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Magic Venn Diagrams

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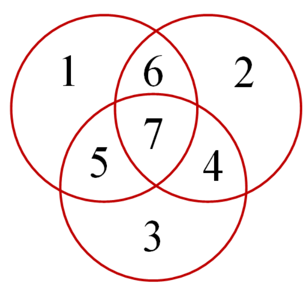
Abstract

A magic Venn diagram (MVD) of order n, size r, degree d and magic sum m is a Venn diagram of n sets $A_1, A_2, \ldots, A_n$, in which r of the 2^n non-overlapping Venn regions are weighted 1-1 by the numbers $1, 2, \ldots, r$ in such a way that each set A_i contains d weights adding up to the constant m. After demonstrating how MVDs arise and how they encompass a broad array of magic figures, we give results on the frequencies of different MVD's of orders 3 and 4, by combinatorial type and magic sum. Illustrations and explanations are given for several of the smaller types.

"Give him three pence, since he must make gain out of all that he learns"
– Euclid



1 Introduction

For over 4,000 years magic squares and other figures of their kind have delighted puzzlers, soothsayers and mathematicians the world over.[1] Their popularity is evidently not

[1]According to Chinese legend, around 2200 B.C. the first magic square, the 3×3 *lo-shu*, appeared to the enlightened sage-Emperor Yii on the back of a tortoise surfing the troubled waters of the Lo River. [See [Caj85], p. 77.]

due to their 'usefulness' for they do not seem to have much 'practical' value other than as magical charms and recreational pastimes. Yet there is no doubting their mathematical attractiveness, due to their unique blend of geometrical and arithmetical symmetry and balance. Add to this the enormous number and variety of such figures[2] and it seems there is no limit to the amount of pleasure and challenge they can provide to anyone who does not feel it necessary to "make gain out of all that he learns."

Venn diagrams have been around only since the 1880's, when John Venn introduced them as a practical tool for depicting relationships among logical propositions in symbolic logic. (See [Ven80].) Unlike magic figures and in spite of their abstractness, Venn diagrams have turned out to be very useful indeed, not only in logic, but in geometry, combinatorics, computer science, etc.

In this paper we combine these two concepts in a way that generalizes the former while adding a bit of 'magic' to the latter.

2 Definitions and Examples

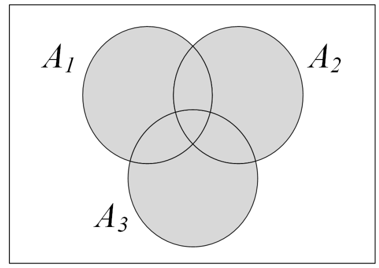
The concept of a magic Venn diagram (MVD) first occurred to me (Robinson) in a mathematical logic course I was teaching several years ago, when students were being introduced to truth tables and the associated Venn diagram representations thereof. Here is how it happened:

As the reader is no doubt aware, the truth value of a compound logical expression $P(S_1, S_2, S_3)$ of three logical variables S_1, S_2 and S_3 depends on which of the $2^3 = 8$ possible binary sequences of truth values TTT, TTF, TFT, ..., FFF is assigned to the sequence S_1, S_2, S_3. This information may be compiled in a table with eight rows and as many columns as necessary to compose the proposition from its constituent parts (variables and operations) and their truth values on each of the eight T/F sequences. The final result, i.e., the list of truth values of P on each T/F sequence, may then be visualized by means of a three set Venn diagram in which the three (abstract) sets A_1, A_2 and A_3 correspond to the three statements S_1, S_2 and S_3 and the eight non-overlapping Venn regions to the eight T/F sequences via the mappings $A_i \to T$ and $A_i' \to F$, where A_i' denotes the complement of A_i in the universal set X. For example, the Venn region $A_1 \cap A_2 \cap A_3'$ corresponds to the sequence TTF, the region $A_1' \cap A_2 \cap A_3'$ to the sequence FTF, etc. Finally, the regions may be shaded two different colors according as their respective truth values on P are T or F.

Example 2.1. *For the logical OR operation $P = S_1 \vee S_2 \vee S_3$, a.k.a., the set-theoretical* union $A_1 \cup A_2 \cup A_3$*, the truth table and associated Venn diagram may be displayed as follows:*

[2]See Example 2.7 below.

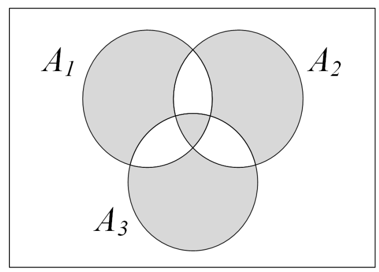
S_1	S_2	S_3	P
T	T	T	T
T	T	F	T
T	F	T	T
F	T	T	T
T	F	F	T
F	T	F	T
F	F	T	T
F	F	F	F



$A_1 \cup A_2 \cup A_3$

Example 2.2. *For the logical* Exclusive OR *operation* $P = S_1 \oplus S_2 \oplus S_3$*, a.k.a., the set-theoretical* symmetric difference *operation* $A_1 \Delta A_2 \Delta A_3$*, the truth table and associated Venn diagram are as follows:*

S_1	S_2	S_3	P
T	T	T	T
T	T	F	F
T	F	T	F
F	T	T	F
T	F	F	T
F	T	F	T
F	F	T	T
F	F	F	F



$A_1 \Delta A_2 \Delta A_3$

The results here express the fact that Exclusive OR applied to n *variables is true if and only if an* odd *number of the variables are true, as can easily be shown by induction on* n*.*

Now in both of these examples, the eight T/F sequences are listed in the same order: first by decreasing numbers of T's, then lexicographically among sequences of the same T-cardinality. But they could have been listed in any order we liked. (There are *only* 8! = 40,320 possibilities!) Naturally, the question arose in class: "What order do you want us to put them in on a test?" Annoyed as I was by the reference to test questions, I first calmed myself, and then thought: If I say "it doesn't matter, so long as you get all eight of them", I would pay for this later when it came time to grade the tests! Therefore, I quickly made up on the spot two orders which I thought they could easily remember, namely the lexicographic ordering with T preceding F and the slightly more sophisticated ordering used in the preceding examples. Once this was done, the corresponding regions of the Venn diagram could be labelled 1 through 8 accordingly.

Then it struck me! What if we could rank the sequences 1 through 8 in such a way that when the ranks were placed in the appropriate regions of the Venn diagram, the resulting sum of the ranks in each of the three sets A_1, A_2, A_3 was the same for all three sets? In short, I had landed on the idea of a *magic Venn diagram*!

The name of course comes from the analogous concept of a *magic square*, i.e., an arrangement of the first n^2 natural numbers in an $n \times n$ array so that every row, every column, and both diagonals have the same *'magic'* sum, namely

$$m = \frac{1}{n}\sum_{j=1}^{n^2} j = \frac{1}{2}n(n^2+1) \tag{1}$$

In fact, as I contemplated my discovery further, I realized that a magic square is a magic Venn diagram! The "sets" of the diagram are the $2n+2$ rows, columns and main diagonals and the Venn regions are the n^2 cells of the $n \times n$ grid upon which the square is based. In other words, such a grid is a Venn diagram in which the empty regions have been suppressed to make it possible to display (in two dimensions) all of the *nonempty* regions, and showing all the regularity and symmetry that that system possesses. Even the smallest magic square, the (unique) 3×3 *Lo-Shu* square shown below, has too many "sets" to be portrayed in the usual "box-and-ovals" scenario of a Venn diagram. That would require a 2-D arrangement of 8 "ovals" exhibiting *all* of the $2^8 = 256$ Venn regions, which, as far as I know, is topologically impossible. (See [BPS19] and [RSW06].)

