

New Approaches to Complexity via Quantum Graphs

Eric Culf¹ and Arthur Mehta^{2,*}

¹ Faculty of Mathematics and Institute for Quantum Computing, University of Waterloo, 200 University Ave W, Waterloo, ON N2L 0A4; eculf@uwaterloo.ca

² Department of Mathematics and Statistics, University of Ottawa, 75 Laurier Avenue East, Ottawa, ON, K1N 6N6

* Corresponding author: amehta2@uottawa.ca

Received date: 7 May 2025; Accepted date: 27 August 2025; Published online: 7 October 2025

Abstract: Problems based on the structure of graphs—for example, finding cliques, independent sets, or colorings—are of fundamental importance in classical complexity. Defining well-formulated decision problems for quantum graphs, which are an operator system generalization of graphs, presents several technical challenges. Consequently, the connections between quantum graphs and complexity have been underexplored. In this work, we introduce and study the clique problem for quantum graphs. Our approach utilizes a well-known connection between quantum graphs and quantum channels. The inputs for our problems are presented as quantum channels induced by circuits, which implicitly determine a corresponding quantum graph. We show that when quantifying over all channels, this problem is complete for QMA(2); in fact, it remains QMA(2)-complete when restricted to channels that are probabilistic mixtures of entanglement-breaking and partial trace channels. Quantifying over a subset of entanglement-breaking channels, this problem becomes QMA-complete, and restricting further to deterministic or classical noisy channels gives rise to complete problems for NP and MA, respectively. In this way, we exhibit a classical complexity problem whose natural quantization is QMA(2), rather than QMA, and provide the first problem that allows for a direct comparison of the classes QMA(2), QMA, MA, and NP by quantifying over increasingly larger families of instances. We use methods that are inspired by self-testing to provide a direct proof of QMA(2)-completeness, rather than reducing to a previously studied complete problem. We also give a new proof of the celebrated reduction of QMA(k) to QMA(2). In parallel, we study a version of the closely related independent set problem for quantum graphs, and provide preliminary evidence that it may be in general weaker in complexity, contrasting to the classical case.

Keywords: Complexity Theory; Quantum Graphs ; Quantum Channels; Unentanglement.

1. Introduction

Shannon’s foundational work on zero-error capacity established a deep connection between noisy channels and graph theory [1]. Shannon showed that to each classical noisy channel N , one can associate a corresponding graph G_N , called a *confusability graph*, which encodes many important properties of the channel. Notably, G_N will have an independent set (cf. a clique) of size k if and only if there is a collection of distinct inputs $x_1, \dots, x_k$ for N which have a zero (cf. non-zero) probability of being confused as outputs.

A *quantum graph* is a generalization of a graph which allows to extend the graph notion of confusability to quantum channels [2]. In brief, given a quantum channel Φ there exists a corresponding *operator system* $\mathcal{S}_\Phi$.

A collection of states $\rho_1, \dots, \rho_k$ forms an *independent set* (cf. clique) for $\mathcal{S}_\Phi$ if the average overlap $\text{Tr}(\Phi(\rho_i)\Phi(\rho_j))$ is sufficiently small (cf. large). Quantum graphs have been studied from various points of view including channel capacity, non-local games, and non-commutative relations [3–10].

Due to the ubiquity of graphs in classical complexity, one might ask if it is also natural to study quantum graphs in the context of quantum complexity theory. In the classical setting, deciding if a graph has a clique (or an independent set) is NP-complete [11]. The complexity class QMA is recognized as a quantum analogue of NP, as the classical proof strings of NP are replaced by general quantum states for this class. Kitaev emphasized this view by

demonstrating QMA-completeness of the local Hamiltonian problem, a quantum analogue of the NP-complete problem k -SAT [12]. Similarly, other NP-complete problems have been shown to quantize to QMA-complete problems, e.g., [13,14]. It is therefore reasonable to expect the clique and independent set problems for quantum graphs to be QMA-complete.

Surprisingly, the complexity of these problems has received relatively little study. An initial treatment was given by Beigi and Shor [15]; however, as we discuss in Section 1.4, their work would require the additional assumption that the input states are product states in order to apply to our setting. An alternative quantum analogue of NP is class QMA(2), which contains QMA and differs from QMA in that the verifier is provided with two poly-sized quantum witness states, with the promise that the two states are unentangled [16,17]. There are relatively few known QMA(2)-complete problems compared to the cornucopia of QMA-complete problems; moreover, known QMA(2)-complete problems like the sparse separable Hamiltonian problem [18], and the product isometry output problem [19], don't appear to be straightforward generalizations of NP-complete problems.

In this work, we study the decision problems of identifying cliques and independent sets in quantum graphs. We present our problem instances as circuit descriptions of quantum channels Φ , which in turn determines quantum graphs (operator systems) $\mathcal{S}_\Phi$. We also consider versions of the clique/independent set problems for graphs, where instead of an explicit description of a graph, the verifier is given a circuit description of a noisy channel which implicitly determines a graph, given as its confusability graph.

Summary of Results

We examine the clique and independent set problems for quantum channels, and we show they are both in QMA(2). Furthermore we prove that the clique problem is QMA(2)-complete and additionally show that, by quantifying over a restricted class of channels, we recover a complete problem for QMA. For deterministic and classical noisy channels, we show that our versions of both the clique and independent set problems are complete for NP and MA, respectively. Our work outlines a hierarchy of decision problems which captures completeness for all four of these important complexity classes and provides evidence for viewing QMA(2) as a natural quantum analogue of NP.

1.1. Background

In this section, we provide a brief overview of quantum graphs. For completeness we highlight that there are several overlapping notions of quantum graphs, which have been discovered in somewhat distinct settings, for example in early works due to Erdős, Katavolos, and Shulman [20] and Weaver [21,22]. More comprehensive introductions to the topics of operator systems and quantum graphs can be found in [23], [24], and [25].

We focus on the motivation of quantum graphs from the point of view of confusability graphs and describe how to include ordinary graphs¹ as quantum graphs. We also discuss the technical challenges of giving a quantum graph as an input for a decision problem. Next, we outline our approach of presenting them as the description of a quantum circuit and then sketch the definitions of the 2-clique and 2-independent set problems. The extension of these definitions to larger sets of states is given in Section 3. Lastly, we discuss how presenting problem instances as circuits descriptions of channels, which implicitly determines a confusability graph, motivates new versions of the clique and independent set problems for classical graphs as well.

