma \Rightarrow \varphi \qquad \psi, \Pi^\sharp, \Lambda \Rightarrow^G \Sigma^\sharp, \Upsilon}{\varphi \rightarrow \psi, G\Gamma^\sharp, \Pi^\sharp, \Lambda \Rightarrow^G \Sigma^\sharp, \Upsilon}$$

$$(\rightarrow\Rightarrow c)\quad \frac{\Gamma^\sharp, \Theta \Rightarrow^G \Delta^\sharp, \Xi, \varphi \qquad \psi, \Pi \Rightarrow}{\varphi \rightarrow \psi, \Gamma^\sharp, \Theta, G\Pi^\sharp \Rightarrow^G \Delta^\sharp, \Xi}$$

and three dual rules for H.

Before we present the extensions of Kt let us provide a simple example of proof:

$$\begin{array}{ll}
(AX) & \underline{p \Rightarrow p} \\
(G1) & \underline{Gp \Rightarrow Gp} \\
(\neg\Rightarrow) & \underline{Gp, \neg Gp \Rightarrow} \\
(H2) & \underline{Gp^\sharp, H\neg Gp \Rightarrow^G} \\
(\Rightarrow\neg) & \underline{Gp^\sharp \Rightarrow^G \neg H \neg Gp} \\
(G3) & \underline{Gp \Rightarrow G\neg H \neg Gp} \\
(\Rightarrow\rightarrow) & \Rightarrow Gq \rightarrow G\neg H \neg Gp
\end{array}$$

Kt is the weakest logic of rather technical value. For Kt4 we need additional temporal rules:

$$(GC1)\quad \frac{G\Gamma, \Delta \Rightarrow \varphi}{G\Gamma, G\Delta \Rightarrow G\varphi} \qquad (HC1)\quad \frac{H\Gamma, \Delta \Rightarrow \varphi}{H\Gamma, H\Delta \Rightarrow H\varphi}$$

$$(GC2)\quad \frac{\Gamma \Rightarrow G\Delta, \Theta}{\Gamma^\sharp \Rightarrow^G G\Delta, \Theta^\sharp} \qquad (HC2)\quad \frac{\Gamma \Rightarrow H\Delta, \Theta}{\Gamma^\sharp \Rightarrow^H H\Delta, \Theta^\sharp}$$

$$(GC3)\quad \frac{\Gamma, G\Sigma, \Lambda \Rightarrow \Delta}{\Gamma^\sharp, G\Sigma, G\Lambda \Rightarrow^H \Delta^\sharp} \qquad (HC3)\quad \frac{\Gamma, H\Sigma, \Lambda \Rightarrow \Delta}{\Gamma^\sharp, H\Sigma, H\Lambda \Rightarrow^G \Delta^\sharp}$$

and two new versions of cut:

$$(Cut4)\quad \frac{G\Gamma, \Theta \Rightarrow \varphi \quad \varphi, \Pi^\sharp, \Lambda \Rightarrow^G \Sigma^\sharp, \Upsilon}{G\Gamma^\sharp, G\Theta^\sharp, \Pi^\sharp, \Lambda \Rightarrow^G \Sigma^\sharp, \Upsilon}$$

with a dual rule having H instead of G.

To obtain extensions adequate for axioms D^+ and D^- we must use the following rules:

$$(GD^+1)\quad \frac{\Sigma \Rightarrow}{G\Sigma \Rightarrow} \qquad (GD^+2)\quad \frac{\Gamma^\sharp, \Sigma \Rightarrow^G \Theta^\sharp}{\Gamma, G\Sigma \Rightarrow \Theta}$$

and dual rules for D^-.

