No π for you!

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Dedicated to the memory of Jonathan and Peter Borwein.

Abstract

In which we serve up several helpings of not-π.

1 Un-Identities for π

Identities for π abound. So prevalent are they that it is easy at times to think we see π even where it is not. In this paper we present four examples of "un-identities" for π, expressions which might seem to be π but on closer examination reveal themselves to be not-π instead. This paper is about curating a few nice examples; we will explain a little of the mathematics but mainly we'll refer the reader to original sources to find the details.

2 "Everybody knows" that π is $\frac{22}{7}$

> [T]he ratio of the diameter and circumference is as five-fourths to four[.]
>
> 1897 Indiana Engrossed House Bill No. 246 ("Indiana Pi Bill")

We all learn the approximation $\pi \simeq 22/7$, and most of us learn that they are not in fact equal. A beautiful demonstration of this, in fact showing $22/7 > \pi$, appeared as a Putnam competition problem in 1968 [AKLoA03]. It relies on the evaluation of an integral which is accessible to students in a first calculus class:

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x$$

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Evaluation of the integral involves expansion of the numerator, followed by polynomial long division to obtain

$$\begin{aligned}\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x &= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 - \frac{4}{1+x^2}\right)\mathrm{d}x \\ &= \left(\frac{x^7}{7} - \frac{4x^6}{6} + x^5 - \frac{4x^3}{3} - 4\arctan x\right)\Bigg|_0^1 = \frac{22}{7} - \pi\end{aligned}$$

Since the original integrand is positive on $(0, 1)$, the integral must be greater than 0, and hence $22/7 > \pi$. Since $3.2 > 22/7$, this approach might have been of some use to amateur mathematician Edward J. Goodwin, whose misguided work on squaring the circle led to the infamous "Indiana Pi Bill" quoted at the start of this section, in which the State of Indiana came very close to declaring by legislative fiat that $\pi = 3.2$ [Sta97].

This idea can be extended to prove other similar inequalities for continued fraction approximations of π, using rational functions of higher degree. For example, the continued fraction approximation

$$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = 355/113$$

can be seen to be not-π using same techniques of expansion, polynomial long division, and evaluation of the definite integral [Luc05]:

$$0 < \int_0^1 \frac{x^8\,(1-x)^8\,(25+816x^2)}{3164\,(1+x^2)}\,\mathrm{d}x = \frac{355}{113} - \pi$$

3 π for a while

> It seems to me there's this grand mathematical world out there, and I am wandering through it and discovering fascinating phenomena that often totally surprise me. I do not think of mathematics as invented but rather discovered.
>
> George Andrews

In 2001, Jonathan Borwein [BD09, p. 138] discovered using a computer algebra package that

$$
\begin{aligned}
\int_0^\infty \frac{\sin x}{x}\mathrm{d}x &= \frac{\pi}{2}\\
\int_0^\infty \frac{\sin x}{x}\frac{\sin(x/3)}{x/3}\mathrm{d}x &= \frac{\pi}{2}\\
\int_0^\infty \frac{\sin x}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\mathrm{d}x &= \frac{\pi}{2}\\
&\vdots\\
\int_0^\infty \frac{\sin x}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\cdots\frac{\sin(x/13)}{x/13}\mathrm{d}x &= \frac{\pi}{2}
\end{aligned}
$$

but, to his surprise and brief consternation, that

$$
\int_0^\infty \frac{\sin x}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\cdots\frac{\sin(x/13)}{x/13}\frac{\sin(x/15)}{x/15}\mathrm{d}x < \frac{\pi}{2}
$$

At first, he assumed that this was a bug in the software package (and the developers of the package agreed with him!) However, it turned out [BB01] that the pattern *does not* continue! One approach to evaluating these integrals involves integrating over multiple variables with $S_n = \sum_{k=1}^{n} 1/(2k+1)$ playing a key role. When $S_n < 1$ the domain of integration is a hypercube but when $S_n > 1$ a 'bite' is taken out of the hypercube [BBG04, p. 98-102]. Taking a bite out of the hypercube corresponds to the integral being less than $\pi/2$. Since

$$
S_6 = \sum_{k=1}^{6} \frac{1}{2k+1} = \frac{43024}{45045} < 1
$$

but

$$
S_7 = \sum_{k=1}^{7} \frac{1}{2k+1} = \frac{46027}{45045} > 1
$$

it is at the 7^{th} term that we begin taking bites out and the value of the integral drops below $\pi/2$ [BBG04, p. 123-124].

