Stern, Gamow, and the Shabbat Elevator Effect

Loukas Georgiou [1]

[1] *University of Pikeville-Kentucky College of Osteopathic Medicine*
147 Sycamore St. Pikeville, KY 41501
loukasgeorgiou@upike.edu

Abstract

In addition to being known for their other work, physicists George Gamow and Marvin Stern also published an analysis of an intriguing problem regarding their perception that although both worked on different floors, it seemed that the first elevator to arrive on their respective floors was traveling in the opposite direction of their destination. In this manuscript, a fictitious conversation occurs between the two in regards to the probability of the elevator stopping on every floor (which they term the Shabbat Effect). One claims that it is the number of elevator occupants that predominates whereas the other claims it is the number of floors in the building that predominates in giving rise to this effect. Equations for estimating the probability of the Shabbat Effect are developed and analyzed. The resulting equations show that the relationship between the number of elevator occupants and floors is surprisingly complicated. Consequently, no general rule can be given for whether the probability of the Shabbat Effect increases or decreases for an arbitrary change in the number of occupants and floors. However, it discovered that for similar changes in the number of occupants and floors, it is the number of floors that predominates in determining the probability of the elevator stopping on every floor.

1 The number of elevant occupants or the number of floors: Which is the dominant factor in causing an elevator to stop on every floor?

In 1958, physicists George Gamow and Marvin Stern first published an analysis of an intriguing situation that they had noticed while working for the Corvair Corporation in San Diego, California. This now famous problem is eponymously known as the *Gamow-Stern Elevator Problem.* The problem originated when Gamow, working on the second floor of the building, would take the elevator up to see Stern, who was working on the sixth floor, and *vice versa.* It seemed to both individuals that the first elevator to arrive at their respective floors was traveling in the opposite direction of their desired travel. The analysis of this paradox was published in their book *Puzzle-Math.*[GaS58] Later, Robert Knuth discovered and corrected an erroneous assumption that the two had made with regards to an infinite set of elevators.[Knu69]

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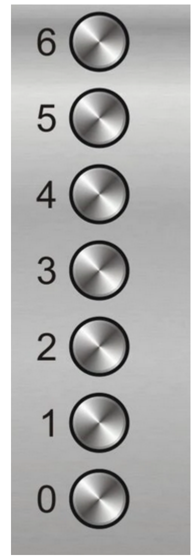

In addition to having the elevator stop on a given floor only to be going the wrong way, a close second to causing impatience is having the elevator simply stop on every floor. Therefore, it isn't hard to imagine the following scenario occurring between the two inquisitve phsyicists.

After completing their work in San Diego, Gamow is sent to an East Coast City and works in a building with more floors than his previous office tower. Stern now finds himself in a city in the Midwest working in a building with less floors than his previous office tower. Both, however, find themselves working near the top floor. It doesn't take long before the two are discussing elevators. Gamow comments that Stern is better off as Stern's smaller building implies fewer elevator occupants and therefore a lesser probability of stopping on all floors on the way to Stern's office whereas Gamow, being in a larger building, would expect a larger number of elevator occupants stopping on lower floors. On the contrary, Stern replies that Gamow is much better off since the larger number of floors in Gamow's building would lower the chance of stopping on every floor whereas Stern, being in a building with fewer floors, would expect a higher chance of stopping on each floor. Clearly, the answer of whether Stern or Gamow is right lies in the relationship between the number of occupants in an elevator and the number of floors to which the elevator travels.

The following is a mathematical analysis of the situation which for conciseness has been named the Shabbat Elevator Effect. A Shabbat elevator is an elevator which works in a special mode in operating automatically by stopping on every floor of the building to satisfy the Jewish law requiring the observant to abstain from operating electrical switches on Shabbat (the Sabbath). Passengers simply step on or off the elevator on their floors of departure or arrival respectively.

The situation is idealized in assuming that a single elevator of infinite capacity is present that travels in a unidirectional trip from the ground floor to the upper floors and it is assumed that no one else will enter the elevator as it ascends through the building.