1.1.1. Quantum Graphs Confusability Graphs

A classical noisy channel $N : X \rightarrow Y$ may be parametrized by a probability transition matrix $N = N(y|x)$, which gives the probability of receiving output y given input x . Given a channel N , the corresponding *confusability graph* G_N is defined on vertex set X with edges $x \sim x'$ whenever $N(y|x)N(y|x') \neq 0$, for some $y \in Y$, that is, whenever there is non-zero probability that two inputs are mapped to the same output [1]. A set $X' \subseteq X$ forms an independent set of G_N if $x \not\sim x'$ for all distinct $x, x' \in X'$, which is if and only if the probability of any pair of distinct $x, x' \in X'$ being sent to the same output is 0. Similarly, X' forms a clique of G_N if $x \sim x'$ for all distinct $x, x' \in X'$, which is if and only if any pair of distinct elements $x, x' \in X'$ have a non-zero probability as being received as the same output.

¹It is sometimes useful to emphasize the distinction between a graph (i.e., vertex and edge set) and a quantum graph (i.e., operator system). In this work, we highlight this by using the terms ordinary or classical graph.

In the quantum setting, classical noisy channels are superseded by quantum channels, represented by completely positive trace-preserving (CPTP) maps $\Phi : \mathbb{M}_n(\mathbb{C}) \rightarrow \mathbb{M}_m(\mathbb{C})$. To each such quantum channel, one can associate an operator system $\mathcal{S}_\Phi \subseteq \mathbb{M}_n$, which is constructed as the linear span of all products $A_i^* A_j$ for some set of Kraus operators $\{A_i\}_i$ of Φ . The operator system $\mathcal{S}_\Phi$ is independent of the choice of Kraus representation and can be used to generalize the role of confusability graphs to study zero-error capacities of quantum channels. In particular, a collection of orthonormal states $|v_1\rangle, \dots, |v_k\rangle$ can be distinguished from one another with certainty after passing through the channel if and only if $|v_i\rangle\langle v_j|$ is orthogonal to $\mathcal{S}_\Phi$, in which case we say $|v_1\rangle, \dots, |v_k\rangle$ is an (*exact*) *independent set* for Φ or $\mathcal{S}_\Phi$ [2]. More generally, one can extend other notions from classical graph theory such as cliques and graph colorings to the setting of operator systems [9, 26–28]. In this context, operator systems are often referred to as quantum graphs or non-commutative graphs. For the purposes of this work we use this same terminology, and a quantum graph refers to an operator system $\mathcal{S}$ inside a matrix algebra $\mathbb{M}_n$. Below, we discuss how classical graphs can be embedded into the set of quantum graphs, and in Section 1.1.2, we discuss how they can be presented using a quantum circuit.

Classical Graphs as Quantum Graphs

It is possible to embed classical graphs into the set of quantum graphs in the following way. Starting with an undirected graph G , define an operator system $\mathcal{S}_G$ by taking the span of $|i\rangle\langle j|$ over vertices i and j in G satisfying $i = j$ or $i \sim_G j$. Graphs G and H are isomorphic if and only if $\mathcal{S}_G$ and $\mathcal{S}_H$ are unitarily equivalent [28]. Similarly, given a noisy channel $N = N(y|x)$, there are several ways to construct a corresponding quantum channel Φ_N which satisfies $\mathcal{S}_{\Phi_N} = \mathcal{S}_{G_N}$. Generalizations of cliques, independent sets, and colorings to quantum graphs respect this construction, i.e., there is a correspondence between k -cliques, k -independent sets, and k -colorings for G and those for the quantum graph $\mathcal{S}_G$ [24].

The Correspondence between Channels and Graphs

Different noisy channels can give rise to the same confusability graph, i.e., the construction $N \mapsto G_N$ is not injective. Similarly, in the quantum setting, the construction $\Phi \mapsto \mathcal{S}_\Phi$ is not injective. On the other hand, given any graph G one can construct N satisfying $G = G_N$ with an appropriate choice of transition probabilities $N = N(y|x)$; likewise given any operator system $\mathcal{S} \subseteq \mathbb{M}_n$, it is possible to define Φ such that $\mathcal{S}_\Phi = \mathcal{S}$ [24]. Consequently, quantifying over all quantum channels is sufficient to quantify over all quantum graphs, and it is hence well-motivated to consider parameters for quantum graphs at the level of channels.

The Complement of a Quantum Graph

Recall that, given a graph G , the complement $\overline{G}$ is a graph on the same vertex set where the adjacency relation is defined as $i \sim_{\overline{G}} j$ if and only if $i \not\sim_G j$. However, there is no generally agreed-upon approach to extending the notion of the graph complement to quantum graphs. One straightforward approach would be to take the orthogonal complement of $\mathcal{S}$ inside $\mathbb{M}_n$ with respect to the Hilbert-Schmidt inner product, but this will not be an operator system as it fails to contain the identity element, and hence fails to correspond to any channel. One can remedy this by making a correction such as $\overline{\mathcal{S}} = \mathcal{S} + \mathbb{C}I$ to account for the identity [9]. Unfortunately, this does not extend the graph complement as $\mathcal{S}_{\overline{\mathcal{S}}} \neq \overline{\mathcal{S}_G}$. Another approach allows both operator systems as well as their orthogonal complements to be considered as quantum graphs [27].

Relevant for this work is the fact that there is no straightforward and unique notion of a graph complement for quantum graphs, and consequently, the complexity of decision problems based on cliques and independent sets for quantum graphs cannot be reduced to one another by taking a complement. Indeed, as we discuss in Section 1.5, it is even possible that they have different complexity.

Quantum Independent Sets and Quantum Cliques

Motivated by the study of non-local games, quantum generalizations of many graph parameters have been found. The existence of k -cliques, k -independent sets, and k -colorings for a classical graph G can be determined by the existence of a perfect classical strategy for a corresponding non-local game. In some cases, *quantum strategies* for these games can outperform classical strategies, and perfect quantum strategies give rise to *quantum cliques* and *quantum independent sets* for classical graphs [29, 30]. By considering non-local games where the questions are quantum states, one can also define quantum cliques and quantum independent sets for quantum graphs [31].