Example 2.3. *Two of the oldest and most famous magic squares:*

4	**9**	**2**
3	**5**	**7**
8	**1**	**6**

Lo-Shu *square* $(m = 15)$
(Emperor Yii, 2200 B.C.)

16	**3**	**2**	**13**
5	**10**	**11**	**8**
9	**6**	**7**	**12**
4	**15**	**14**	**1**

The Melancholia *square* $(m = 34)$
(Albrecht Dürer, 1514 A.D.)[3]

But this is just the beginning! A survey (cf. [BT09]) of other kinds of "magic" figures devised over the centuries—magic circles, magic stars, magic graphs, magic polyhedra, etc.—reveals that most such figures are in fact examples of magic Venn diagrams, according to the following.

Definition 2.4. *A* Venn diagram of order n *is a two-dimensional depiction of a sequence* $\mathcal{A}$ *of (distinct) nonempty subsets* $A_1, A_2, \ldots, A_n$ *of some universal set* X*, showing some or all of the* 2^n *non-overlapping* Venn regions

$$R_I = R_I(\mathcal{A}) = (\cap_{i\in I} A_i) \cap (\cap_{i\notin I} A_i') \tag{2}$$

where $I \subseteq [n] = \{1, 2, \ldots, n\}$ *and* A_i' *denotes the complement of* A_i *in* X*. The number* r *of* nonempty *Venn regions is called the* size *of the diagram. The number of nonempty Venn regions contained in a given set* A_i *is called the* degree *of* A_i*. The diagram is* d-regular *if* *every set has the same degree d, in which case we may refer to it as an (n, r, d)*-diagram. *Finally, the diagram is* magical *if it is regular and if the r nonempty regions are weighted (one-to-one) by the numbers* $1, 2, \ldots, r$ *in such a way that the total weight m of the regions constituting a set* A_i *is the same for all* $i = 1, 2, \ldots, n$. *We call such a diagram a* magic Venn diagram *of* order n, size r, degree d *and* magic sum m, *or more briefly, an (n, r, d, m) - MVD.*

For a given Venn diagram, it is easy to see from (2) that distinct Venn regions are disjoint. Hence the nonempty Venn regions form a partition of the underlying set X and the nonempty Venn regions within a given set A_i form a partition of A_i. As to which regions are nonempty, technically this is determined by the set X and the subsets A_i thereof. But for our purposes, we are only interested in the combinatorial (or logical) structure of the diagram, as determined by the *regional support system* (or hypergraph)

$$\{I \subseteq [n] \,|\, R_I(\mathcal{A}) \neq \emptyset\} \tag{3}$$

So, two Venn diagrams with same support system, after permutation of the sets if necessary, shall be regarded as (structurally) *equivalent* regardless of how the nonempty regions come about. This leads us to the following.

Definition 2.5. *Two MVD's are said to be* isomorphic *if they have the same combinatorial structure and if corresponding sets in their support systems are assigned the same labels.*

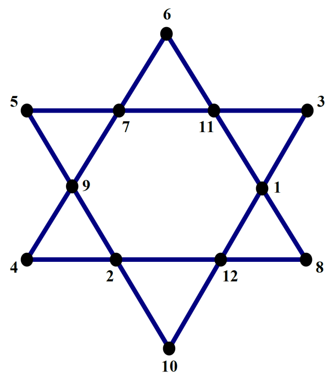
Example 2.6. *The symmetric difference diagram of Example 2.2 is equivalent to the diagram arising from the subsets* $A_1 = \{1, 4\}$, $A_2 = \{2, 3, 4\}$ *and* $A_3 = \{4, 5\}$ *of the set* $X = [5] = \{1, 2, 3, 4, 5\}$, *since in both cases the nonempty regions* R_I *are precisely those for which* $|I| = 1$ *or 3, i.e., both have support system* $\{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$. *Neither of these (3, 4, 2)-diagrams can be magically labelled since, in any MVD, d must be greater than 2 (else the common weight of any two sets would involve a common label).*

Example 2.7. *An* $n \times n$ *grid may be seen as a* $(2n, n^2, n)$*-diagram, with its rows and columns as the sets and its cells as the nonempty Venn regions. It becomes a* $(2n + 2, n^2, n)$*-diagram if we include the main diagonals among the sets, as we observed earlier. Thus, a magic square is a* $(2n + 2, n^2, n, m)$*-MVD with m given by* (1). *It is well known that, up to isomorphism, there is only one magic square of order 3, 880 of order 4, 275,365,224 of order 5, and about* 1.8×10^{19} *of order 6. Little is known about the number of magic squares of higher orders. (See [PW98] and [SG76].)*

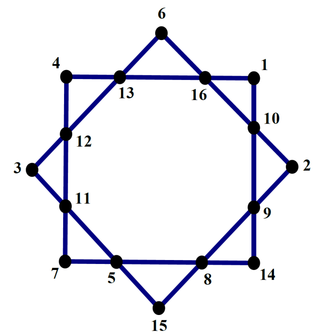
Example 2.8. *A* magic n–pointed star *of order* $n \geqslant 5$ *is formed by joining every other vertex of a regular n-gon and numbering the resulting* $r = 2n$ *points of intersection of the n sides by the numbers* $1, 2, \ldots, 2n$ *so that the* $d = 4$ *numbers on each side have the same sum*

$$m = \frac{1}{n}\sum_{j=1}^{2n} j = 2(2n + 1) \tag{4}$$

This is possible for all $n > 5$. *[Cf. [Tre04] or [BT09] pp. 47- 48.] Examples for the cases* $n = 6$ *and 8 are shown below.*



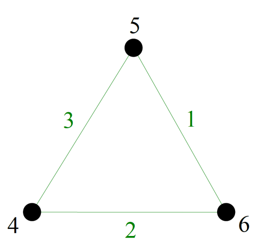
$n = 6,\ r = 12,\ d = 4,\ m = 26$



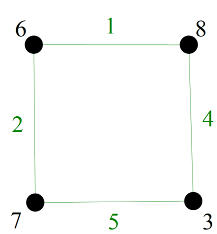
$n = 8,\ r = 16,\ d = 4,\ m = 34$

Such a star is an $(n, 2n, 4, m)$*-diagram in which the sets are the* n *sides, the Venn regions are the* $2n$ *vertices of the star, four on each side, and* m *is given by* (4)*. For* $n = 8$*, this diagram has the same parameters as a* 4×4 *grid-diagram; i.e., both are* $(8, 16, 4)$*-diagrams. However, they are not structurally equivalent since, e.g., the grid diagram contains two parallel classes of 4 sets (i.e., 4 rows and 4 columns), while the star diagram contains four parallel classes of 2 sets. Thus, the parameters alone do not determine the combinatorial structure of a Venn diagram.*