The probability of having the elevator stop on every floor as seen by an outside observer is first calculated. Then, the probability of having the elevator stop on every floor as seen by an occupant of the elevator is derived. It is noted that the two probabilities are not necessarily the same as an occupant will perceive the elevator stopping on every floor if the elevator stops on every floor beneath the passenger's disembarkation although it may not stop on every floor above this level. Finally, an analsysis is made with regard to the change in probability of the Shabbat Effect when the number of floors and passengers is altered in order to determine the dominance of either the number of floors or number of passengers.

2 The probability that the elevator will stop on every floor of the building as seen by an outside observer

Assume that a building has a total of F number of floors above the main floor (see Figure 1). Futhermore, let each floor of the building be sequentially numbered in an ascending fashion with the main floor (synonymous with naming it the ground floor) being numbered as 0. As an example, a building with 10 floors above the main (or ground floor) would have $F = 10$. The author has noted that this method of numbering of the floors is not only more conducive to the mathematical analysis but also agrees with the traditional numbering of floors in most countries with the United States being the notable excption in which the main floor is considered the first floor. In addition, let P represent the number of occupants on the elevator when it

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starts its ascent from the main floor. Furthermore, it stands to reason that $P \geq F$ otherwise the probability of stopping on every floor is *a priori* equal to 0 since otherwise there will not be enough passengers to distribute to every floor.

At this point, we are now interested in the total number of ways that all P individuals can be distributed to a total of F floors with each floor (above the gound floor) having at least one individual. This is similar to the number of ways of distributing a total n distinguishable balls into k distinguishable boxes with each box containing at least one ball. The total number of ways in which this can be done is given by $k!\mathbf{S}_2\left\{{n \atop k}\right\}$ with $\mathbf{S}_2\left\{{n \atop k}\right\}$ representing the Stirling number of the second kind.[Stirl] Similarly, the total number of ways D_s (with the subscript s being shorthand for the word “Shabbat" in describing the effect of stopping on every floor) to distribute P distinguishable individuals among F distinguishable floors with at least one person getting off on each floor is given by:

$$D_s = F!\ \mathbf{S}_2\left\{{P \atop F}\right\} \tag{1}$$

The total number of ways D_t (with the subscript t being used as shorthand for “total") to distribute P individuals among F floors without restriction is given by:

$$D_t = F^P \tag{2}$$

The probability S that the elevator stops on every floor is given by the ratio of the number of the total distributions in which the elevator stops on every floor D_s, given by Eq. (1), to the total number of distributions D_t, given by Eq. (2), as:

$$S = \frac{D_s}{D_t} = \frac{F!\ \mathbf{S}_2\left\{{P \atop F}\right\}}{F^P} \tag{3}$$

3 The probability that the elevator will stop on every floor as seen by the occupant of the elevator

After confirming the validity of Stern's analysis, Gamow realizes that Eq. (3) only applies to the probability of the elevator stopping on every floor of the building and not necessarily to the Shabbat Effect as experienced by an occupant of the elevator. An elevator occupant alighting on a middle floor may be annoyed that the elevator stopped on each prior floor and perceives the elevator as stopping on every floor when in reality it is possible that the elevator bypass floors higher up. From an outside observer's point of view, stopping on every floor means *stopping on every floor of the building*. From an elevator occupant's point of view, stopping on every floor means *stopping on every floor during his or her duration of travel*. Therefore, the probability of this latter effect occuring will be designated as S^* (with the superscript * designating the point of view of the elevator passenger) and requires a more complicated derivation.

Assume that Mr. Able (wearing a black suit) will be alighting on upper floor F_a (with the subscript a being shorthand for the word “Able") in a building with total number of upper floors F (See Figure 2). From Mr. Able's point of view, any of the other passengers alighting on his floor (i.e. floor F_a) or above become irrelevant in contributing to the Shabbat Effect. Once again, let P represent the total number of occupants getting on the elevator on the ground floor (note