It is known that determining the quantum value—or commuting operator value—of a non-local game is an undecidable problem [32–34]. These results can be captured using the non-local games associated to the graph parameters mentioned above, and thus it is undecidable to determine the existence of quantum cliques/quantum independent sets in either classical or quantum graphs [35]. In this work, we take a different perspective and do not consider quantum cliques or quantum independence sets. Instead, we exclusively study the complexity of determining the existence of cliques and independent sets in graphs and quantum graphs.

1.1.2. The Clique/Independent Set Problems for Quantum Graphs

In Section 4, we define approximate k -cliques and approximate k -independent sets at the level of quantum channels. For convenience, we restate these definitions here. To keep the presentation concise, we only include the case $k = 2$. In fact, restricting to $k = 2$ is sufficient for all of our complexity results in this paper, and in Appendix A we describe an explicit reduction of the corresponding decision problems from $k > 2$ to $k = 2$.

Definition 1. Let $\alpha \in [0, 1]$. We say a pair of orthogonal states ρ_1, ρ_2 are an $(\alpha, 2)$ -clique for Φ if $\text{Tr}[\Phi(\rho_1)\Phi(\rho_2)] \geq \alpha$, and an $(\alpha, 2)$ -independent set if $\text{Tr}[\Phi(\rho_1)\Phi(\rho_2)] \leq 1 - \alpha$.

We remark on a few details motivating Definition 1. Firstly, we require the input state to be exactly orthogonal. This agrees well with the classical noisy channel setting, which considers sets of distinct inputs $x_1 \neq x_2$. Second, we express the probability that the outputs of the channel are confused by means of the overlap under the Hilbert-Schmidt inner product $\text{Tr}[\Phi(\rho_1)\Phi(\rho_2)]$. This also agrees well with classical notions of confusability since, if the output states are classical, i.e., $\Phi(\rho_1) = \sum_x \lambda_x |x\rangle\langle x|$ and $\Phi(\rho_2) = \sum_x \mu_x |x\rangle\langle x|$, then the overlap $\sum_x \lambda_x \mu_x$ becomes simply the probability of the channels providing the same input on independent evaluations. Further, using the overlap as a measure of confusability is also natural in the computational setting, as the overlap can be efficiently sampled from the output states by means of the swap test, unlike other measures of closeness such as the trace distance. We note that we do not require the input states to be pure, though we can always assume the states in a clique or independent set are pure by convexity. Neither do we assume the output states are pure, as we quantify over general, not necessarily unitary, channels. Lastly, the definition reduces to the exact case for particular values of the parameters: $(1, k)$ -independent sets of Φ are k -independent sets of $\mathcal{S}_\Phi$, and $(1, k)$ -cliques are k -independent sets of $\mathcal{S}_\Phi^\perp$.

What may be seen as a flaw of our definition is that it is possible for a mixed state ρ to have small overlap with itself. This observation may seem somewhat contradictory to our measure of confusability, as this means that two inputs that have the same output state may nevertheless be close to independent, or *pseudo-independent*. However, by seeing mixed states as samples from an ensemble of pure states, what this really means is that, upon taking two independent samples from the ensemble, they are unlikely to be equal. As we discuss in Section 1.3, this property actually plays an important role in our proof approach establishing QMA(2)-completeness of the clique problem.

Presenting Quantum Graphs as Quantum Circuits

Problems based on finding cliques, independent sets, or graph colorings are well-studied in classical complexity; they all give rise to NP-complete problems [11]. In these cases, the input to the problem includes an explicit description of the vertices and edges of a graph. Providing an explicit description of a quantum graph raises some challenges: notably quantum graphs are linear subspaces $\mathcal{S} \subseteq \mathbb{M}_n$ and are thus determined by a basis of operators. As such, given a quantum channel Φ on an n -qubit system, the corresponding quantum graph $\mathcal{S}_\Phi$ is a subspace with up to 2^{2n} dimensions, so specifying it in this way requires a description of an exponentially sized basis. Note that this problem also arises in the classical case, if the graphs in question are seen as arising from a noisy channel on an n -bit input space. For our work, we will instead consider quantum graphs as being determined by an underlying quantum channel Φ , and give a problem instance as a quantum circuit C describing the action of the channel $\Phi = \Phi_C$.

Clique and Independent Set Problems

The *quantum channel 2-clique problem* with completeness c and soundness s is the promise problem with yes instances consisting of circuits C such that Φ_C has a $(c, 2)$ -clique, and no instances consisting of circuits for which Φ_C has no $(s, 2)$ -clique. We always assume $c \geq s$, and in the cases we consider that there is some probability gap between them. We define the *quantum channel 2-independent set problem* in a parallel way. In Definition 7, we express these problems more formally and extend them to larger k . These promise problems are written $\text{qClique}(k)_{c,s}$ and $\text{qIS}(k)_{c,s}$. In this approach, a verifier given a problem instance C may perform computations by running the circuit

C in order to implement channel Φ_C , which determines properties of an underlying quantum graph $\mathcal{S}_{\Phi_C}$. The size of a problem instance is then the size of the quantum circuit, which is used to run this computation, rather than the size of an explicit description for the operator system $\mathcal{S}_{\Phi_C}$.

We also consider the restricted problems where the circuits C are only taken from a specific family of quantum circuits $\mathcal{C}$, which we write as $\text{qClique}(k, \mathcal{C})_{c,s}$ and $\text{qIS}(k, \mathcal{C})_{c,s}$. This allows us to determine hardness results with respect to particular families of channels/quantum circuits such as entanglement-breaking channels and channels that implement partial traces. Importantly, in Section 5, we specify a family of circuits for which these problems are QMA-complete.

Revisiting the Classical Case

In light of the above discussion, we can revisit problems related to cliques and independent sets of classical graphs. More concretely, we consider the input given as a description of a classical circuit which implements a deterministic or probabilistic (noisy) function f on input set X . Note that if $X = \{0, 1\}^n$, then the confusability graph G_f will be an exponentially sized graph. We then define our decision problems as deciding if the corresponding confusability graph G_f has a clique or independent set of size k . In this approach, the size of the problem instance is related to the runtime needed to compute the adjacency relations in G_f , rather than the size of an explicit description of G_f itself. We also note that our approach differs from graph problems for succinctly presented graphs [36], and to the best of our knowledge has not been studied before from this point of view.

In the case that f is deterministic, the confusability graph consists of a disjoint union of complete subgraphs. In this case, deciding if G_f has a clique of size $k = 2$ is equivalent to the well-studied collision problem for functions. It asks whether, given some function $f : X \rightarrow Y$, is there a pair of elements in the domain $x, y \in X$ with the same image $f(x) = f(y)$ —a collision. The presumed computational difficulty of this problem on certain classes of functions underlies the security of hash functions, which serve as a primary building block of much of modern cryptography.