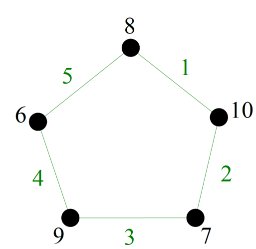
Example 2.9. *A* magic n-cycle *is formed by labelling the* n *vertices and* n *edges of a regular* n*-gon by the numbers* $1, 2, \ldots, 2n$ *so that the 3 numbers on each edge-endpoint triple have common sum* m*. These are special cases of Kotzig's* edge-magic graphs *[KR70]. They can be viewed as* $(n, 2n, 3, m)$*-MVD's in which the sets are the* edge-endpoint *triples* $\{i, i+1\}, \{i\}, \{i+1\} (mod\ n), i = 1, 2, \ldots, n$*, and the regions are the* n *edges and* n *vertices of the graph. The support system for such a diagram consists of the* n *singletons* $\{i\}$ *and* n *pairs* $\{i, i+1\} (mod\ n), i = 1, 2, \ldots, n$*. Unlike magic squares and stars, for a fixed value of* n *there is more than one possible value of* m*. In fact, it is not hard to prove that* $\frac{5n+3}{2} \leqslant m \leqslant \frac{7n+3}{2}$ *and that every integer in this interval is the magic sum of some magic* n*-cycle. (See [GS98] or [PR17].) Three examples attaining the upper bound for* m *are shown below.*



$n = 3,\ m = 12$ (max.)



$n = 4,\ m = 15$ (max.)



$n = 5,\ m = 19$ (max.)

Figure 1: Maximal magic n-cycles

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To get min(m), subtract every label from $r+1 = 2n+1$. *This is called* conjugation. *In general, the* conjugate of a magic figure *with sum* m *is a magic figure of the same type with magic sum* $m^* = d(r+1) - m$. *Edge-magic* complete *graphs* K_n *exist if and only if* $n = 2, 3, 5$, *or 6. Edge-magic* bipartite *graphs* $K_{s,t}$ *exist for all* s, t. *(See [MW13]).*

Example 2.10. *A* d*-regular graph may also be made magical by numbering the* n *vertices and* p *edges by the numbers* $1, 2, \ldots, r = n+p$ *in such a way that each vertex and all of its incident edges have the same sum* m. *Such a* vertex-magic *graph is an* $(n, n+p, d+1, m)$*-diagram in which a set is a vertex together with its incident edges, a region is a vertex or an edge and*

$$\frac{p(n+p+1)}{n} + \frac{n+1}{2} \leqslant m \leqslant \frac{(n+p)(n+p+1)}{n} - \frac{n+1}{2}$$

(See Kotzig and Rosa, [KR70].) The support system consists of the singletons (one for each vertex) and edges (viewed as 2-element sets).

If the graph is an n*-cycle, the diagram is isomorphic to the corresponding edge-magic* n*-cycle diagram with the roles of vertices and edges interchanged. For instance, the following pictures show a magic cycle of order 4 depicted in three different ways. All three are isomorphic to a (4, 8, 3, 15)-MVD with support set* $\{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$ *and corresponding region weights* $\{1, 2, 5, 4, 6, 7, 3, 8\}$ *(in that order).*

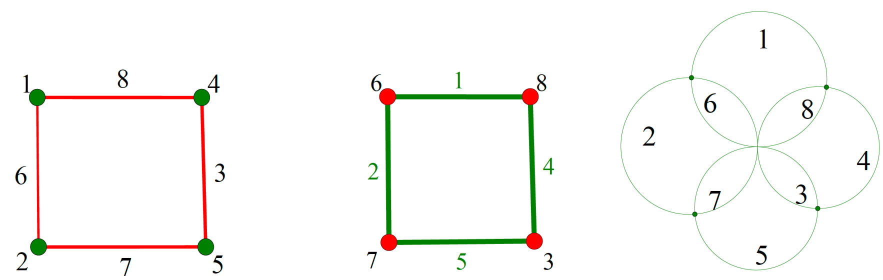


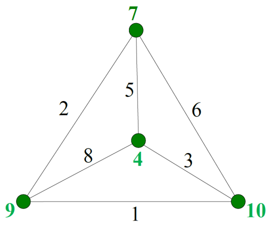
Figure 2: Isomorphic (4, 8, 3, 15) cycle MVD's

Vertex-magic complete *graphs* K_n *exist if and only if* $n \neq 2$. *(For* $n = 4$*, there are 14 of them: two with* $m = 20$ *and five with* $m = 21$*, plus the conjugates of these.) Vertex-magic complete bipartite graphs* $K_{s,t}$ *exist if and only if* $s = t-1$, t, *or* $t+1$. *If only the edges are labeled, then a vertex-magic bipartite* $K_{n,n}$ *is equivalent to an* $n \times n$ *row-column magic square. (For these results and more, see [MW13].) All of these are MVD's.*

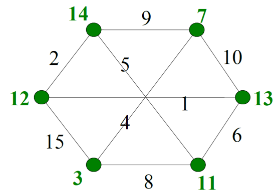
Example 2.11. *A* complete *diagram is one for which the support system is a complete hypergraph, i.e., every region is nonempty. These diagrams have structural parameters* $n, r = 2^n$ *and* $d = 2^{n-1}$ *and magic sum* m *satisfying*

$$2^{n-2}(2^n+1) - \left\lfloor \frac{1}{2}\binom{2n-1}{n} \right\rfloor \leqslant m \leqslant 2^{n-2}(2^n+1) + \left\lfloor \frac{1}{2}\binom{2n-1}{n} \right\rfloor \qquad (5)$$

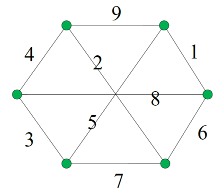
The lower (and conjugate upper) bounds here occur when the magical labelling is chosen so that the weights decrease (resp. increase) with increasing sizes of the support sets (with



Vertex-magic K_4
$n = 4$, $r = 10$, $d = 4$,
$m = 20$



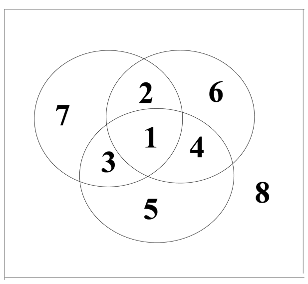
Vertex-magic $K_{3,3}$
$n = 6$, $r = 15$, $d = 4$,
$m = 30$



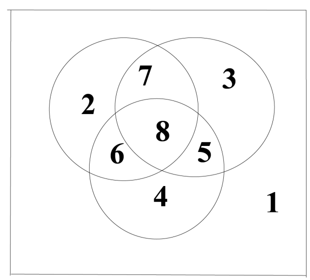
0-Vertex-magic $K_{3,3}$
$n = 6$, $r = 9$, $d = 3$, $m = 15$
$\cong$ Lo-Shu square

Figure 3: Vertex-magic complete graphs

slight adjustments in the cases where $\binom{2n-1}{n}$ *is odd, i.e., when* n *is a power of 2). This is always possible, i.e., the* bounds are sharp for all n, *though we haven't room here to present the proof. (See [RR] and our comments at the end of this paper.) Examples of orders 3 and 4 are shown below (figures 4 and 5).*



$n = 3$, $r = 8$, $d = 4$, $m = 13$ (min.)