The most involved modifications are needed to obtain a formalisation of the logic of continuous time; we need not only new rules but also two new types of temporal sequents: $\Rightarrow_G, \Rightarrow_H$. Structural rules (except (Cut)) and logical rules with one premiss apply to these new sequents as well. Additionally we have temporal rules:

$$(GD1)\quad \frac{\Gamma \Rightarrow G\varphi}{G\Gamma \Rightarrow G\varphi} \qquad (HD1)\quad \frac{\Gamma \Rightarrow H\varphi}{H\Gamma \Rightarrow H\varphi}$$

$$(GD2)\quad \frac{\Gamma^\sharp, \Sigma \Rightarrow^G \Delta, G\Sigma, \Theta^\sharp}{\Gamma, G\Sigma \Rightarrow G\Delta, G\Sigma, \Theta^\sharp} \qquad (HD2)\quad \frac{\Gamma^\sharp, \Sigma \Rightarrow^H \Delta, H\Sigma, \Theta^\sharp}{\Gamma, H\Sigma \Rightarrow H\Delta, H\Sigma, \Theta^\sharp}$$

$$(GD3)\quad \frac{\Gamma, G\Sigma \Rightarrow \Delta}{\Gamma^\sharp, G\Sigma \Rightarrow_G \Delta^\sharp} \qquad (HD3)\quad \frac{\Gamma, H\Sigma \Rightarrow \Delta}{\Gamma^\sharp, H\Sigma \Rightarrow_H \Delta^\sharp}$$

$$(GD4)\quad \frac{\Gamma^\sharp, \Sigma \Rightarrow_G \Theta^\sharp}{\Gamma, G\Sigma \Rightarrow \Theta} \qquad (HD4)\quad \frac{\Gamma^\sharp, \Sigma \Rightarrow_H \Theta^\sharp}{\Gamma, H\Sigma \Rightarrow \Theta}$$

and additional rules of cut:

$$(Cut-a)\quad \frac{\Gamma \Rightarrow \varphi \quad \varphi, \Pi^\sharp, \Lambda \Rightarrow_G \Sigma^\sharp, \Upsilon}{\Gamma, \Pi^\sharp, \Lambda \Rightarrow_G \Sigma^\sharp, \Upsilon}$$

$$(Cut-b)\quad \frac{\Gamma^\sharp, \Theta \Rightarrow_G \Delta^\sharp, \Xi, \varphi \quad \varphi, \Pi \Rightarrow}{\Gamma^\sharp, \Theta, \Pi \Rightarrow_G \Delta^\sharp, \Xi}$$

$$(Cut-c)\quad \frac{\Gamma^\sharp, \Theta \Rightarrow^G \Delta^\sharp, \Xi, \varphi \quad \varphi, \Pi^\sharp, \Lambda \Rightarrow_G \Sigma^\sharp, \Upsilon}{\Gamma^\sharp, \Theta, \Pi^\sharp, G\Lambda^\sharp, G\neg\Upsilon^\sharp \Rightarrow^G \Delta^\sharp, \Sigma^\sharp, \Xi}$$

$$(Cut-d)\quad \frac{\Pi^\sharp, \Lambda \Rightarrow_G \Sigma^\sharp, \Upsilon, \varphi \quad \varphi, \Gamma^\sharp, \Theta \Rightarrow^G \Delta^\sharp, \Xi}{\Gamma^\sharp, \Theta, \Pi^\sharp, G\Lambda^\sharp, G\neg\Upsilon^\sharp \Rightarrow^G \Delta^\sharp, \Sigma^\sharp, \Xi}$$

as well as four dual rules with H instead of G in all schemata.

All presented formalisations are adequate in the sense of satisfying the following theorems for every considered logic:

Theorem 1

If $\vdash_L \varphi$, then $\vdash_{GL} \Rightarrow \varphi$

Theorem 2

If $\vdash_{GL} \Gamma \Rightarrow \Delta$, then $\vdash_L \bigwedge \Gamma \rightarrow \bigvee \Delta$

Moreover, for all systems it holds:

Theorem 3

If $\vdash_{GL} \Gamma \Rightarrow \Delta$, then it is cut-free provable.

6. MSC – Multiple Sequent Calculus

In the 1990s Indrzejczak started systematic development of the multi-sequent approach to several modal logics called MSC. Some of them required the addition of a small and fixed number of additional sequents; for example, in the case of S5, to obtain its cut-free formalisation we need only one modal sequent, and in the case of such logics as KB, KTB, two types of modal sequents are enough. Temporal logics appeared particularly difficult; The calculus MSC for them needs a countable number of types of sequents of the form:

$\Gamma \ < i, j > \Rightarrow \Delta$, for $i, j \geq 0$

in particular $\Gamma < 0, 0 > \Rightarrow \Delta$ represents standard sequent.