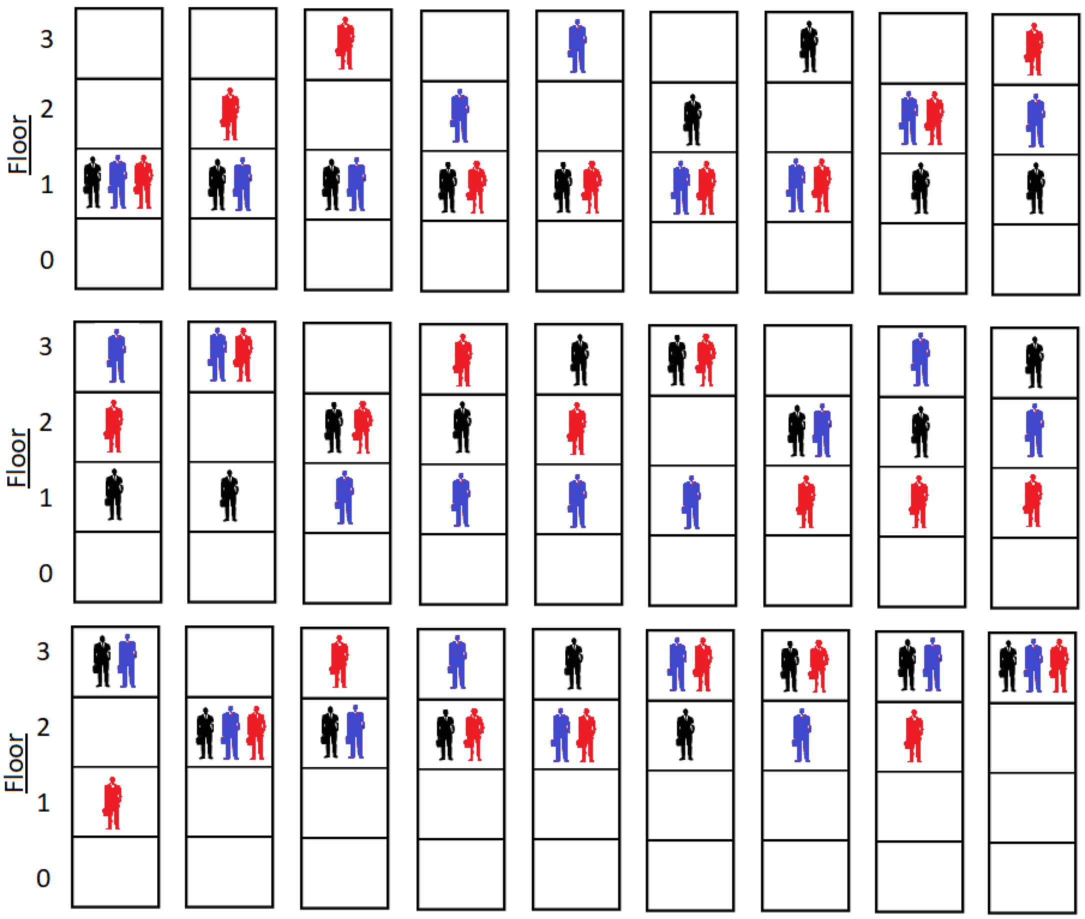

Figure 1: In the above graphic, there are three elevator occupants with $P = 3$ and a total of three upper floors with $F = 3$. There are a 27 possible distributions of passengers alighting on the three floors as given by Eq. (2). Of these, only 6 distributions involve the elevator stopping on every floor of the building as given by Eq. (1). The probability of the elevator stopping on every floor is 2/9 as given by Eq. (3).

that this number includes Mr. Able). Also, let i represent the number of individuals getting off on the elevator prior to floor F_a. Therefore, there are a total of:

$$(F_a - 1)!\ \mathbf{S}_2\left\{ \begin{matrix} i \\ F_a - 1 \end{matrix} \right\}$$

ways in which at least one person is deposited on the floors preceding floor F_a. Since each passenger is distinguishable, there are a total of $\binom{P-1}{i}$ ways that this distribution can happen for i individuals. This now gives

$$\binom{P-1}{i}(F_a-1)!\ \mathbf{S}_2\left\{\begin{matrix} i \\ F_a-1 \end{matrix}\right\}$$

as the number of ways in which i distinguishable individuals may be distributed to the floors below floor F_a with at least one person alighting on each floor. Note that in calculation of the combination function, a total of $P-1$ individuals are available since Mr. Able is not available to get off on the prior floors.