More formally, we say a deterministic function f has a k -clique, or a k -independent set, if there exists distinct inputs $x_1, \dots, x_k$ such that $f(x_1) = \dots = f(x_k)$ or $f(x_1) \neq \dots \neq f(x_k)$ respectively. The corresponding decision problems, whose instances are classical circuits describing functions, are denoted $\text{Clique}(k)$ and $\text{IS}(k)$. We study this case in Section 6.1.

In the case that the underlying function f is probabilistic, i.e., f is a noisy classical channel, the corresponding confusability graph G_f can be more interesting. Indeed, by suitably picking the transition probabilities, it can be observed that any graph on a vertex set V can arise as the confusability graph of some noisy channel f with input set V . In Section 6.2, we study the complexity of deciding the existence of approximate k -cliques and approximate k -independent sets for probabilistic functions. We restate the relevant definitions for the case $k = 2$ here.

Definition 2. Let $\alpha \in [0, 1]$. A probabilistic function $f : X \rightarrow Y$ has an $(\alpha, 2)$ -clique if there exist distinct $x, x' \in X$ such that $\Pr[f(x) = f(x')] \geq \alpha$, and an $(\alpha, 2)$ -independent set if there exist distinct $x, x' \in X$ such that $\Pr[f(x_i) = f(x_j)] \leq 1 - \alpha$.

The *probabilistic 2-clique problem* and the *probabilistic 2-independent set problem* are the promise problems that consist of deciding the existence of approximate 2-cliques or 2-independent sets, up to some specified promise gap $c - s$. In Section 6.2, we extend these definitions to larger k and we write $\text{pClique}(k)_{c,s}$ and $\text{pIS}(k)_{c,s}$ for the corresponding promise problems.

1.2. Main Results

Our contributions focus on the complexity of the k -clique and k -independent set problems across four different settings: the quantum case, a restricted quantum case where we only consider entanglement-breaking channels, and we conclude by addressing the classical deterministic and probabilistic cases. We show that deciding 2-cliques across these settings is complete for the classes QMA(2), QMA, NP, and MA, respectively. Further, we show that the k -independent set problems in the deterministic, probabilistic, and restricted quantum cases are complete for the same classes.

In the quantum setting, we study the complexity of deciding approximate k -cliques ($\text{qClique}(k)_{c,s}$) and approximate k -independent sets ($\text{qIS}(k)_{c,s}$) for a quantum channel Φ . The inputs for these problems are given as the description of a quantum circuit which implements Φ . In Section 4, we show that for $k = 2$ both problems are in QMA(2), and establish the completeness of $\text{qClique}(2)_{c,s}$. Our hardness result is obtained when quantifying over a specific

subset of quantum circuits, $\mathcal{C}_{\text{Tr}} \oplus \mathcal{C}_{\text{Tr}} \oplus \mathcal{C}_{\text{EB}}$, as defined in Definitions 9, 10, and 11. Informally, this corresponds to the set of circuits which implement channels that are probabilistic mixtures of entanglement-breaking channels and partial trace maps. This is stronger than the hardness of the problem quantified over all channels and implies it.

Theorem 1 (Quantum Case). *There exist $c, s : \mathbb{N} \rightarrow (0, 1)$ with constant gap such that the clique problem $\text{qClique}(2, \mathcal{C}_{\text{Tr}} \oplus \mathcal{C}_{\text{Tr}} \oplus \mathcal{C}_{\text{EB}})_{c,s}$ is QMA(2)-complete.*

In Section 4.2, we show that, by slightly extending our proof of hardness of the quantum clique problem for QMA(2), we can show that it is also hard for QMA(k). Hence, we provide an alternate proof that $\text{QMA}(k) = \text{QMA}(2)$ than the original proof of Harrow and Montanaro [37].

In Section 5, we show that by further restricting the set of channels to a particular subset $\mathcal{C}_{\text{EB}'}$ of the entanglement-breaking channels, we can obtain QMA-completeness for $\text{qClique}(2, \mathcal{C}_{\text{EB}'})_{c,s}$. This suggests any potential distinction between QMA and QMA(2) may be reflected in the difference between the properties of these channels and those considered above.

Theorem 2 (Restricted Quantum Case). *There exist $c, s : \mathbb{N} \rightarrow (0, 1)$ with polynomial gap such that $\text{qClique}(2, \mathcal{C}_{\text{EB}'})_{c,s}$ is QMA-complete.*

In Section 6, we review the complexity of deciding k -cliques and k -independent sets for deterministic functions—the problems $\text{Clique}(k)$ and $\text{IS}(k)$. These problems can be viewed as a rephrasing of the well-studied collision problem for functions. Our contribution here is to provide new proofs for NP-completeness of these problems and to state them in a way that generalizes to the problems considered in Section 6.2, Sections 4, and Section 5.

Theorem 3 (Deterministic case). *$\text{Clique}(2)$ and $\text{IS}(2)$ are NP-complete.*

In Section 6.2, we consider the problems of deciding approximate k -cliques ($\text{pClique}(k)_{c,s}$) and approximate k -independent sets ($\text{pIS}(k)_{c,s}$) for probabilistic functions.

Theorem 4 (Probabilistic case). *There exist $c, s : \mathbb{N} \rightarrow (0, 1)$ with constant gap such that $\text{pClique}(2)_{c,s}$ and $\text{pIS}(2)_{c,s}$ are MA-complete.*

Finding natural MA-complete problems is in itself an active area of research, and few examples are known [38, 39].

If one begins with a deterministic or classical noisy channel f , and considers its inclusion as a quantum channel Φ_f , as described in [24], then f will have an (α, k) -clique if and only if Φ_f does. Hence, the classical decision problems analyzed in Section 6 can naturally be viewed as a restriction of the problems in Theorem 5, which in turn are a restriction of the problems considered in Section 4. Our results can thus be viewed as a natural hierarchy of problems which are complete for all four of the proof classes NP, MA, QMA, and QMA(2), and whose complexity may be tuned by varying the collection of channels.