Intuitively, indices of the form $< i, j >$ characterising a type of a sequent convey information that formulae in the antecedent are satisfied in the time point which is the ith successor of some starting point, and the formulae from the succedent are unsatisfied in the jth successor of the starting point. In particular $< 0, 0 >$ means that formulae from the antecedent and the formulae from the succedent are satisfied/unsatisfied in the same starting point in the attempted model.

Additionally, formulae are signed in the course of the proof with the help of a polarisation operator $-$. Formulae of the shape $-\varphi$ are called S-formulae. Let us note that $-$ cannot be iterated and S-formulae cannot be combined by means of connectives (expressions like $--\varphi, \varphi \vee -\psi$ are not admissible). S-formulae cannot also occur in the root sequent. The union of ordinary formulae and S-formulae is the set of MSC-formulae. The following abbreviations will be used in the description of rules:

for any MSC-formula ψ, $\psi^* = -\psi$ and $(-\psi)^* = \psi$;

$\Gamma^* = \{\varphi^* : \varphi \in \Gamma\}$;

$F^i\varphi$ means that φ is preceded ith times with F; the same applies to G, H, P.

The system for Kt contains the following structural rules:

(AX) $\quad \varphi \ < 0,0 >\Rightarrow \varphi$

$(W\Rightarrow)$ $\quad \dfrac{\Gamma \ < i,j >\Rightarrow \Delta}{\varphi, \Gamma \ < i,j >\Rightarrow \Delta}$ $\qquad$ $(\Rightarrow W)$ $\quad \dfrac{\Gamma \ < i,j >\Rightarrow \Delta}{\Gamma \ < i,j >\Rightarrow \Delta, \varphi}$

$(\Rightarrow^*)$ $\quad \dfrac{\varphi, \Gamma \ < 0,0 >\Rightarrow \Delta}{\Gamma \ < 0,0 >\Rightarrow \Delta, \varphi^*}$ $\qquad$ $(^*\Rightarrow)$ $\quad \dfrac{\Gamma \ < 0,0 >\Rightarrow \Delta, \varphi}{\varphi^*, \Gamma \ < 0,0 >\Rightarrow \Delta}$

(TR) $\quad \dfrac{\Gamma \ < i,j >\Rightarrow \Delta}{\Delta^* \ < j,i >\Rightarrow \Gamma^*}$

(CUT) $\quad \dfrac{\Gamma \ < 0,0 >\Rightarrow \Delta, \varphi \qquad \varphi, \Pi \ < 0,0 >\Rightarrow \Sigma}{\Gamma, \Pi \ < 0,0 >\Rightarrow \Delta, \Sigma}$

(KF) $\quad \dfrac{\Gamma \ < i,0 >\Rightarrow}{\Gamma \ < i+1,0 >\Rightarrow}$ $\qquad$ (KG) $\quad \dfrac{< 0,j >\Rightarrow \Delta}{< 0,j+1 >\Rightarrow \Delta}$

(KP) $\quad \dfrac{\Gamma \ < 0,j >\Rightarrow}{\Gamma < 0,j+1 >\Rightarrow}$ $\qquad$ (KH) $\quad \dfrac{< i,0 >\Rightarrow \Delta}{< i+1,0 >\Rightarrow \Delta}$

Extensional rules:

$(\neg\Rightarrow)$ $\quad \dfrac{-\varphi, \Gamma \ < i,j >\Rightarrow \Delta}{\neg\varphi, \Gamma \ < i,j >\Rightarrow \Delta}$ $\qquad$ $(\Rightarrow\neg)$ $\quad \dfrac{\Gamma \ < i,j >\Rightarrow \Delta, -\varphi}{\Gamma \ < i,j >\Rightarrow \Delta, \neg\varphi}$

$(\wedge\Rightarrow)$ $\quad \dfrac{\varphi, \psi, \Gamma \ < i,j >\Rightarrow \Delta}{\varphi\wedge\psi, \Gamma \ < i,j >\Rightarrow \Delta}$

$(\Rightarrow\wedge)$ $\quad \dfrac{\Gamma \ < i,j >\Rightarrow \Delta, \varphi \qquad \Pi \ < i,j >\Rightarrow \Sigma, \psi}{\Gamma, \Pi \ < i,j >\Rightarrow \Delta, \Sigma, \varphi\wedge\psi}$