If a total of i individuals get off on the floors below Mr. Able, then this leaves a total of $P-1-i$ individuals to be distributed on floors between F_a through F inclusive. Consequently, the number of distributions giving the Shabbat Effect up to and including floor F_a-1 needs to be multiplied by the factor:

$$[F-(F_a-1)]^{(P-1-i)}$$

It should be noted that at least F_a-1 individuals must be available to alight on the floors prior to Mr. Able's floor of F_a. The most that can alight on the previous floors is given as $P-1$ (i.e. all but one with the one being Mr. Able). These restrictions serve as the lower and upper limits for variable i giving $F_a-1 \leqslant i \leqslant P-1$. The total number of distributions is given by summing over all acceptable values of i. Furthermore, it should be noted that if Mr. Able alights on the first floor above ground level (i.e. $F_a=1$), he will never experience the effect of stopping on every floor. The Shabbat Effect can occur only when he has at least one floor underneath his floor of disembarkation onto which the other pasengers may force the elevator to stop. Therefore, the restriction is placed that $F_a \geqslant 2$.

Therefore, the total number of distributions D_s^* giving the Shabbat Effect is:

$$D_s^* = \sum_{i=F_a-1}^{P-1} \binom{P-1}{i}(F_a-1)!\ \mathbf{S}_2\left\{\begin{matrix} i \\ F_a-1 \end{matrix}\right\}[F-(F_a-1)]^{(P-1-i)} \qquad (F_a \geqslant 2) \tag{4}$$

With the exception of Mr. Able, the number of ways to distribute the elevator's passengers among F floors is given by:

$$D_t^* = F^{(P-1)} \tag{5}$$

Finally, the probability of Mr. Able as an occupant of the elevator experiencing the Shabbat Effect now designated by S^* is given by the ratio of Eq. (4) and Eq. (5):

$$S^* = \frac{\sum_{i=F_a-1}^{P-1} \binom{P-1}{i}(F_a-1)!\ \mathbf{S}_2\left\{\begin{smallmatrix} i \\ F_a-1 \end{smallmatrix}\right\}[F-(F_a-1)]^{(P-1-i)}}{F^{(P-1)}} \qquad (F_a \geqslant 2) \tag{6}$$

4 Determining the dominant factor between elevator occupants and upper floors on the Shabbat effect.

The crux to the answer that resolves the discussion between Gamow and Stern rests with whether the number of occupants is the dominant factor or whether the number of upper floors is the dominant factor. Alternatively, it may be found that the dominant factor alternates between both depending on the actual number of occupants or floors. Unfortunately, equation

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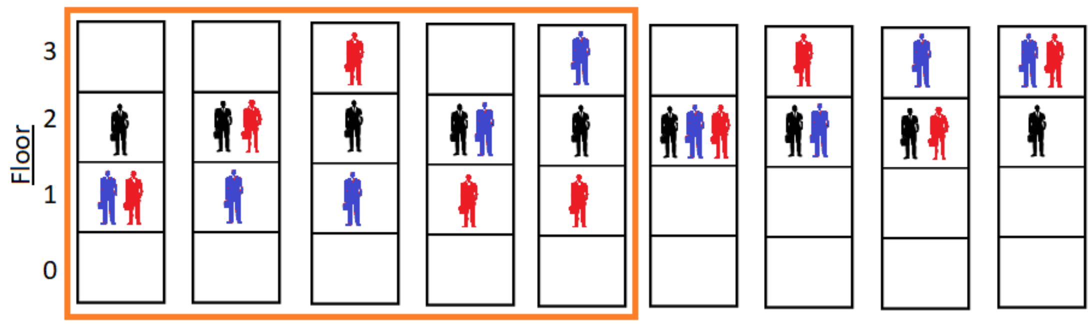

Figure 2: A total of $P = 3$ passenger are on the elevator in a building with a total of $F = 3$ upper floors. Mr. Able in the black suit alights on the second upper floor. There are a total of 9 distributions as given by Eq. (5) that allow for this possibility. However, from his point of view, the elevator stops on every floor for 5 of these distributions as given by Eq. (4). From his perspective, the probability of the elevator stopping on every floor prior to him getting off is 5/9.

