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Numbers Without Ones (Revisited)

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Dedicated to the memory of Andrew Vogt

Abstract

We discuss some open problems about occurrences of the digit one.

1 Introduction

The main problem to be discussed here occurred to me many years ago as I was explaining to a class of philosophy students why the rational numbers are countable. What more graphic way to map positive rationals one-one to natural numbers, I thought, than to replace the line between numerator and denominator by the digit 1? Thus, 2/3 would map to 213, 3/4 to 314. More precisely, each rational number n/m would be mapped to the smallest natural number that can be represented in base 10 as $d_1 \dots d_j 1 e_1 \dots e_k$ where $d_1 \dots d_j$ and $e_1 \dots e_k$ represent numbers n' and m' such that $n'/m' = n/m$ and none of the digits $d_1, \dots, d_j$ or $e_1, \dots, e_k$ is a one. Of course this technique would only work in general if every rational can itself be expressed without any occurrences of the digit one. So the question that emerged was:

Question 1.1. *For every natural numbers a and b is there a natural number m such that both ma and mb lack the digit* 1*?*

Lets call a number m that provides an affirmative answer to Question 1.1 a 1-*eliminating multiplier* for a and b. An affirmative answer to Question 1.1 is sufficent to show that the graphic map is feasible, but it is not necessary. It is conceivable that there is a fraction a/b that has a 1-free equivalent in "lower terms" even though there is no 1-eliminating multiplier for a and b. This raises another question:

Question 1.2. *Are there natural numbers a, b and n such that a and b have a* 1*-eliminating multiplier, but an and bn do not?*

I was unable to find any examples to refute a positive answer to Question 1.1 or confirm a positive answer to question Question 1.2, so I consulted my talented and versatile colleague in the mathematics department, Andrew Vogt. Andy and I combed through the literature and, when we found no answer our question, we worked on the issue ourselves for some time. The results of our investigations (which answer neither question above) appear in [KuV92]. In the 30 years since that publication I have heard of no answers to these questions; indeed [KuV92] has elicited little, if any response. I like to think that this is at least partly because, in adopting what we thought of as the appropriate tone for readers and referees of "serious" mathematical journals, we removed all the playfulness and joy from our manuscript, including the motivation that precipitated the inquiry. It seems about time to present some of these and related ideas in a more suitable venue and format.

2 Some Easy Variations

It is easy to see that the idea of representing rationals like this would not get off the ground if we were to use 0 as the separating digit. There is no 0-eliminating multiplier for 1 and 10, for example, because 10 itself has no 0-free multiple. Furthermore, Question 1.1 has a negative answer for numbers in base 3 notation.

Observation 2.1. *In base-*3 *notation there exist natural numbers a and b that have no* 1*-eliminating multiplier.*

To see this, we need only recall the multiplication algorithm for many-digit numbers. Let a be any number whose last non-zero base-3 digit is 1 and let b be any number whose last non-zero base-3 digit is 2. Now consider mulitplying both a and b by n according to the familiar algorithm. If the last non-zero digit of n is a 1, then the last non-zero digit of an will also be 1 (because $1 \times 1 = 1$). If the last non-zero digit of n is a 2, then the last non-zero digit of bn will be 1 (because 2×2 is 11 in base-3 notation, and we enter the second 1 and "carry" the first).

Observation 2.1 is a special case of result 2(ii) in [KuV92]: If $b = p^n$ for prime p and $k \geqslant n(p^n - p^{n-1})$, then there exists sets $\{x_1, \ldots, x_k\}$ such that for any m, at least one of $mx_1, \ldots mx_k$ has rightmost non-zero digit 1. Since 10 is not the power of a prime this, result tells us nothing about base 10. For base 10, we can use a different argument to at least obtain a negative answer for sets of numbers of size four.

Observation 2.2. *In base-*10 *notation there exist natural numbers a, b, c, d for which there is no* 1*-eliminating multiplier.*

The argument this time relies on leftmost digits. Take $1, 2, 3, 5$ as a, b, c, d and consider any multiplier m. If m begins with 1, then ma begins with 1. If m begins with 2 then m is an integer greater than or or equal to 2×10^q and less than 3×10^q, where q is the number of digits in m. So md is an integer greater than or equal to 10×10^q and less than 15×10^q, and so it must also begin with a 1. Similarly, if m begins with a 3 then

md must be greater than or equal to 15×10^q and less than 20×10^q for some q, and so it must also begin with a 1; if m begins with 4 or 5, then mc must begin with a 1, and if m begins with 6, 7, 8, or 9 then mb must begin with a 1.