$(\rightarrow\Rightarrow)$ $\quad \dfrac{-\varphi, \Gamma \ < i,j >\Rightarrow \Delta \qquad \psi, \Pi \ < i,j >\Rightarrow \Sigma}{\varphi\rightarrow\psi, \Gamma, \Pi \ < i,j >\Rightarrow \Delta, \Sigma}$

$(\Rightarrow\rightarrow)$ $\quad \dfrac{\Gamma \ < i,j >\Rightarrow \Delta, -\varphi, \psi}{\Gamma \ < i,j >\Rightarrow \Delta, \varphi\rightarrow\psi}$

Temporal rules:

$(\mathrm{G}\Rightarrow)$ $\quad \dfrac{\varphi \ < 0,j >\Rightarrow \Delta}{\mathrm{G}\varphi \ < 0,j+1 >\Rightarrow \Delta}$ $\qquad$ $(\Rightarrow\mathrm{G})$ $\quad \dfrac{\Gamma \ < i,j+1 >\Rightarrow \varphi}{\Gamma \ < i,j >\Rightarrow \mathrm{G}\varphi}$

$$(\mathrm{F}\Rightarrow)\quad \frac{\varphi\ <i+1,j>\Rightarrow \Delta}{\mathrm{F}\varphi\ <i,j>\Rightarrow \Delta} \qquad\qquad (\Rightarrow\mathrm{F})\quad \frac{\Gamma\ <i,0>\Rightarrow \varphi}{\Gamma\ <i+1,0>\Rightarrow \mathrm{F}\varphi}$$

$$(\mathrm{H}\Rightarrow)\quad \frac{\varphi\ <i,j>\Rightarrow \Delta}{\mathrm{H}\varphi\ <i+1,j>\Rightarrow \Delta} \qquad\qquad (\Rightarrow\mathrm{H})\quad \frac{\Gamma\ <i+1,0>\Rightarrow \varphi}{\Gamma\ <i,0>\Rightarrow \mathrm{H}\varphi}$$

$$(\mathrm{P}\Rightarrow)\quad \frac{\varphi\ <0,j+1>\Rightarrow \Delta}{\mathrm{P}\varphi\ <0,j>\Rightarrow \Delta} \qquad\qquad (\Rightarrow\mathrm{P})\quad \frac{\Gamma\ <i,j>\Rightarrow \varphi}{\Gamma\ <i,j+1>\Rightarrow \mathrm{P}\varphi}$$

$$(\mathrm{G}^j \Rightarrow)\quad \frac{\Gamma\ <0,j>\Rightarrow \Delta, -\varphi}{\mathrm{G}^j\varphi, \Gamma\ <0,j>\Rightarrow \Delta} \qquad\qquad (\Rightarrow\mathrm{F}^i)\quad \frac{-\varphi, \Gamma\ <i,0>\Rightarrow \Delta}{\Gamma\ <i,0>\Rightarrow \Delta,\ \mathrm{F}^i\varphi}$$

$$(\mathrm{H}^i \Rightarrow)\quad \frac{\Gamma\ <i,0>\Rightarrow \Delta, -\varphi}{\mathrm{H}^i\varphi, \Gamma\ <i,0>\Rightarrow \Delta} \qquad\qquad (\Rightarrow\mathrm{P}^j)\quad \frac{-\varphi, \Gamma\ <0,j>\Rightarrow \Delta}{\Gamma\ <0,j>\Rightarrow \Delta,\ \mathrm{P}^j\varphi}$$