(3) does not allow for a concise rearrangment to calculate the impact of changing the number of occupants versus the number of floors.

However, Gamow is able to answer the question in a more narrow setting. He notes that a change in the number of elevator occupants has an effect opposite to that of a change in the number of floors. For example, increasing the number of elevator occupants increases the probability of the Shabbat Effect whereas increasing the number of floors decreases the probability. Alternatively, decreasing the number of occupants decreases the probability of the Shabbat Effect whereas decreasing the number of floors increases the probabilty. Therefore, Gamow reasons that he would be able to determine whether the number of occupants or number of floors is dominant by seeing the net change in the Shabbat Effect by simultaneously altering the two values by a unit change in similar directions (i.e increase the number of occupants and number of floors by the same amount or, alternatively, decreasing the number of occupants and number of floors by the same amount).

Gamow proposes calculating the probability of the Shabbat Effect for a given P and F as given by Eq. (3) and calls this value S_o . He then proposes increasing the number of elevator occupants by one to $P+1$ and increasing the number of floors by one to $F+1$ and calculating the new probability of the Shabbat Effect and letting this quantity be referred to as S_{+1} (with "+1" referring to a unit increase in both the number of passengers and number of floors). He then takes the ratio of S_o and S_{+1}. A ratio of $S_o/S_{+1} > 1$ cannotes the elevator has less of a probability of stopping on every floor whereas a ratio of $S_o/S_{+1} < 1$ cannotes a greater probability. The former implies that the number of floors carries more weight in determing the probability of the elevator stopping on every floor whereas the latter implies that the number of passengers carries more weight.

In a similar fashion, Gamow once again calculates the probability of the Shabbat Effect for a given P and F as given by Eq. (3) and calls this value S_o . However, he now proposes decreasing the number of elevator occupants by one to $P-1$ and decreasing the number of floors by one to $F-1$ and calculating the new probability of the Shabbat Effect and letting this quantity be referred to as S_{-1} (with "-1" referring to a unit decrease in both the number of passengers and number of floors). He then takes the ratio of S_o and S_{-1}. A ratio of $S_o/S_{-1} > 1$ cannotes

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the elevator has less of a probability of stopping on every floor whereas a ratio of $S_o/S_{-1} < 1$ cannotes a greater probability. The former implies that the number of passengers carries more weight in determing the probability of the elevator stopping on every floor whereas the latter implies that the number of floors carries more weight.

Unfortunately, Eq. (3) is also resistant to this general mode of analysis since there is no simple correlation between S_o and S_{+1} nor S_o and S_{-1}.

However, this ratio can be calculated for a two specific cases. The first case is that in which the number of elevator occupants is equal to the number of floors. The second case is in which the number of elevator occupants is one more than the number of floors.

4.1 The number of elevator occupants and floors are equal

A scenario that can be explicitly solved involves the situation in which the number of elevator occupants and floors are equal to the number n with $n \geqslant 2$. Therefore, with $P = F = n$ as the baseline, the probability of stopping on every floor S_o is given by Eq. (3) as

$$S_o = \frac{n!\ \mathbf{S}_2\left\{ {n \atop n} \right\}}{n^n} = \frac{n!}{n^n} \tag{7}$$

Increasing the number of occupants and floors by one gives the probability of stoppinng on every floor as

$$S_{+1} = \frac{(n+1)!\ \mathbf{S}_2\left\{ {n+1 \atop n+1} \right\}}{(n+1)^{n+1}} = \frac{(n+1)!}{(n+1)^{n+1}} \tag{8}$$

Dividing S_o by S_{+1} gives

$$\begin{aligned} \frac{S_o}{S_{+1}} &= \frac{(n+1)^n}{n^n} \\ \frac{S_o}{S_{+1}} &> 1 \end{aligned} \tag{9}$$

thereby cannoting that, for a unit increase in both occupants and floors, the probability of stopping on every floor decreases which implies that the number of floors is the dominant factor.