$n = 3$, $r = 8$, $d = 4$, $m = 23$ (max.)

Figure 4: Conjugate and optimal complete MVD's of order 3

Proof of (5). The formulas in (5) are proved by summing the n set weights (all equal to m) in two different ways, then simplifying and dividing by n to get m. For a complete MVD, this leads to the inequalities $\frac{L_n}{n} \leqslant m \leqslant \frac{U_n}{n}$, where L_n and U_n are respectively the least and greatest possible sums with repetition of the labels $1, 2, \ldots, r = 2^n$ as they occur in the 2^n Venn regions in decreasing and increasing order of the corresponding support set cardinalities. For instance, in the order 3 case, $L_3 = 0{\cdot}8+1{\cdot}(7+6+5)+2{\cdot}(4+3+2)+3{\cdot}(1) = 39$ and $U_3 = 0 \cdot 1 + 1 \cdot (2 + 3 + 4) + 2 \cdot (5 + 6 + 7) + 3 \cdot (8) = 69$, giving $\min(m) = \frac{39}{3} = 13$ and $\max(m) = \frac{69}{3} = 23$.

In general, we have

$$L_n = \sum_{k=0}^{n-1} (n-k) \sum_{j=1}^{\binom{n}{k}} \left[1 + \binom{n}{1} + \cdots + \binom{n}{k-1} + j\right]$$

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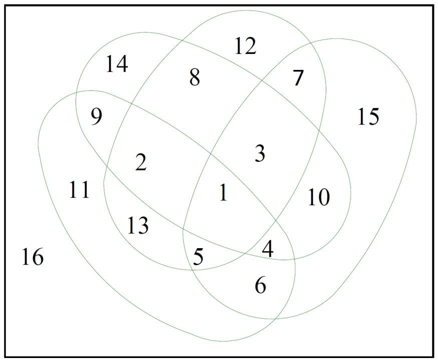
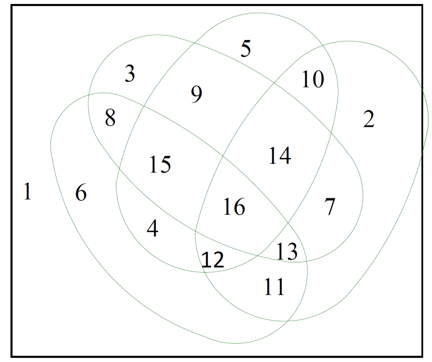


$n = 4$, $r = 16$, $d = 8$, $m = 51$ (min.)



$n = 4$, $r = 16$, $d = 8$, $m = 85$ (max.)

Figure 5: Conjugate and optimal complete MVD's of order 4

and

$$U_n = \sum_{k=1}^{n} k \sum_{j=1}^{\binom{n}{k}} \left[1 + \binom{n}{1} + \cdots + \binom{n}{k-1} + j\right]$$

where $\binom{n}{k}$ denotes the binomial coefficient = the # of k-element support sets in a complete diagram. To simplify these formulas, first we use conjugation to get

$$L_n + U_n = nd(r+1) = n2^{n-1}(2^n + 1) \tag{$*$}$$

Next, we rewrite the expressions for L_n and U_n by putting

$$b(n,k) = \sum_{i=0}^{k} \binom{n}{i} \text{ for } 0 \leqslant k \leqslant n \text{ and } T_p = \sum_{j=1}^{p} j = \frac{p(p+1)}{2} \qquad \forall p \geqslant 1$$

Thus,

$$L_n = \sum_{k=1}^{n-1} (n-k)\binom{n}{k} b(n,k-1) + \sum_{k=0}^{n-1} (n-k) T_{\binom{n}{k}}$$

and

$$U_n = \sum_{k=1}^{n} k\binom{n}{k} b(n,k-1) + \sum_{k=1}^{n} k T_{\binom{n}{k}}$$

Reversing the indexing and using the symmetry property $\binom{n}{k} = \binom{n}{n-k}$ give

$$\sum_{k=0}^{n-1} (n-k) T_{\binom{n}{k}} = \sum_{k=1}^{n} k T_{\binom{n}{n-k}} = \sum_{k=1}^{n} k T_{\binom{n}{k}}$$

Thus, we obtain

$$U_n - L_n = \sum_{k=1}^{n} k\binom{n}{k} b(n,k-1) - \sum_{k=1}^{n-1} (n-k)\binom{n}{k} b(n,k-1)$$

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But $k\binom{n}{k} = n\binom{n-1}{k-1}$ and $(n-k)\binom{n}{k} = n\binom{n-1}{k}$, so

$$\begin{aligned}
U_n - L_n &= n\sum_{k=1}^{n}\tbinom{n-1}{k-1}b(n,k-1) - n\sum_{k=1}^{n-1}\tbinom{n-1}{k}b(n,k-1) \\
&= n\left[\sum_{k=0}^{n-1}\tbinom{n-1}{k}b(n,k) - \sum_{k=1}^{n-1}\tbinom{n-1}{k}b(n,k-1)\right] \\
&= n\left[1 + \sum_{k=1}^{n-1}\tbinom{n-1}{k}\big(b(n,k) - b(n,k-1)\big)\right] \\
&= n\left[1 + \sum_{k=1}^{n-1}\tbinom{n-1}{k}\tbinom{n}{k}\right] = n\sum_{k=0}^{n-1}\tbinom{n-1}{k}\tbinom{n}{k} = n\sum_{k=0}^{n-1}\tbinom{n-1}{k}\tbinom{n}{n-k}
\end{aligned}$$

This last sum is a convolution representing the coefficient of x^n in the expansion of the binomial product $(1+x)^n(1+x)^{n-1} = (1+x)^{2n-1}$, which is $\binom{2n-1}{n}$. Thus, $U_n - L_n = n\binom{2n-1}{n}$. Combining this with $(*)$, dividing through by n, and rounding when n is a power of 2 (the only time $\binom{2n-1}{n}$ is odd), we arrive at the integer bounds given by (5). □

Other 'nearly complete' MVD's can easily be obtained from complete ones by deleting one or both of the regions $R_\emptyset$ and $R_{[n]}$ and adjusting the weights accordingly. For instance, a minimal *union* MVD can be obtained from a minimal complete MVD by simply deleting the inert region $R_\emptyset$ and its weight. Conjugation of the resulting MVD yields a maximal union MVD.

3 Enumeration of order 3 and order 4 MVD's

Enumeration of $n \times n$ magic squares is a notoriously difficult problem for $n > 5$. The situation is no better for most other classes of MVD's of even relatively small orders. Ordinarily the best we can hope to obtain for any given class of diagrams are results on the existence or nonexistence of MVD's of that type under certain restrictions on the parameters—usually on the order and the magic sum. And then only under the severest of such restrictions is it possible to give a complete account of all possible (nonisomorphic) MVD's, and this usually only with the aid of a computer.