The example of proof in Kt:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p\ <0,0>\Rightarrow p\ (AX)}{p\ <1,0>\Rightarrow \mathrm{F}p}\,(\Rightarrow\mathrm{F})}{p\ <2,0>\Rightarrow \mathrm{F}p}\,(4)}{\mathrm{F}p\ <1,0>\Rightarrow \mathrm{F}p}\,(\mathrm{F}\Rightarrow) \qquad \dfrac{\dfrac{q\ <0,0>\Rightarrow q\ (AX)}{q\ -q\ <0,0>\Rightarrow}\,(^*\Rightarrow)}{q\ -q\ <1,0>\Rightarrow}\,(KF)}{\mathrm{F}p\vee q\ -q\ <1,0>\Rightarrow \mathrm{F}p}\,(\vee\Rightarrow)}{-\mathrm{F}p\ <0,1>\Rightarrow q\ -(\mathrm{F}p\vee q)}\,(TR)}{-\mathrm{F}p\ \mathrm{G}(\mathrm{F}p\vee q)\ <0,1>\Rightarrow\ q}\,(\mathrm{G}^1\Rightarrow)}{\mathrm{G}(\mathrm{F}p\vee q)\ <0,1>\Rightarrow\ q\ \mathrm{PF}p}\,(\Rightarrow\mathrm{P}^1)}{\mathrm{G}(\mathrm{F}p\vee q)\ <0,2>\Rightarrow\ q\ \mathrm{PF}p}\,(4)}{\mathrm{PG}(\mathrm{F}p\vee q)\ <0,1>\Rightarrow\ q\ \mathrm{PF}p}\,(\mathrm{P}\Rightarrow)}{\mathrm{PG}(\mathrm{F}p\vee q)\ <0,1>\Rightarrow\ q\vee\ \mathrm{PF}p}\,(\Rightarrow\vee)}{\mathrm{PG}(\mathrm{F}p\vee q)\ <0,0>\Rightarrow \mathrm{G}(q\vee\ \mathrm{PF}p)}\,(\Rightarrow\mathrm{G})
$$

To obtain an adequate formalisation of some extensions of Kt we have to add the following structural rules:

$$(41)\quad \frac{\Gamma\ <i,0<j>\Rightarrow \Delta}{\Gamma\ <i,0<j+1>\Rightarrow \Delta} \qquad\qquad (42)\quad \frac{\Gamma\ <0<i,j>\Rightarrow \Delta}{\Gamma\ <0<i+1,j>\Rightarrow \Delta}$$

$$(D^{+}1)\quad \frac{\Gamma\ <i,j+1>\Rightarrow}{\Gamma\ <i,j>\Rightarrow} \qquad\qquad (D^{+}2)\quad \frac{<i+1,j>\Rightarrow \Delta}{<i,j>\Rightarrow \Delta}$$

$$(D^{-}1)\quad \frac{\Gamma\ <i+1,j>\Rightarrow}{\Gamma <i,j>\Rightarrow} \qquad\qquad (D^{-}2)\quad \frac{<i,j+1>\Rightarrow \Delta}{<i,j>\Rightarrow \Delta}$$

$$(4C1)\quad \frac{\Gamma\ <i>0,j+1>\Rightarrow \Delta}{\Gamma\ <i>0,j>\Rightarrow \Delta} \qquad\qquad (4C2)\quad \frac{\Gamma\ <i+1,j>0>\Rightarrow \Delta}{\Gamma\ <i,j>0>\Rightarrow \Delta}$$

$$(3F) \quad \frac{\Gamma \ <0,1>\Rightarrow \Delta \quad \Gamma \ <1,0>\Rightarrow \Delta}{\Gamma \ <1,1>\Rightarrow \Delta}$$

$$(LF) \quad \frac{\Gamma \ <0,0>\Rightarrow \Delta \quad \Gamma \ <0,1>\Rightarrow \Delta \quad \Gamma \ <1,0>\Rightarrow \Delta}{\Gamma \ <1,1>\Rightarrow \Delta}$$

To show the adequacy of MSC with respect to considered logics we need the following translation functions τ and $\mathcal{T}$ from the set of formulae and sequents into the set of formulae:

$\tau(-\varphi) = \neg\varphi \quad \tau(\varphi) = \varphi$

$\mathcal{T}(\Gamma < i, j >\Rightarrow \Delta) = \mathrm{F}^i(\bigwedge \tau(\Gamma)) \rightarrow \mathrm{G}^j(\bigvee \tau(\Delta))$ where $\tau(\Gamma) = \{\tau(\varphi) : \varphi \in \Gamma\}$

By induction on the height of proof and by provability of axioms we obtain:

Theorem 4

MSC-L $\vdash\Rightarrow \varphi$, iff $\vdash \varphi$, for any logic L, axiomatised by means of axioms listed in section 4.