In Appendix A, we provide a direct reduction from clique and independent set problems in the deterministic, probabilistic, and quantum settings from larger k to $k = 2$. In Appendix B, we provide a more straightforward proof establishing QMA(2)-hardness for $\text{qClique}(2)_{c,s}$ when quantifying over all quantum circuits than the stronger one given in Theorem 1.

1.3. Proof Techniques

Our proof techniques were inspired by techniques in self-testing and protocols for delegating quantum computation, such as in [40, 41]. Other recent work of Bassirian, Fefferman, and Marwaha considers different connections between self-testing and QMA(2) [42]. In short, if we know a channel has or does not have cliques over a particular set of states—for example, the separable states—then we can combine it with another channel to guarantee that possible cliques among the remaining states are not important. The proofs use two important constructions, which we will comment on here. We also discuss different techniques we use in the proof of QMA-completeness of a restricted version of the problem.

Non-unitary Circuits and Direct Sums

First, we devise a new technique to take direct sums of quantum circuits (10). In order to be able to parameterize arbitrary non-unitary channels by means of circuit diagrams, we work with an extended circuit model that appends partial trace and state preparation maps to a basic set of unitary gates. In this way, we extend the universality of a gate set to CPTP maps by means of the Stinespring dilation theorem. At the level of channels, there are a variety of ways to compose the maps to get new ones, but it is not always immediate how that translates to composition at the level of circuits—importantly, whether it is computationally efficient. In our constructions, the way we will combine channels is by means of the direct sum: given two channels Φ_1 and Φ_2 and a probability p , the direct sum is $p\Phi_1 \oplus (1 - p)\Phi_2$, the convex combination of the channels Φ_1 and Φ_2 with orthogonal output spaces. Of course, the circuit model is well-adapted for composition and direct products of maps, but not for sums. To get the wanted superposition of circuits, we might add a qubit in state $\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$, conditionally act by either Φ_1 or Φ_2 based on this qubit, and then measure the added qubit. This, however, has a technical obstruction: conditioning a non-unitary channel can put the number of output qubits in superposition or probabilistic mixture, which is untenable. Instead, we use the fact that the circuits decompose into the basic gates to separate the unitary from the non-unitary gates. Then, the unitary gates act conditionally on the additional modulating qubit as wanted, and the non-unitary gates can be aligned to act the same way for both Φ_1 and Φ_2 . In this way, we get a circuit that implements a direct sum of channels of size polynomial (in fact linear) in the size of the circuits describing the original channels. See also Figure 2 for an illustration of this construction. We are optimistic that this direct sum construction might be able to find various other applications in quantum information.

Self-Testing for Cliques

Next, the direct sum construction allows us to port ideas from self-testing over to questions about quantum cliques. The main technical tool we prove to use this technique is Lemma 5. As noted above, the technique takes the direct sum of two channels in order to expand the set of pairs of states on which some property of one of the channels holds. To illustrate how this works, we outline the proof of QMA(2)-hardness of the quantum 2-clique problem (6), which is based on this technique. Given some instance x of a QMA(2) promise problem (Y, N) , we know that there is some efficiently constructable circuit Φ_x that accepts with high probability on some separable state if x is a yes instance, and with low probability on every separable state if x is a no instance. In the first step, we construct an entanglement-breaking channel Ψ_x that runs this verification, measures, and outputs a pure state if it accepts and a highly mixed state if it rejects (8). Note that this channel acts on one more qubit than Φ_x , which it immediately traces out, in order to have that qubit guarantee orthogonality of any clique. If Φ_x accepts with high probability, then Ψ_x outputs a state close to pure, and thus, the overlap of two input states differing only in the orthogonality qubit will be high. As such, if x is a yes instance, Ψ_x has a clique in the separable states. On the other hand, if Φ_x accepts with low probability, then the output state will be far from pure, resulting in low overlap—as such, a no instance leads to a channel Ψ_x with no cliques among the separable states. The trick however is to lift this property to all the states. To do so, we use the channel Φ_1 constructed in Lemmas 6 and 7, consisting of a direct sum of partial traces on the subregisters we wish to be unentangled. For this channel, we uncover a property reminiscent of self-testing: if states form a near optimal clique, then they must be separable, since partial traces of entangled states are mixed and hence have low overlap. Finally, in the convex combination of Φ_1 and Ψ_x , we find via Lemma 5 that if we weight Φ_1 sufficiently, then the testing for separability overcomes any potential non-separable cliques of Ψ_x , and thus that for a no instance the channel has no cliques. Since the weighting can be chosen to preserve polynomial or constant gaps, this proves the hardness of the 2-clique problem for QMA(2). We believe that this technique is readily extendable to prove other properties of channels; in fact, in Appendix A, we use Φ_1 as well as a channel that guarantees orthogonality among the tensor terms of a separable state to reduce the quantum k -clique problem directly to the 2-clique problem.

Independent Sets

It is interesting to note that, though this proof technique is very useful in determining properties of cliques, it seems not to be well-adapted to the study of independent sets. First, Lemma 5 is used to expand collections of states where there is no clique, whereas this would require us to be able to expand collections of states where there is no independent set. A more serious technical obstruction, however, is that where cliques may arise from essentially one source, independent sets arise from two. The overlap of two states is high if they are approximately equal *and* pure. However, the overlap of two states is low if they are approximately orthogonal *or* highly mixed. The first corresponds to an approximate version of the independent sets for an operator system, but the second does not. This case of low

overlap, which we refer to as a *pseudo-independent* set, underpins the proofs of hardness of the clique problem, but seems to be an obstruction for the independent set problem. In particular, if we want to guarantee small overlap, it is not required to output orthogonal states—it suffices to output a highly mixed state. Also, high overlap is critical in guaranteeing separability of states by means of Lemma 7. This raises an interesting question: does there exist a channel where the independent sets are separable states? Resolving this would serve to illuminate the complexity of the clique problem greatly.