Order 3

For order 3 Venn diagrams, there are only four inequivalent combinatorial types which can be magically labeled, according as each of the two regions $R_\emptyset$ and $R_{[3]}$ is non-empty or empty. The total number of non-isomorphic MVD's of each type and each possible magic sum has been calculated in [FR14] (using systems of equations). The results are shown below. Altogether there are 140 different order 3 MVD's, 100 of which are complete. We leave it as a fun exercise for the reader to construct the four non-isomorphic MVD's of order 3 with $r = 6$, as indicated in the fourth table below.

Number of order 3 *complete* (3, 8, 4, m)-MVD's:

m	13	14	15	16	17	18	19	20	21	22	23	Total
Frequency	1	8	4	15	10	24	10	15	4	8	1	100

Number of order 3 *union* (3, 7, 4, m)-MVD's ($R_\emptyset$ empty):

m	13	14	15	16	17	18	19	Total
Frequency	1	3	2	6	2	3	1	18

Number of (3, 7, 3, m)-MVD's ($R_{[3]}$ empty):

m	9	10	11	12	13	14	15	Total
Frequency	1	3	2	6	2	3	1	18

Number of (3, 6, 3, m)-MVD's with $R_\emptyset = \emptyset$ and $R_{[3]} = \emptyset$:

m	9	10	11	12	Total
Frequency	1	1	1	1	4

Note that any MVD of the third type here can be obtained from a union MVD by *complementation*; i.e., by replacing each set A_i with its complement in X and moving the labels accordingly. The following diagrams illustrate this relationship:

$m = 13$ (min.) $m = 15$ (max.)

Figure 6: Optimal union and complementary union MVD's of order 3

Order 4

There are many more combinatorial types of order 4 which can be magically labelled: 99 to be exact! These too have been enumerated by type and magic sums, the smaller ones by hand and the rest by using a Java computer program. (See [Rem19] or [RR].) Altogether, there are **35,783,330** non-isomorphic MVD's of order 4, the vast majority of which are complete or nearly complete. The following tables show the frequencies by magic sum of just the very large ones ($r = 16, 15, 14$) and the very small ones ($r = 8, 7$ and both $R_\emptyset$ and $R_{[4]}$ empty).

Table (i). *Number of complete (4, 16, 8, m)-MVD's*

m	51	52	53	54	55	56	57	58	59
#	327	3,506	11,457	34,568	70,930	148,256	230,288	381,686	503,732

m	60	61	62	63	64	65	66
#	829,020	903,036	1,203,986	1,371,212	1,843,868	1,775,342	2,083,910

m	67	68	...	85	Total
#	2,015,348	2,492400	...	327	***29,313,344***

Table (ii). *Number of Union (4, 15, 8, m)-MVD's and complementary (4, 15, 7, $m' = 120 - m$)-MVD's*

m	51	52	53	54	55	56	57	58
m′	69	68	67	66	65	64	63	62
#	194	1,487	4,425	11,491	21,436	39,551	57,785	83,979

m	59	60	61	62	63	64	...	77	Total
m′	61	60	59	58	57	56	...	43	
#	103,440	155,775	152,960	188,674	185,882	230,042	...	194	***2,244,200***

Table (iii). *Number of (4, 14, 7, m)-MVD's with only $R_\emptyset$ and $R_{[4]}$ empty*

m	43	44	45	46	47	48	49	50
#	106	629	1,691	3,460	6,608	9,736	16,558	15,978

m	51	52	...	62	Total
#	19,228	24,250	...	106	***196,488***

Table (iv). *Number of (4, 14, 7, m)-MVD's with only $R_{\{4\}}$ and $R_{\{1,2,3\}}$ empty*

m	40	41	42	43	44	45	46	47	48
#	40	426	1,784	3,994	8,034	13,488	21,462	29,490	38,530

m	49	50	51	52	53	...	65	Total
#	49,952	54,676	59,830	63,066	63,066	...	40	***681,544***

Table (v). *Number of (4, 14, 7, m)-MVD's with only $R_{\{1,3\}}$ and $R_{\{2,4\}}$ empty*

m	38	39	40	41	42	43	44	45	46	47	48
#	12	204	812	2,064	4,170	6,866	10,524	14,816	19,862	25,496	31,008

m	49	50	51	52	53	...	67	Total
#	35,800	40,652	44,294	46,188	46,188	...	12	***565,536***

Table (vi). *Number of (4, 8, 4, m)-MVD's with support sets 1, 4, 12, 13, 24, 34, 123, 234*

m	16	17	18	19	20	Total
#	6	12	0	12	6	***72***

Table (vii). *Number of (4, 8, 4, m)-MVD's with support sets 1, 4, 12, 13, 24, 34, 123, 234*

m	*18*	*Total*
#	*36*	***36***

Table (viii). *Number of* Δ *(4, 8, 4, m)-MVD's with support = all subsets of [4] of odd cardinalities*

m	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*Total*
#	*1*	*0*	*1*	*1*	*0*	*1*	*1*	*0*	*1*	***6***

Table (ix). *Number of Cyclic (4, 8, 3, m)-MVD's with support 1, 2, 3, 4, 12, 23, 34, 14 and the complementary (4, 8, 5,* $m' = 36 - m$*)-MVD's: support 23, 12, 14, 34, 123, 124, 134, 234*

m	*12*	*13*	*14*	*15*	*Total*
m'	*24*	*23*	*22*	*21*	
#	*1*	*2*	*2*	*1*	***6***

Table (x). *Number of (4, 7, 3, m)-MVD's with support 1, 2, 3, 14, 24, 34, 123 and the complementary (4, 7, 4,* $m' = 28 - m$*)-MVD's: support 4, 12, 13, 23, 124, 134, 234*

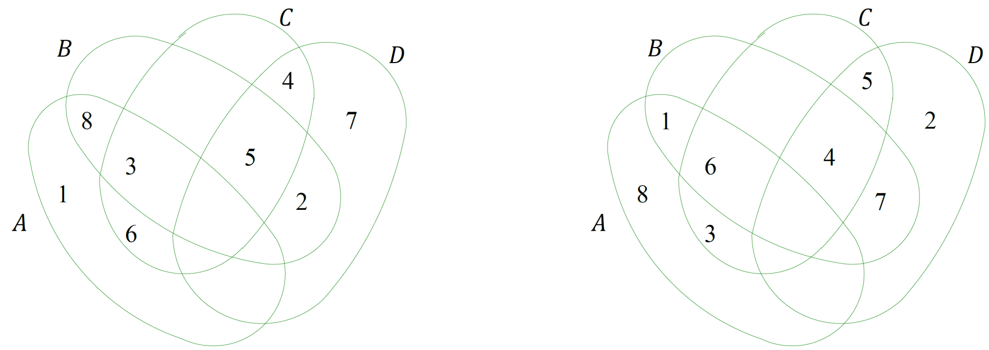
m	*10*	*11*	*12*	*13*	*14*	*Total*
m'	*18*	*17*	*16*	*15*	*14*	
#	*1*	*0*	*2*	*0*	*1*	***4***

Table (xi). *Number of (4, 7, 3, m)-MVD's with support 1, 4, 12, 13, 23, 24, 34 and the complementary (4, 7, 4,* $m' = 28 - m$*)-MVD's: support 12, 13, 14, 24, 34, 123, 234*

m	*11*	*12*	*13*	*Total*
m'	*17*	*16*	*15*	
#	*1*	*3*	*1*	***5***

We conclude this paper with some examples and explanations of the relatively few order 4 MVD's enumerated in Tables (vii)–(xi).