Schmid subsequently used Fourier transforms and convolutions with unit-area rectangles to introduce a factor of $2\cos x$ that allowed him to extend the pattern much further. Indeed, he showed that

$$
\begin{aligned}
\int_0^\infty 2\cos(x)\frac{\sin(x)}{x}\mathrm{d}x &= \frac{\pi}{2} \\
\int_0^\infty 2\cos(x)\frac{\sin x}{x}\frac{\sin(x/3)}{x/3}\mathrm{d}x &= \frac{\pi}{2} \\
\int_0^\infty 2\cos(x)\frac{\sin x}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\mathrm{d}x &= \frac{\pi}{2} \\
&\vdots \\
\int_0^\infty 2\cos(x)\left(\prod_{i=1}^{111}\frac{\sin(\frac{x}{2i-1})}{2i-1}\right)\mathrm{d}x &= \frac{\pi}{2}
\end{aligned}
$$

but

$$
\int_0^\infty 2\cos(x)\left(\prod_{i=1}^{k}\frac{\sin(\frac{x}{2i-1})}{2i-1}\right)\mathrm{d}x < \frac{\pi}{2}
$$

for $k = 113$ and beyond [Sch14]. In addition to the original papers in which further mathematical details of these astounding series of not-π integrals may be found, we direct the gentle reader to John Baez's blog in which he takes a moving averages approach to intuitively understand why such promising patterns break down at a certain point [Bae18].

4 π for a long while ...

Whose says pie are square? Pie are round!

Unknown source

... where "long while" is somewhat more than 40 digits.

Many of us will have taught the need for proof to verify conjectures based on observation of patterns, often using examples such as the quadratic polynomial x^2-x+41, which takes prime values for non-integers $-39 \leqslant x \leqslant 40$, but clearly is composite when $x = 41$. In this section, we provide a somewhat more sophisticated example of an approximation for π that works for a very long time ... until it doesn't.

Consider the following expression[BB92]:

$$
P = \left(\frac{1}{10^5}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{n^2}{10^{10}}\right)\right)^2
$$

The inner summand $S_n = \exp(-n^2/10^{10})$ is varying extremely slowly; for example, $S_{100} = 0.999999000000500$ and $S_{1000} = 0.999900004999833$. If we approximate the sum by an integral, we see that we expect the expression to evaluate to something very close to π. The inner infinite sum is approximately equal to

$$\frac{1}{10^5}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{n^2}{10^{10}}\right)\simeq\int_{-\infty}^{\infty}\exp(-x^2)\mathrm{d}x=\sqrt{\pi}$$

so P should be a reasonable approximation to π. Indeed it is! P is not only equal to π to more than 40 digits, it is correct to more than 42 *billion* digits!

When the first author encountered this statement in [BBB97, p. 689] while teaching a history of math course on Pi Day, it was unaccompanied by an explanation or even a citation. Fortunately, during a pleasant walk back across campus from class to office, he was able to figure out what techniques might be useful. If we recognize that this is a special type of Jacobi theta function, and recall that theta functions satisfy some lovely identities (see for example [Odl95, p. 1093]) then we can write

$$\theta(t)=\sum_{n=-\infty}^{\infty}\exp(-t\pi n^2)$$

and observe that $\theta(t)=\frac{1}{\sqrt{t}}\theta(\frac{1}{t})$. If we now set $t=\frac{1}{10^{10}\pi}$, we have

$$\begin{aligned}P &= \left(\frac{1}{10^5}\theta(t)\right)^2\\ &= \frac{1}{\sqrt{t}}\theta(\frac{1}{t})\\ &= \left(\frac{10^5\pi}{10^5}\sum_{-\infty}^{\infty}\exp(-\pi n^2 10^{10}\pi)\right)^2\\ &\simeq \pi(1+4\exp(-\pi^2 10^{10})+\varepsilon)\end{aligned}$$

where $\varepsilon\simeq 4\exp(-2\pi^2 10^{10})$. Hence we see that P exceeds π by approximately $3.07354655934306\cdot 10^{-42863147299}$. Clearly, by replacing 10^{10} by something larger, we can make the approximation even better: for example, if we replace it with 10^{20} we obtain an approximation correct to 428631472996115661548 digits!