Entanglement-Breaking Channels and QMA

Finally, our proof of the QMA-completeness of the clique problem for a subclass of the entanglement-breaking channels uses significantly different techniques. The main difficulty lies in showing that this problem actually lies in QMA. This relies on an interesting result of Brandão [43], which shows that the subclass of QMA(2) where the verifier may measure with a Bell measurement—measure each of the factors in the separable state separately, and then classically compute whether or not to accept—lies in QMA. We find that computing the clique value by running the channel and then doing the swap test is in fact a Bell measurement when the channel is entanglement-breaking. Nevertheless, we must still further restrict, because there is another check needed to verify a clique: checking orthogonality of the states. In our proof that the clique problem is in QMA(2), we also do this via the swap test. However, since this swap test is not prefixed by an entanglement-breaking channel, it cannot be evaluated by a Bell measurement. In fact, Harrow and Montanaro [37] show that the swap test cannot even be run by an LOCC measurement, which is a larger class than the Bell measurements. As such, we need a different method to guarantee orthogonality; we work over channels where the output is independent of one of the qubits, i.e., that qubit is immediately traced out. This is the same method as we have used to guarantee existence of an orthogonal clique in the QMA(2)-hard clique problem. As such, the prover need not even send as a proof a clique where the states are orthogonal—the verifier may nevertheless be certain that there exists one. Finally, we note that, unlike the QMA(2)-completeness result, the QMA-completeness result generalizes immediately to the independent set problem as well.

1.4. Relation to Other Works

Closely related to our work is that given in the unpublished work of Beigi and Shor [15]. In their work, a promise problem referred to as the *quantum clique problem* is proven to be complete for QMA. This problem is similar to but distinct from $\text{qIS}(k, \mathcal{C})_{c,s} = (Y, N)$ as we define it in Definition 6. Firstly, in terms of terminology, what Beigi and Shor refer to as a *quantum clique* for a channel is more closely related to what is commonly called an independent set in the quantum graph literature. The standardization of this terminology was established after the release of their work, and we chose to model our terminology following more current conventions.

There are also a few more substantial differences that make our results and theirs incomparable in a direct sense. In order to highlight some of these differences, we restate their definition below.

Definition 3. [Definition 2.9 from [15]] *Quantum clique problem* (Φ, k, a, b)

- **Input** Integer numbers n and k , non-negative real numbers a, b with an inverse polynomial gap $b - a > n^{-c}$, and an entanglement-breaking channel Φ that acts on n -qubit states.
- **Promise** Either there exists $\rho^1 \otimes \cdots \otimes \rho^k$ such that $\sum_{i,j} \text{Tr}(S \Phi(\rho^i) \otimes \Phi(\rho^j)) \leq a$ or for any state $\rho^{1,2,\dots,k}$ we have $\sum_{i,j} \text{Tr}(S \Phi^{\otimes 2}(\rho^{i,j})) \geq b$.
- **Output** Decide which one is the case.

One key difference between Definitions 7 and 3 is that their promise problem makes explicit reference to the SWAP operator S —see Section 2.5 for more details on this and the well-known swap test. Even when quantifying over only entanglement-breaking maps, it is not clear that the two problems are the same, since the swap test can fail if the input state is not given as a product state. That is, the promise made in the work of Beigi and Shor is more significant than we use: no instances are those where the swap test rejects with high probability on all, potentially entangled, states and not just on separable states, as we require.

To give more detail, we consider the case $k = 2$. No instances of Definition 3 are channels Φ satisfying $\text{Tr}(S \Phi^{\otimes 2}(\rho^{1,2})) \geq b$ for all possibly entangled states $\rho^{1,2}$, whereas no instances of Definition 7 are channels Φ satisfying $\text{Tr}(\Phi(\rho_i)\Phi(\rho_j)) \geq b$ for all pairs of orthogonal states ρ_1 and ρ_2 . That is, no instances of Definition 3 will be no instances of Definition 7 but the converse does not clearly hold, even in the case of restricting to entanglement-breaking maps.

Another difference is with regard to how the classical description for a quantum channel Φ is given as input to the problems. Beigi and Shor establish QMA-completeness for Definition 3 when considering entanglement-breaking quantum channels given in terms of an operator sum, or Kraus, representation

$$\Phi(\rho) = \sum_{i=1}^r E_i \rho E_i^\dagger, \quad (1)$$

whereas in our case we consider inputs given as a description of a quantum circuit C .

Despite some differences in our approaches we took much inspiration from the ideas present in [15]. Separate from our work here, another active project is investigating the complexity of quantum graph problems using an approach more closely following that given in [15].

1.5. Open Problems

In this section, we outline a selection of open problems motivated by our work.

Deciding Independent Sets in Quantum Graphs

The existence of graph complementation provides a straightforward way to reduce the problems of deciding cliques and independent sets for graphs. This also extends to the versions of these problems introduced in Section 6.

In the quantum setting, our proof technique establishing QMA(2)-hardness for the clique problem $\text{qCliques}(k)_{c,s}$ could not readily be extended to the problem $\text{qIS}(k)_{c,s}$. As discussed in Section 1.1.1, there is no canonical way to take the complement of a quantum graph as there is classically. Instead, several differing approaches have been considered in the literature. Even when one considers the different approaches available, it is not clear how to use these maps on the level of operator systems to obtain an efficient reduction between $\text{qCliques}(k)_{c,s}$ and $\text{qIS}(k)_{c,s}$. Indeed, in an informal sense, it would appear that under any appropriate notion of complementation the complement of a quantum graph can be a significantly more complicated object than the original quantum graph. On the one hand, we know that the quantum independent set problem is contained in QMA(2). On the other hand, it is hard for QMA. However, we feel it is unlikely that there exists a QMA verification circuit for $\text{qIS}(k)_{c,s}$. This is a similar situation to the pure state version of the QMA-complete consistency of local density matrices problem, which has long been conjectured to be complete for QMA(2) [13]. It is possible that neither of these problems is QMA or QMA(2)-complete, and a reduction from one of the two problems to the other would be enlightening.

Deciding Colorings

Famously, Lovász showed a reduction between the k -coloring problem and the 3-coloring problem in ordinary graphs [44]. In this case, the instances are given using an explicit description of the graph. In Section 6, we introduce new variants of the k -clique and k -independent set problems motivated from the study of noisy channels and confusability graphs. It is straightforward to similarly define a new version of the k -coloring problem. We did not investigate the complexity of this problem but suggest it as worthy area of exploration. Since a coloring on exponentially many vertices cannot be described in polynomial time, it is altogether possible that the class where this coloring problem lives is significantly larger than NP.

In the quantum setting, various approaches to define a k -coloring for a quantum graph have been considered [9,27]. Informally, a k -coloring of a quantum graph $\mathcal{S} \subseteq \mathbb{M}_k$ consists of a partition of an orthogonal basis for $\mathbb{C}^k$ into independent sets. As in the classical case, for an n -qubit quantum channel, this would require specifying a basis for a 2^n dimensional space. It seems unlikely that the corresponding problem could be decided in QMA(2).