Example 3.1. *A conjugate pair of type (vii) MVD's is shown below (Figure 7). According to the table there are 36 non-isomorphic MVD's of this type all with the same magic sum* $m = 18$*. That only one magic sum is possible here is due to the fact that in this combinatorial structure the nonempty regions fall into 'parallel' classes of sets: In this case, two of the sets (A and D in the figure) have no regions in common – i.e., they are parallel – and together they contain all of the regions of the diagram. So, the sum of all the weights, which is a constant (*$= r(r+1)/2$*), is a multiple of the magic sum, in this case* $2m$*; this forces* m *to be a function of* r *alone, hence the same for all MVD's of this type, namely,* $m = r(r+1)/4 = (8 \times 9)/4 = 18$*.*



$m = 18$ (min.) $m^* = 18$ (max.)

Figure 7: Conjugate MVD's of type (vii)

Example 3.2. *The MVD's counted in Table (viii) are* symmetric difference *MVD's of order 4. According to this table there are only 6 of them, one for each of the magic sums between 14 and 22, excluding 15, 18 and 21. As the support sets are the one-element sets and their 3-element complements, each such MVD can be viewed as a* vertex-magic *or* face-magic tetrahedron, *i.e., a tetrahedron whose 4 vertices and 4 faces are weighted 1 to 8 in such a way that the sum of the weights either of a vertex and its 3 incident faces or a face and its 3 vertices is a constant. Following is a depiction of the* Δ*-MVD of (minimal) sum 14 and the associated magic tetrahedron. Conjugation yields the one of (maximal) sum 22.*

$m = 14$ (min.) $m_{\text{vertex}} = 14$, $m_{\text{face}} = 22$

Figure 8: Δ-MVD of order 4 and associated magic tetrahedron

A more efficient way to display these MVD's is to list the support sets I and the associated labels $\lambda(I)$ in a table, as follows:

I	$\lambda(I)$	I'	$\lambda(I')$
1	*5*	*234*	*1*
2	*6*	*134*	*2*
3	*7*	*124*	*3*
4	*7*	*123*	*4*

(4, 8, 4, 14) Δ*-MVD*

I	$\lambda(I)$	I'	$\lambda(I')$
1	*1*	*234*	*5*
2	*2*	*134*	*6*
3	*3*	*124*	*7*
4	*4*	*123*	*8*

(4, 8, 4, 22) Δ*-MVD*

This reveals a relationship between the labels $\lambda_i = \lambda(\{i\})$ *and the corresponding labels* $\lambda'_i = \lambda(\{i\}')$*, which happens to hold in any order 4* Δ*-MVD, namely,*

$$\lambda'_i - \lambda_i = s_3 - m, \quad \forall i \tag{6}$$

where s_3 *is the sum of all the labels on the 3-element sets. (E.g., In the second table above,* $s_3 - m = 26 - 22 = 4$*.) The reason for this is that, for any given singleton* $\{i\}$*, the complementary set* $\{i\}'$ *is the* only *3-element support set not containing* i*. Since* m *is the sum of the label* λ_i *and all* $\lambda(I')$ *for which* $i \in I'$*, it follows that* $s_3 = \lambda'_i + m - \lambda_i$*, as claimed. We may now use this result to explain the rest of the entries in Table (viii).*

Suppose we have a magic labelling of an order 4 Δ*-diagram. The labels are the numbers 1 through 8, in some order. The support sets are the singletons* $\{i\}$ *and their complements* $\{i\}'$ *in the set [4]* $= \{1,2,3,4\}$*. As above, let* $\lambda_i = \lambda(\{i\})$ *and* $\lambda'_i = \lambda(\{i\}')$ $\forall i$*, and put* $s_1 = \sum_i \lambda_i$ *and* $s_3 = \sum_i \lambda'_i$*. Summing the weights of the four sets in the diagram gives* $4m$*. But in this sum, each* λ_i *occurs once and each* λ'_i *occurs three times, so the sum also equals* s_1+3s_3*; i.e.,* $s_1+3s_3 = 4m$*. Combining this with the fact that* $s_1+s_3 = 1+2+\cdots+8 = 36$*, we get*

$$s_1 = 54 - 2m \text{ and } s_3 = 2m - 18 \tag{7}$$

From the second of these two equations it follows that $14 \leqslant m \leqslant 22$*, since the smallest and largest possible values of* s_3 *are 10 and 26, occurring respectively when the set of* λ'_i*'s is* $\{1,2,3,4\}$ *or* $\{5,6,7,8\}$*. These two extremes give the unique optimal* Δ*-MVD's (*$m = 14$ *or 22) discussed above. (Uniqueness is due to the symmetry among the support sets.) For the remaining values of* m*, we may proceed by means of* (6) *and* (7) *and as follows:*

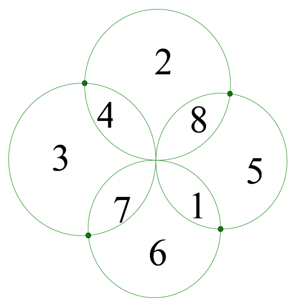
If $m = 18$*, then* $s_1 = s_3 = 18$ *and* (6) *implies* $\lambda'_i = \lambda_i$ $\forall i$*, which is impossible since labels may not be duplicated. Hence there is no order 4* Δ*-MVD with* $m = 18$*.*

If $m = 21$*, then* $s_1 = 12$*,* $s_3 = 24$ *and, by* (6)*,* $\lambda'_i - \lambda_i = 24 - 21 = 3$ $\forall i$*, which implies* $\lambda'_i \geqslant 4$ $\forall i$*. Hence the labels* λ_i *are 1, 2, 3, and 6 in some order. This leads to the duplication* $\lambda'_i = \lambda_i + 3 = 3 + 3 = 6 = \lambda_j$ *for some* i*,* j*. (It also leads to* $\lambda'_j = 6 + 3 = 9$*, another contradiction.) Thus, no order 4* Δ*-MVD's exist with* $m = 21$ *(or the conjugate value* $m^* = 15$*).*

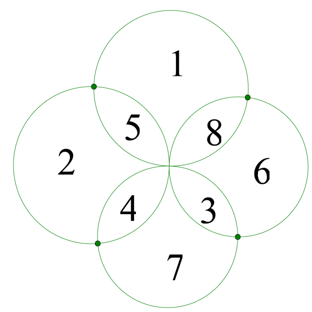
If $m = 20$*, then* $s_1 = 14$*,* $s_3 = 22$ *and, by* (6)*,* $\lambda'_i - \lambda_i = 22 - 20 = 2$ $\forall i$*, which implies* $\lambda'_i \geqslant 3$ $\forall i$*. In this case the labels* λ_i *must be 1, 2, 5, and 6 in some order and the corresponding labels* λ'_i *are 3, 4, 7 and 8. By symmetry, it follows that there is only one order 4* Δ*-MVD with* $m = 20$ *(or its conjugate* $m^* = 16$*).*