Let us note that all rules, except cut, satisfy the subformula property of the form: if formula φ or $-\varphi$ occurs in the premisses, then φ or $-\varphi$ occurs in the conclusion, either as a formula or as a subformula φ of some other formula in the conclusion.

The problem of cut elimination in MSC is rather complex. (CUT) restricted to sequents with the index $< 0, 0 >$ is sufficient for proving the simulation of MP, hence to show the syntactic equivalence of MSC with the axiomatic formulation of respective logic. To show cut elimination syntactically such a rule is too restrictive, however. Instead we should use two rules of the form:

$$(CUT - r) \quad \frac{\Gamma \ \ < i, j >\Rightarrow \Delta, \varphi \quad \Gamma \ \ < i, j >\Rightarrow \Delta, -\varphi}{\Gamma \ \ < i, j >\Rightarrow \Delta}$$

$$(CUT - l) \quad \frac{\varphi, \Gamma \ \ < i, j >\Rightarrow \Delta \quad -\varphi, \Gamma \ \ < i, j >\Rightarrow \Delta}{\Gamma \ \ < i, j >\Rightarrow \Delta}$$

Such rules avoid the problems and allow us to apply the modified strategy of cut elimination proof used in [Indrzejczak 19]. The other solution is to show the completeness directly for the system without cut based on the procedure of proof search for the arbitrary sequent. The procedure specified in [Indrzejczak 97] may be adapted for this aim in the case of logics which do not satisfy conditions $(3F)$ or (LF). To capture linearity of time flow requires some other, more involved solutions (their examination can be found in [Indrzejczak 10]).

7. Concluding remarks

Both approaches to the formalisation of temporal logics presented above are based on the same technical device of multiplying the number of types of sequents, but they use this technique in significantly different ways. The calculus of Trzęsicki is based rather on syntactical intuitions, and as a result it needs a small and fixed number of different sequent types; three in general and five for the case of dense time flow.

In the case of MSC, rather semantical intuitions, based on the interpretation 4, are in the background. This interpretation is split here into possibly different time points represented by the antecedent and the succedent of the sequent. Since both points may be at different distances to the starting point, the temporal rules allow us to change the distance between these time points at least with respect to temporal formulae.

Both approaches can also be compared with other kinds of generalised sequent calculi applied to the formalisation of temporal logics. Indices applied in MSC provide the tool for encoding in the syntax some minimal semantical information necessary to obtain an attempted falsifying model which we are trying to provide for an unprovable sequent. From this standpoint MSC resembles labelled systems mentioned in section 3. The main difference is that in the latter this semantic information is attributed to each formula, whereas in MSC it is a kind of global qualification of a sequent. As a result, labelled systems are strongly dependent on semantics, while in the case of MSC this semantical commitment is very weak. In the case of Trzęsicki's calculus there is no straightforward dependence of the rules on relational semantics.

Since both in the case of Trzęsicki's calculus and in MSC some other structural constants are applied to formulae, we can observe some resemblance to display calculus. In the latter the whole family of additional structural constants is introduced, which is the main technique for modelling intensional elements. In the former, such additions have only an auxiliary and very modest character.

Let us finish our considerations with the remark that providing well-behaved sequent calculi for temporal logics is a particularly troublesome task. The small number of proposed systems is a good witness for this claim (see the survey in [Indrzejczak 10]). Part of the problem is connected with the symmetry of past and future expressed by the axioms $\varphi \to GP\varphi$, $\varphi \to HF\varphi$. It is not incidental that in the case of modal logics those characterised by models with symmetric accessibility relation always generated the most significant problems. It should be emphasised that the technical devices applied by Trzęsickiego or in MSC help to avoid these problems. Additional problems are generated by some other properties of time flow, like linearity. Let us note that in MSC we can formalise temporal logics in which the future branching is excluded; it is also possible to obtain the dual variant for logics with past linearity. But the apparatus of MSC does not allow us to formalise the logic with linearity holding for future and past together. The solution to this problem was provided by the author in the framework of modified hypersequent calculus ([Indrzejczak 19]).

NOTE

[1] Let me underline here that reading this translation, during my philosophical studies, was the first contact with Gentzen's writings and had a decisive influence on the development of my later interests and scientific research. Therefore I am greatly indebted to Professor Trzęsicki for his work.

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