Other Graph Problems

There are a variety of decision problems for graphs which can possibly be extended to the quantum setting via a similar approach to the one we use for the clique and independent set problems. We suggest two more here. Firstly, in our problems, the size k of a clique or independent set is a parameter which defines the problem. One can instead consider the value k to be specified along with the circuit description as an input. In the classical case of graphs described by their adjacency matrices, this takes the complexity of finding a clique or independent set from P to NP-hard [11]. One might imagine a similar jump happens in the cases we have considered, from NP, MA, QMA, or QMA(2) to larger classes.

Secondly, one can consider versions of the well-studied graph isomorphism and non-isomorphism problems. In particular, consider a quantum verifier who receives as input a pair of circuits C_1 and C_2 which implement classical or

quantum channels Φ_1 and Φ_2 . The corresponding promise problems would then be to decide if the quantum graphs $\mathcal{S}_{\Phi_1}$ and $\mathcal{S}_{\Phi_2}$ are approximately isomorphic or far from isomorphic. The quantum analogues of these problems may also provide new approaches to zero-knowledge proof protocols in the quantum setting, since graph isomorphism/non-isomorphism played an important role in early work on zero-knowledge proof systems [45,46]. The area of zero-knowledge proof systems in the quantum setting has seen a flurry of recent advances [13,40,47–49], and it would be interesting if the study of quantum graph isomorphisms can provide new approaches to zero-knowledge in some of these settings.

Another possible area of exploration is Ramsey theory for quantum graphs. Ramsey theory is a branch of mathematics initiated by F. Ramsey which examines the existence of ordered substructures arising in graphs of a sufficient size [50]. Ramsey theory has some interesting connections with computational complexity, such as studying the computational complexity of computing the Ramsey numbers [51]. Several of the motivating themes of Ramsey theory have been explored for quantum graphs [7,52], and it may be worthwhile to explore connections to complexity inspired by these works that incorporate our approaches for presenting quantum graphs as circuits.

Applications

While the problems studied here present natural problems which are expressive enough to give complete problems for a variety of well-studied classes, we do not present any concrete applications. It would be interesting to explore how the qClique and qIS problems may have applications in other areas of quantum information theory.

Another potential application may be found using the construction given in Section 4.1.2. This presents a new tool that allows one to take direct sums of quantum circuits as opposed to the more standard tensor product. It would be interesting to see if this has applications in areas such as quantum algorithms or quantum linear algebra.

1.6. Acknowledgments

We thank David Cui, Joshua Nevin, Seyed Sajjad Nezhadi, Laura Mančinská, Se-Jin Kim, and Adrian She for lengthy discussions during the early stages of this project. We also thank Adrian She and David Cui for detailed discussions and for sharing extensive notes on [15].

1.7. Outline

In Section 2, we outline the important notation we use and give some brief background information on complexity theory and aspects of quantum information we require. In Section 3, we recall the standard definitions of cliques and independent sets for graphs, and connect them to our definitions of these properties for quantum channels. Next, in Section 4, we define and study the clique and independent set problems for quantum channels. We show in Section 4.1 that the quantum 2-clique problem is QMA(2)-complete, and in Section 4.2 that this completeness implies a new proof that $\text{QMA}(k) = \text{QMA}(2)$. In Section 4.3, we discuss the quantum independent set problem for quantum channels. Finally, in Section 5, we present a restriction of the quantum clique and independent set problems that is QMA-complete. In Section 6, we consider the classical deterministic (Section 6.1) and probabilistic (Section 6.2) versions of the clique and independent set problems. We show their completeness for the complexity classes NP and MA, respectively. We also collect some additional results in appendices: first, in Appendix A, we give reductions from k -clique and independent set problems to the corresponding problem with $k = 2$; also, in Appendix B, we give a simpler but weaker proof of the QMA(2)-completeness of the quantum 2-clique problem.

2. Preliminaries

2.1. Notation

We use italic capitals H, K, L to represent Hilbert spaces, which we always assume to be finite-dimensional. Given a finite set S , write $\mathbb{C}^S$ for the $|S|$ -dimensional Hilbert space with canonical (computational) basis $\{|s\rangle \mid s \in S\}$. As a special case, we denote the space of one qubit $Q = \mathbb{C}^{\{0,1\}}$. Fix some Hilbert space H . Write $H_\perp$ for the superspace $H + \mathbb{C}|\perp\rangle$ where $\|\perp\| = 1$ and $\langle \perp | \psi \rangle = 0$ for all $|\psi\rangle \in H$. For any subspace $K \subseteq H$, write μ_K for the maximally mixed state on K . Write $\mathcal{B}(H)$ for the space of linear operators on H , write $\mathcal{P}(H)$ for the space of positive operators, and $\mathcal{D}(H)$ to be the set of density operators (trace 1 positive operators). We use quantum state interchangeably with density operator, specifying to pure quantum states when necessary; we use quantum channel interchangeably with completely positive trace-preserving (CPTP) map.

We denote the set of all bitstrings $\{0,1\}^* := \bigcup_{n=0}^{\infty} \{0,1\}^n$. Given some $x \in \{0,1\}^*$, write $|x| \in \mathbb{N}$ for its length. For a function $f : \mathbb{N} \rightarrow [0,1]$, write $f : \mathbb{N} \rightarrow (0,1)_{\text{exp}}$ to mean that there exist $N, k > 0$ such that $2^{-n^k} < f(n) <$

$1 - 2^{-n^k}$ for all $n \geq N$. Write the set of polynomially bounded functions $\text{poly} = \{f : \mathbb{N} \rightarrow \mathbb{R} \mid \exists k, N \geq 0. f(n) \leq n^k \forall n \geq N\}$. For a natural number $n \in \mathbb{N}$, write $[n] := \{1, 2, \dots, n\}$.

2.2. Complexity Theory

A *language* is a subset $L \subseteq \{0, 1\}^*$. An element $x \in L$ is called a *yes instance*, and an element $x \notin L$ is called a *no instance*. A *promise problem* is a pair (Y, N) of disjoint subsets $Y, N \subseteq \{0, 1\}^*$. Here, $x \in Y$ is called a *yes instance* and $x \in N$ is called a *no instance*, while $x \notin Y \cup N$ is *indeterminate*. Promise problems generalize languages in the sense that every language L corresponds to the promise problem (L, L^c) . A *complexity class* is a collection of languages or promise problems.