If $m = 19$*, then* $s_1 = 16$*,* $s_3 = 20$ *and, by* (6)*,* $\lambda'_i - \lambda_i = 20 - 19 = 1$ $\forall i$*, which implies* $\lambda'_i \geqslant 2$ $\forall i$*. In this case the labels* λ_i *must be 1, 3, 5, and 7 in some order and the* *corresponding labels λ_i' are 2, 4, 6 and 8. By symmetry, it follows that there is only one order 4 Δ-MVD with $m = 19$ (or its conjugate $m^* = 17$).*

Example 3.3. *The six $(4, 8, 3, m)$-MVD's enumerated in Table (ix) are magic cycles of order 4. One of these, the one with $m = 15$, was described in Examples 2.9 and 2.10 above. Its conjugate together with the two magic cycles shown below and their conjugates constitute all of the non-isomorphic magic cycles of order 4.*



$m = 14$



$m = 14$

To see why these are the only order 4 cyclic MVD's, we may proceed as we did in Example 3.2 above. The support sets are the singletons $\{i\}$ and the pairs $\{i, i+1\}$ (mod 4). Magically labelling these with the numbers 1 through 8, and summing the resulting four set weights in two different ways, we find that $s_1 + 2s_2 = 4m$, where s_k denotes the sum of the labels on the k-element support sets, $k = 1, 2$. Combining this with $s_1 + s_2 = 36$ gives $s_1 = 72 - 4m$ and $s_2 = 4m - 36$. Thus, since $10 \leqslant s_2 \leqslant 26$, we have $12 \leqslant m \leqslant 15$, in agreement with Table (ix).

If $m = 15$, then $s_1 = 12$ and $s_2 = 24$, forcing the set of labels on the 1-element support sets to be $1, 2, 3, 6$ in some order or $1, 2, 4, 5$ in some order. The first case is impossible, as it requires the label 6 to be duplicated among the 2-element support sets (6 and 8 being required among these labels to get a magic sum of 15 in the Venn set containing the region labelled 1). The second case is possible, as we know, but in only one way, up to isomorphism, as can easily be checked by trial-and-error.

If $m = 14$, then $s_1 = 16$ and $s_2 = 20$, forcing the set of labels on the 1-element support sets to be among the six possible subsets of [8] of sum 16: $\{1, 4, 5, 6\}$, $\{1, 3, 5, 7\}$, $\{1, 3, 4, 8\}$, $\{1, 2, 5, 8\}$, $\{1, 2, 6, 7\}$, $\{2, 3, 4, 7\}$ and $\{2, 3, 5, 6\}$. The first two of these are impossible, since assigning label 1 to a 1-element support set requires that labels 5 and 8 or 6 and 7 be among the 2-element support sets to get the magic sum of 14 in the Venn set containing label 1. Likewise, $\{2, 3, 4, 7\}$ and $\{1, 2, 5, 8\}$ are impossible, since the labels 4 and 8 or 5 and 7 must be among the 2-element support sets to get the magic sum of 14 in the Venn set containing label 2. Finally, we can rule out $\{1, 3, 4, 8\}$ since it contains 1 and 4, at least one of which must be assigned to a 2-element support set in order to get the magic sum of 14 in the Venn set containing label 8. Thus, the only possibilities for labels on the one-element support sets are $\{1, 2, 6, 7\}$ and $\{2, 3, 5, 6\}$. By trial-and-error again, we find that each of these is possible in exactly one way, up to isomorphism. These are *the two magic 4-cycles shown above. Thus, invoking conjugation and complementation, we have accounted for all of the entries in Table (ix).*

Arguments like those used in Examples 3.1–3.3 can be employed to verify the counts in Tables (x) and (xi). However, in the interest of space, we shall omit the details and content ourselves with just a few pictures and miscellaneous remarks to stimulate the reader's intellect (and imagination!)

Example 3.4. *The four* $(4, 7, 3, m)$*-MVD's enumerated in Table (x) are depicted below in conjugate pairs, with the four sets represented as three 'lines' and a 'triangle' (or circle), each containing three of the seven points, labeled 1 to 7 in such a way that each line and the triangle has weight* m*. The two of sum 12 each have the additional property that the endpoints of the three lines form another triangle of weight 12. This yields a (5, 7, 3, 12)-MVD, sometimes portrayed as 'magic concentric circles', i.e., two circles and three radii of the same sum, as shown below. (See [Wen].) Alternatively, this figure can be represented by five lines, two of which are parallel, as also shown.*

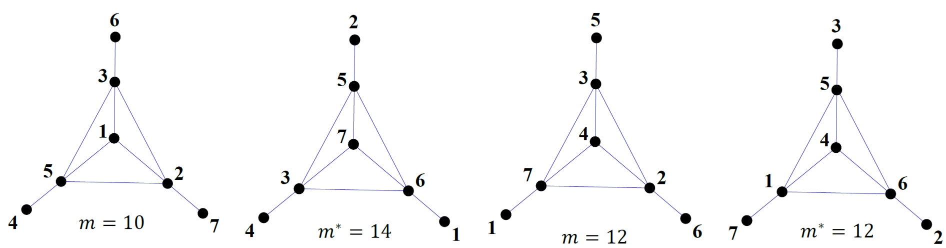


Figure 9: Depictions of the four (non-isomorphic) (4, 7, 3, m)-MVD's listed in Table (x)

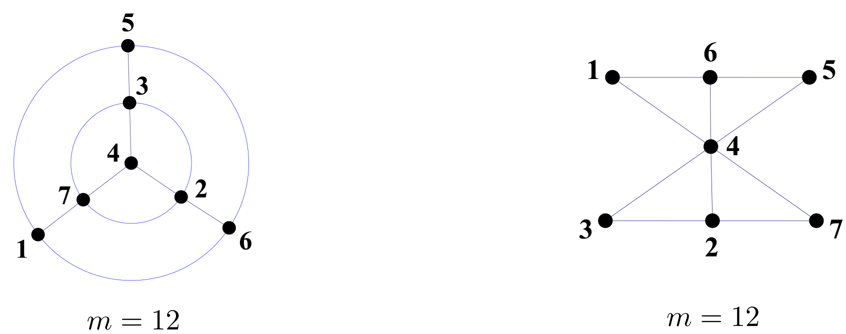


Figure 10: 'Magic Concentric Circles' ≅ (5, 7, 3, 12)-MVD obtained from a (4, 7, 3, 12)-MVD

Example 3.5. *The five* $(4, 7, 3, m)$*-MVD's enumerated in Table (xi) are isomorphic to five corresponding vertex-magic graphs (with only some of the vertices labelled), since all the support sets have cardinalities 1 or 2. Depictions of these are shown below (Figure 11),*

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including the graphical representation of one of them (Figure 12). Three of them are self-conjugate, i.e., isomorphic to their own conjugates. Any one of these three can be extended to an order 5 MVD by adjoining the set made up of the regions $R_{\{1\}}$, $R_{\{4\}}$ and $R_{\{2,3\}}$. In all three cases, this larger MVD is the same as the (5, 7, 3, 12) concentric circles diagram of Example 3.4.