5 Not-π in the right place

> Computers are pencils with power steering.
>
> George Andrews

Those who have memorized the decimal expansion of π to enough digits will, on careful examination, notice that

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$$3.14159\underline{0}6535897932404626433832695028841 97\ldots$$

is *not quite* π: indeed it is incorrect in the sixth decimal place. Those who have memorized a few more digits, though, might also notice that the following ten digits are correct. In fact, of the forty digits shown, 36 are correct. It's easy enough to create a string of digits with this property; there are $\binom{40}{4} \cdot 9^4$ of them. This particular string of digits, though, happens to be a truncation to 500,000 terms of the Gregory series for π:

$$4\sum_{k=1}^{500{,}000} \frac{(-1)^{k-1}}{2k-1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$$

The proof of the alternating series test for convergence also gives information about the oscillation of error around the partial sums S_n generated by truncations of the series. When the sequence of general terms a_n is smoothly decreasing in absolute value, as occurs with the Gregory series, we can expect that

$$|S - S_{2k-1}| \approx |S_{2k} - S| \approx \frac{1}{2}|S_{2k} - S_{2k-1}| \approx \frac{1}{2}a_{2k}$$

When we take the partial sum $S_{2k=500{,}000}$, we can thus estimate the error as

$$\frac{1}{2}a_{250000} = \frac{1}{2}\left(\frac{1}{500000-1}\right) \approx 10^{-6}$$

It should now be unsurprising that the sixth digit in the decimal expansion of the truncated series is incorrect. What *is* surprising is where the other parts of the error appear. The following indicates the positions of the first few incorrect digits.

$$3.14159\underline{0}6535897932\underline{40}46264338 32\underline{6}9502884197\ldots$$

This intriguing observation was made by R. D. North, an amateur mathematician (but perhaps rather more serious than Goodwin, who nearly persuaded the Indiana legislature to pass the bill cited in section 2). North sent his observation to Jonathan Borwein with a request for an explanation. In [BBD89], Borwein, Borwein and Dilcher proved that this is no accident. Here we provide a high-level overview of the basic ideas behind their analysis, referring the reader to the original for full details.

Some gentle readers will have encountered Euler Maclaurin Summation. (Those who have not are encouraged to explore this beautiful area of mathematics.) Euler Maclaurin summation is a powerful tool for estimating sums by integrals; it can be especially powerful at estimating truncated sums of convergent series of positive terms [Odl95, p. 1090-1092].

For sums of alternating series Boole developed a similar technique (although Euler probably had an equivalent formula). We wish to understand the error

$$4\sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1} - \pi = 2\left(2\sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1} - \frac{\pi}{2}\right)$$

so we wish to estimate the tail

$$2\sum_{k=n}^{\infty}\frac{(-1)^k}{2k+1}$$

Boole's formula gives

$$2\sum_{k=n}^{\infty}\frac{(-1)^k}{(2k+1)} = (-1)^n\sum_{l=0}^{N}\frac{E_{2l}}{(2n)^{2l+1}} + R$$

in which

$$|R| < \frac{E_{2N}}{(2n)^{2N+1}}$$

and E_{2l} is an Euler number. This explains why the errors in the truncation of the Gregory series are so sparse: n is 5 times a power of 10 so $2n$ is a power of 10 and the errors align with a sparse subsequence of the digits, at least for a while.

Since the Euler number E_{2N} grows somewhat faster than $\left(\frac{4N}{\pi e}\right)^{2N}$ we can't take N to be too big. Indeed, if N is larger than $n\pi/2$ then R goes to infinity! Of more interest, perhaps, is how many terms can we take before the errors overlap? Heuristically, if $2n = 10^k$, then the errors start to overlap when l is around $k/\log_{10}(k)$. If we compute the values to high precision, when $k = 6$, we see that the error for $l = 8$ doesn't overlap the error for $l = 7$: but the error for $l = 9$ overlaps the error for $l = 8$, and subsequent terms all overlap.

For a unified approach to Euler Maclaurin and Boole summation, see [BCM09]. As a final observation that amuses us and, we hope, will also amuse others, we note that while Euler Maclaurin summation involves Bernoulli numbers, ironically Boole summation involves Euler numbers. This shows again that Euler had a finger in every pie and also in some slices of not-π.

6 Conclusion

Sometimes it seems as though π can be found everywhere. We hope the gentle reader will have enjoyed this serving of not-π.

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