Alternatively, a unit decrease in both the number of occupants and floors can be calculated. The new probability of stopping on every floor S_{-1} is given by

$$S_{-1} = \frac{(n-1)!\ \mathbf{S}_2\left\{ {n-1 \atop n-1} \right\}}{(n-1)^{n-1}} = \frac{(n-1)!}{(n-1)^{n-1}} \tag{10}$$

Dividing S_o by S_{-1} gives

$$\begin{aligned} \frac{S_o}{S_{-1}} &= \left(\frac{n-1}{n}\right)^{n-1} \\ \frac{S_o}{S_{-1}} &< 1 \end{aligned} \tag{11}$$

thereby cannoting that, for a unit decrease in both occupants and floors, the probability of stopping on every floor increases which again implies that the number of floors is the dominant factor.

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4.2 The number of elevator occupants is one more than the number of floors

The change in the Shabbat Effect can be explicitly calculated for another special case by taking note of the the identity

$$\mathbf{S}_2\left\{ {n \atop n-1} \right\} = \binom{n}{2} \tag{12}$$

which then allows for the probability of the Shabbat Effect to be calculated when the number of occupants is one unit larger than the number of floors.

Let S_a represent the probability of stopping on every floor as given by Eq. (3) when $P = n$ and $F = n-1$ with $n \geqslant 2$. Therefore,

$$S_a = \frac{(n-1)!\ \mathbf{S}_2\left\{ {n \atop n-1} \right\}}{(n-1)^n} = \frac{(n-1)!\binom{n}{2}}{(n-1)^n} \tag{13}$$

Let S_b represent the probability of stopping on every floor when there is a unit increase in the number of occupants with $P = n+1$ and a unit increase in the number of floors gives $F = n$. The value of S_b can be calculated through the use of Eq. (3) as

$$S_b = \frac{n!\ \mathbf{S}_2\left\{ {n+1 \atop n} \right\}}{n^{n+1}} = \frac{n!\binom{n+1}{2}}{n^{n+1}} \tag{14}$$

Let R represent the ratio of S_a and S_b in which case

$$R = \frac{S_a}{S_b} = \left(\frac{n}{n-1}\right)^n \cdot \frac{n-1}{n+1} \tag{15}$$

An examination of Eq. (15) reveals that it monotonically increases as the value of n increases. Furthermore, taking the limit of R as $n \to \infty$ gives

$$\lim_{n\to\infty}\left[\left(\frac{n}{n-1}\right)^n \cdot \frac{n-1}{n+1}\right] = \infty \tag{16}$$

Therefore, $R > 1$ for all valid values of n and therefore the Shabatt Elevator Effect decreases even though both the number of elevator occupants and the number of floors is increased (with the number of elevator occupants being one unit more than the number of floors in this scenario). This signifies that the ameliorating effect of increasing the number of floors by one unit outweighs the detrimental effect of increasing the number of elevator occupants by one unit for any number of floors present in the building.

4.3 The general impact of a change in the number of occupants versus the number of floors on the probability of stopping on every floor

Based on the two above special circumstances, initially it appears to Stern and Gamow that the number of floors outweighs the number of passengers in all cirumstances but they realize that this conclusion may not hold since, unfortunately, equation (3) does not allow for a concise rearrangment to calculate the impact of changing the number of occupants and/or floors by an amount greater than one. Nonetheless, a pattern on whether the Shabbat Effect becomes more or less probable can still be determined. A Python program was developed that allowed for the calculation of the Shabbat Effect for a given number of occupants P and upper floors F. The

Shabbat Effect was then calculated when the number of occupants or floors were then sequentially changed. The ratio of these two numbers was then taken and plotted as shown in Figure 3. If the ratio was greater than one (i.e. the probability of the Shabbat Effect was greater) that particular change was colored as a red square. If the ratio was less than one (i.e. the probability of the Shabbat Effect was lesssened), the particular change was colored as blue.