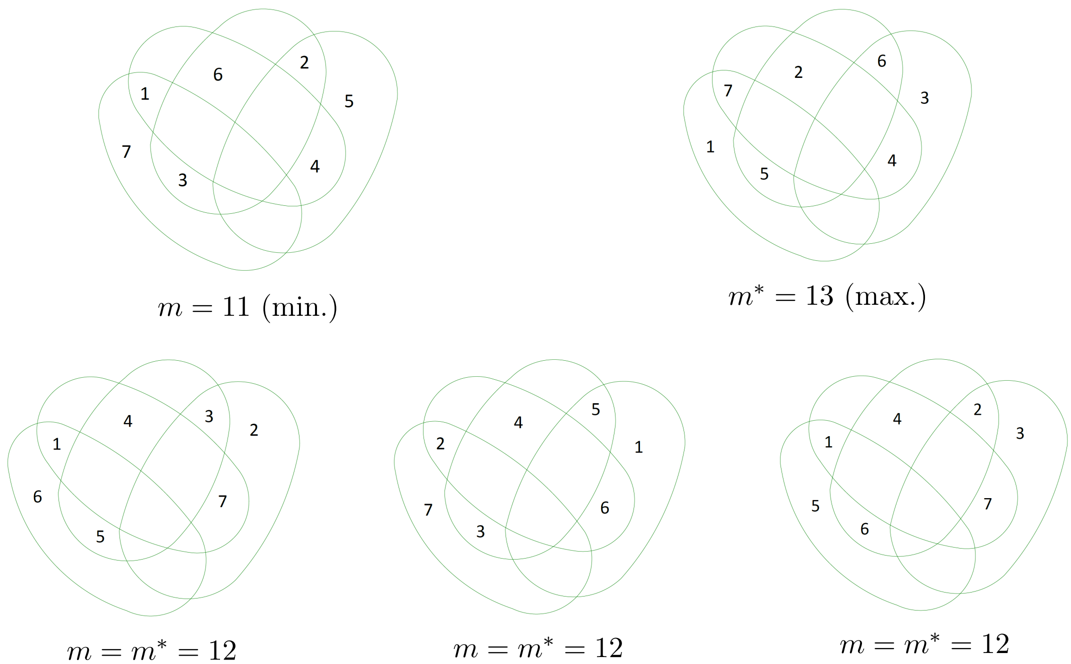

Figure 11: The five (non-isomorphic) (4, 7, 3, m)-MVD's listed in Table (xi)

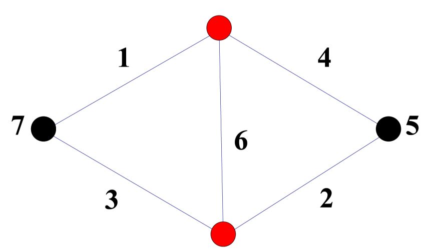

Figure 12: Graphical depiction of the (4, 7, 3, 11)-MVD in Table (xi)

4 Summary and Directions for Further Research

We hope this paper has aroused the interest of the reader enough to further pursue some aspect of the subject. For detailed proofs of all of the assertions made herein, we refer

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the reader to our more extensive paper [RR]. In particular, we show there how to construct optimal (and other) complete (and nearly complete) MVD's of any order $n \geqslant 3$, by piecing together certain *mod n orbital* MVD's (of which the order 4 Δ-MVD's of Example 3.2 are examples).[4] We also prove similar results for Δ-MVD's.

In [Rem19], Remshagen provides an in-depth explanation of the Java program she used to count order 4 MVD's. Based on the results for orders 3 and 4, we feel confident that complete MVD's of all orders exist for every possible magic sum. However, since the size of a complete MVD grows exponentially with its order, we feel powerless to count them![5] Even identifying and counting the different combinatorial *types* of Venn diagrams of relatively small orders is a challenging, and perhaps insurmountable, problem, though it should be of interest to all combinatorialists. A more productive approach would be to identify and study MVD's of particularly "nice" (i.e., highly symmetrical) types for which the size r is "small" relative to the order n. This is exactly what has been done historically, though without consciousness of the connection with Venn diagrams. That connection, we believe, opens up a wide vista for further variations on the magic figure concept. For those who are so inclined, it may even be "useful" for educational and recreational purposes!

References

[BPS19] T. Jean Brice, G. Easwara Prasad, and G. Selvan. Topological Venn Diagrams and Graphs. *International Journal of Science and Research*, 8(5):1436–1440, 2019.

[BT09] Seymour S. Block and Santiago A. Tavares. *Before Sudoku*. Oxford University Press, 2009.

[Caj85] Florian Cajori. *A History of Mathematics*. AMS Chelsea Publishing, New York, 4 edition, 1985.

[FR14] Meredith Foster and David G. Robinson. Magic Venn Diagrams of Order 3. Senior Project, Math. Dept., University of West Georgia, 2014.

[GS98] R. D. Godbold and P. J. Slater. All cycles are edge-magic. *Bull. Inst. Combin. Appl.*, 22:93–97, 1998.

[KR70] A. Kotzig and A. Rosa. Magic Valuations of Finite Graphs. *Canad. Math. Bull.*, 13:451—461, 1970.

[MW13] Alison M. Marr and W. D. Wallis. *Magic Graphs*. Birkhäuser Basel, Springer Science+Business Media, New York, 2 edition, 2013.

[4]The idea is to magically label the sets in the mod n additive orbits of each k-element subset of $[n]$ and its $(n-k)$-element complement, as we did in Example 3.2. Then adjust the labels by appropriate constants to "lift up" all these orbital MVD's into a magical labelling of the whole set of all subsets of $[n]$. A similar approach works for constructing optimal Δ-MVD's of all odd orders.

[5]Even the 275 million 5×5 magic squares outnumber the 35 million MVD's of order 4, and such squares are only a tiny fraction of all the order 12 MVD's, of which they are but one type!

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[PR17] Bridgette Puckett and David G. Robinson. Magic Circle Venn Diagrams. Senior Project, Math. Dept., University of West Georgia, 2017.

[PW98] K. Pinn and C. Wieczerkowski. Number of Magic Squares from Parallel Tempering Monte Carlo. *International Journal of Modern Physics C*, 9(4):541—546, 1998.

[Rem19] Anja Remshagen. Counting Magic Venn Diagrams. *Congressus Numerantium*, 234:261–272, 2019.

[RR] David G. Robinson and Anja Remshagen. Magic Venn diagrams. (preprint) July, 2023 (75 pages).

[RSW06] Frank Ruskey, Carla D. Savage, and Stan Wagon. The Search for Simple Symmetric Venn Diagrams. *Notices of the AMS*, 53(11):1304–1311, 2006.

[SG76] R. Schroeppel and M. Gardner. Mathematical Games. *Scientific American*, 249:118, 1976.

[Tre04] Marián Trenkler. Magic Stars. *The PME Journal (USA)*, 11(10):549–554, 2004.

[Ven80] John Venn. On the diagrammatic and mechanical representation of propositions and reasonings. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 9:1–18, 1880.

[Wen] Wu Wenjun, editor. *Grand Series of History of Chinese Mathematics*, volume 6. Part 6 Yang Hui, section 2 Magic circle.