Observation 2.2 is a special case of Result 1(ii) in [KuV92]: If $b \geqslant 2^k$ then there is a set $\{x_1, \ldots x_k\}$ such that for any m at least one of $mx_1, \ldots, mx_k$ has leftmost digit 1 in base-b notation. Since the base 10 is smaller than 2^4, this result does nothing to settle Question 1.1, nor do any of the other results in that paper. Indeed, they leave room for a positive or negative answer to the following:

Question 2.3. *Are there natural numbers a, b, c for which there is (in base 10) no 1-eliminating multiplier?*

3 1-eliminating Multipliers for Single Numbers

Investigations into the properties of digits of numbers have a surprisingly long history. Many early results are chronicled in chapter XX of L.E. Dickson's authoritative *History of the Theory of Numbers* ([Dik19] pp 453-465). Dickson reports (p 454) that that in the *Annales de Gergonne* issue of 1811-1812, "anonymous writers" proved that every natural number has an integer multiple of the form $9 \ldots 90 \ldots 0$ and that in 1829-30 A.L. Crelle showed that the digits '9' in this result can be replaced by any repeating sequence of digits. There is a clever and simple demonstration of this. Let k be any natural number and $\bar{d} = d_1 \ldots d_n$ be any sequence of digits. Now consider the sequence of numbers represented in base 10 by increasing repetitions of $\bar{d}$, i.e., $\bar{d}, \bar{d}\bar{d}, \ldots$, until we reach $k+1$ repetitions. Two of these will have to be the same mod k. Their difference will then be 0 mod k, i.e., it will be a multiple of k. The situation is illustrated below.

$$\begin{array}{rl} \bar{d} \ldots \bar{d}\bar{d} \ldots \bar{d} & = j \ mod \ k \\ -\bar{d} \ldots \bar{d} & = j \ mod \ k \\ \hline \bar{d} \ldots \bar{d}0 \ldots 0 & = 0 \ mod \ k \end{array}$$

Since numbers of this special form are a small proportion of the numbers without ones, we should perhaps not be surprised that the 1-free multiples (and 1-eliminating multipliers) found by the above proof are quite large. Even if we take a lone 2 as our repeating sequence of non-unit digits, the 1-free multiple must lie between 20 (which we get when the first two items in the sequence are a mod k match) and the number represented (in base 10) by a sequence of k 2's followed by a 0 (which we get when the first and last items match). As k increases the latter number approaches, but never exceeds, $(2\frac{2}{9} \times 10^k) - 2$, and so the 1-eliminating multiplier approaches, but never exceeds, $(2\frac{2}{9} \times \frac{10^k}{k}) - \frac{2}{k}$. Let's call the least 1-free multiplier of k found by comparing repetitions of 2's the *Crelle multiplier* for k and the upper bound just given the *Crelle bound* for k. Figure 1 shows the smallest 1-eliminating multipliers, Crelle multipliers and Crelle bounds (the latter rounded for display purposes to three significant digits) for numbers 1 through 25.

Number	Least Multiplier	Crelle Multiplier	Crelle Bound
1	2	20	20.2
2	1	10	110.0
3	1	740	740.0
4	1	5	5560.0
5	1	4	44400.0
6	1	370	370000.0
7	1	317,460	3.17e+06
8	1	25	2.78e+07
9	1	246,913,580	2.47e+08
10	2	2	2.22e+09
11	2	20	2.02e+10
12	2	185	1.85e+11
13	2	170,940	1.71e+12
14	2	158,730	1.59e+13
15	2	148	1.48e+14
16	2	125	1.39e+15
17	2	1,307,189,542,483,660	1.31e+16
18	2	123,456,790	1.23e+17
19	2	116,959,064,327,485,380	1.17e+18
20	1	1	20.2
21	2	105,820	1.06e+20
22	1	10	1.01e+21
23	1	966,183,574,879,227,053,140	9.66e+21
24	1	925	9.26e+22
25	1	8	8.89e+23

Figure 1: One-Eliminating Multipliers for 1-25

The differences are dramatic. Every number on the table has a 1-eliminating multiplier of one or two. The Crelle multipliers surpass 10^{20} and the Crelle bounds, 10^{23}. Further calculations suggest that the differences do not diminish for larger numbers. The smallest number requiring a one-eliminating multiplier as large as 3 is 81. ($81 \times 1 = 81$; $81 \times 2 = 162$; $81 \times 3 = 243$). The Crelle multiplier and the Crelle bound for 81 are both greater than 2.7×10^{79}. The largest 1-eliminating multiplier required any for any number between 1 and ten million is 49, required to eliminate ones from $4,580,102$. ($4,580,102 \times 49 = 224424998$). To find the largest Crelle multiplier within those bounds would require an unreasonable amount of patience and computer power, but just for numbers between one and one thousand, Crelle numbers exceed 2×10^{979}.[1] All of this suggests that there must be a strongly affirmative answer to the following:

Question 3.1. *Is there a smaller upper bound for the least* 1*-eliminating multiplier for* k *than the Crelle bound?*

[1] This is the Crelle number of 979. This and all other calculations for the paper were made by Julia programs in a Jupyter Notebook that is available at [Kuh22]

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4 The Distribution of Levels

The distribution of least 1-eliminating multipliers is intriguing. We can think of these as *levels* (the level of k indicating level of difficulty of removing ones from k.) Not all numbers are levels. If 10n is a 1-eliminating multiplier of k, then so is n. Hence no multiple of ten can be a level. On the other hand, we can use a leftmost digit argument to show that

Observation 4.1. *There is no largest level.*

For suppose the number k represented in base 10 by the digits $d_1 \ldots d_n$ has level λ. Let d be the leftmost digit of λ. Consider the number represented by $d'0 \ldots 0 d_1 \ldots d_n$, where the interior string of 0's is of length larger than the product of (1+the number of digits in k) and (1+the number of digits in λ) and d' is: 1 if d is 1; 5 if d is 2 or 3; 3 if d is 4 or 5; and 2 if d is 6, 7, 8, or 9. When k is multiplied by any number smaller than λ, the result will have a one among its non-leading digits and when it is multiplied by λ it will have a 1 as its leading digit. So k's level must be larger than λ.

Table 2 shows the smallest number of level λ (which we have labeled the *least exemplar* of λ) for λ between one and one hundred.

Level	Least_Exemplar	Level	Least_Exemplar	Level	Least_Exemplar	Level	Least_Exemplar
1	2	26	7701303	51	40050102	76	26800501
2	1	27	7428017	52	39005301	77	260050102
3	81	28	7142859	53	39008631	78	30060134
4	51	29	701303	54	38008513	79	257700501
5	531	30	none	55	37007018	80	none
6	5421	31	6516354	56	36513007	81	250050102
7	3157	32	6253014	57	35131007	82	244005301
8	2571	33	6253013	58	35007013	83	241005401
9	25517	34	5890103	59	50251301	84	239005301
10	none	35	5780102	60	none	85	24005401
11	25157	36	5701027	61	328007013	86	232900501
12	20531	37	5420102	62	40060201	87	231004009
13	26571	38	540102	63	321006015	88	227900501
14	143053	39	530102	64	32006013	89	229100501
15	15703	40	none	65	40050201	90	none
16	130305	41	5010268	66	303900501	91	2200400915
17	140257	42	5240103	67	30200501	92	240060301
18	125703	43	5130103	68	30050102	93	220045801
19	105302	44	4550102	69	30050201	94	2130040501
20	none	45	4570102	70	none	95	2110040084
21	109027	46	5010269	71	290050102	96	229900501
22	91027	47	5026901	72	277890051	97	210045801
23	90271	48	43008018	73	274006012	98	22900501
24	102053	49	4580102	74	270900501	99	210043008
25	839031	50	none	75	30134006	100	none

Figure 2: Least Exemplars

A quick inspection of the table reveals that, except for multiples of ten, every number less than 100 is the level of some number less than 3×10^{10}. This suggests the following

question.

Question 4.2. *Is every number that is not a multiple of ten a level?*

The argument for observation 4.1 shows that if k has level λ there is some number less than $10^{k+\lambda+1}$ that is of level greater than λ. In the table, however, the least exemplar of a level λ, is always less than 11 times the least exemplar of the immediately preceding level (and frequently it is less than that exemplar itself). This makes one wonder about the "growth rate" of levels. In particular:

Question 4.3. *Can one find a better bound on the gap between the least exemplars of successive levels than that given in the proof of 4.1?*

References

[Dik19] Dickson, L.E. (1919) *History of the Theory of Numbers, (Vol. 1: Divisibility and Primality)*, Carnegie Institution of Washington.

[KuV92] Kuhn, S., Vogt, A. (1992) 'Numbers Without Ones' *Fibonacci Quarterly*, **30** (1), pp. 48–53.

[Kuh22] Kuhn, S. (2022) 'Numbers-Without-Ones-Revisited', `https://github.com/steven-t-kuhn/Numbers-Without-Ones-Revisited`.