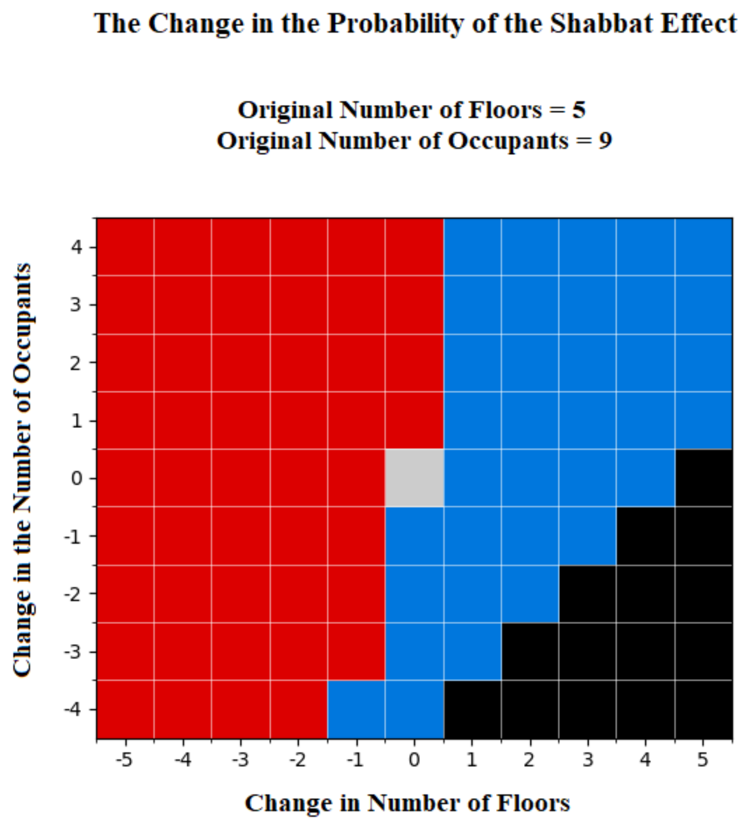

Figure 3: The figure above denotes the change in the Shabbat effect for an elevator starting its ascent with a total of 9 occupants and servicing a total of 5 upper floors. A red square denotes that the Shabbat effect worsens for that combination of changes in the number of floors and occupants. A blue square denotes that the Shabbat effect lessens for that particular change in the number of floors and occupants. A black square denotes a combination in which the number of floors exceeds the number of occupants and therefore the Shabbat effect is an impossibility. Note that for the squares surrounding the center, an increase in floor number by one results in a blue square even if the number of occupants is increased by one as predicted by equation (9). Conversely, a decrease in the number of floors by one results in a red square even if the number of occupants is decreased by one as predicted by equation (11). It can be seen that in order to compensate for a decrease in the number of floors by one, the number of occupants must decrease by four in order have an ameliorating effect on the Shabbat effect.

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They note that for small unit changes in the number of occupants or floors from baseline the number of change in floor number is the dominant factor. However, as shown in Fig. 3, this is not always the case. For example, in comparison to a building with a total of $P = 9$ elevants occupants and a total of $F = 5$ upper floors, an elevator in a building with one less floor (i.e. $F = 4$) but four less occupants (i.e. $P = 4$) has a less chance of stopping on everyfloor despite the decrease in floor number.

5 Conclusion

For small changes from the baseline number of people and floors in a building, the number of floors is the dominant factor in determining the probability of the elevator stopping on every floor. However, this is not universally the case as a large change in either parameter may increase or decrease the probability. Although formulas were developed to determine the probability of an elevator stoppinng on every floor, comparison between different scenarios requires that the calculations be performed numerically since the equations cannot be rearranged to give an answer in closed form.

The author notes that the preceding analysis is based upon no other passengers entering the elevator on floors above the ground floor (i.e. no interfloor travel is allowed). The effect of having interfloor travel with one or more passengers alighting and disembarking on floors above the ground floor piques the interest of the two physicists but preliminary analysis shows that it is a much more complicated problem requiring a lengthy derivation that is best presented at a later date. Of even greater interest is how to determine the probability of the Shabbat Effect for an infinite number of passengers in an elevator servicing an infinite number of upper floors. Stern and Gamow conclude that in order to even attack the latter problem, they need to make a trip to seek the advice of the mathematician David Hilbert who currently is residing in his *Infinite Hotel*.

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