A function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ is *computable in time $T(n)$* if there exists a Turing machine M that, on input x , outputs $M(x) = f(x)$ in $T(|x|)$ steps. We say f is *polynomial-time computable* if there exists a polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ such that f is computable in time $p(n)$.

A language L_1 is *polynomial-time Karp-reducible* to another language L_2 if there exists a polynomial-time computable function f such that $x \in L_1$ iff $f(x) \in L_2$. For promise problems, a similar definition holds, the only difference being that the function must map no instances to no instances as well. Given a complexity class C , L is *C-hard* (under polynomial-time Karp reductions) if every element of C is polynomial-time Karp reducible to L . L is *C-complete* if additionally $L \in C$.

2.3. Quantum and Classical Circuits

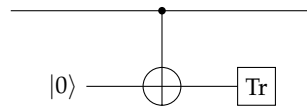
In its usual usage, the circuit model of quantum computation is used to describe unitary channels. To describe unitaries, we fix a (finite) universal gate set $S \in \mathcal{U}_{n_0}(\mathbb{C})$ —in this work, we do not need to specify the choice of gate set, but one can, for example, always assume that it is the Clifford+ T gate set $\{\text{CNOT}, H, T\}$. We can approximate any unitary to arbitrary precision by a composition of finitely many tensors of the basic gates.

In order to describe all quantum channels, we need to add two non-unitary operations, that change the number of qubits, to the basic gate set:

- The *partial trace* on one qubit $\text{Tr} : \mathcal{B}(\mathbb{C}^2) \rightarrow \mathbb{C}$;
- *State preparation* of one qubit in the computational basis $\mathbb{C} \rightarrow \mathcal{B}(\mathbb{C}^2)$, $\lambda \mapsto \lambda|0\rangle\langle 0|$.

Due to the Stinespring dilation theorem, the augmented gate set $S \cup \{\text{Tr}, |0\rangle\langle 0|\}$ can approximate any quantum channel to arbitrary precision. Stinespring dilation also guarantees that the circuit can be expressed in a canonical form where qubit preparation happens first, followed by unitary gates, followed by partial traces. In fact, given a circuit description, it can be taken to this canonical form efficiently simply by sliding the state preparations backwards and the partial traces forwards. As such, we will assume that the circuits are always in this form.

Note that POVM measurements can in particular be realized in this model. For example, measurement of a qubit in the computational basis can be realized by the following circuit:



Other measurements can be realized similarly—diagonalizing for PVMs or using Naimark’s theorem for general POVMs—although the circuit representations may consist of many gates.

For the computational problems we consider, the instances take the form of circuit diagrams describing some quantum channel. It is perhaps simplest from the theory standpoint to present the diagrams as pictures, but it is wholly equivalent to represent the circuits as an encoding of the diagram as a bitstring, which is more natural from a computational standpoint. For a circuit diagram C , write Φ_C for its implementation as a quantum channel. We take the length of the circuit $|C| = \text{in}(C) + \text{out}(C) + \text{gates}(C)$, the sum of the number of input qubits, the number of output qubits, and the number of gates. Note that it is possible to choose an encoding as a bitstring whose length is polynomial in the circuit length described above, and vice versa.

We can also treat classical circuits in a similar way. Here, circuits describe not channels but functions $f : \{0, 1\}^m \rightarrow \{0, 1\}^n$. But, as in the quantum case, we can choose a finite universal gate set, for example, $\{\text{NAND}\}$. Given a classical circuit diagram C , write f_C for the function it represents and $|C| = \text{in}(C) + \text{out}(C) + \text{gates}(C)$ for the circuit size.

2.4. Distances between States

There are a variety of natural distance measures between quantum states; we use three: the Euclidean norm, the trace norm, and the Frobenius (or Hilbert-Schmidt) norm. The Euclidean distance is only defined on pure states and is simply the natural norm induced by the inner product on the Hilbert space $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$. The trace norm is defined for all $A \in \mathcal{B}(H)$ as $\|A\|_{\text{Tr}} = \frac{1}{2} \text{Tr}[\sqrt{A^\dagger A}]$; and the Frobenius norm is defined similarly as $\|A\|_2 = \sqrt{\text{Tr}[A^\dagger A]}$. The following simple lemma relates the metrics that these norms induce on the pure states. We use this extensively and without mention.

Lemma 1. *Let $|\psi\rangle, |\phi\rangle$ be pure states. Then,*

$$\|\psi\rangle\langle\psi| - \phi\rangle\langle\phi|\|_2 = \sqrt{2}\|\psi\rangle\langle\psi| - \phi\rangle\langle\phi|\|_{\text{Tr}}, \quad (2)$$

$$\|\psi\rangle\langle\psi| - \phi\rangle\langle\phi|\|_2 \leq \sqrt{2}\|\psi\rangle - \phi\rangle\|. \quad (3)$$

Proof. For the first relation, let $X = |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|$. If $X = 0$, then this holds trivially. Otherwise there exists an eigenvalue $\lambda \neq 0$ of X . Since $\text{rank}(X) = 2$ and $\text{Tr}(X) = 0$, the only nonzero eigenvalues of X are λ and $-\lambda$. This gives that $\|X\|_2 = \sqrt{\lambda^2 + (-\lambda)^2} = \sqrt{2}|\lambda|$; and $\|X\|_{\text{Tr}} = \frac{1}{2}(|\lambda| + |-\lambda|) = |\lambda|$.

For the second relation, note that

$$\begin{aligned} \|\psi\rangle\langle\psi| - \phi\rangle\langle\phi|\|_2^2 &= 2 - 2|\langle\psi|\phi\rangle|^2 \leq 2(2 - 2|\langle\psi|\phi\rangle|) \\ &\leq 2(2 - 2\text{Re}\langle\psi|\phi\rangle) = 2\|\psi\rangle - \phi\rangle\|^2. \end{aligned} \quad (4)$$

□

2.5. The Swap Test

The swap test, first introduced in [53], provides a computationally efficient way to project on the symmetric subspace of a tensor product Hilbert space—the $+1$ eigenspace of the swap operator that exchanges the two tensor terms. The swap test proceeds on a state $\rho \in \mathcal{D}(H \otimes H)$ as follows:

1. Prepare an auxiliary qubit in the state $|+\rangle$.
2. Act with the swap operator on ρ conditionally on the auxiliary qubit.
3. Measure the auxiliary qubit in the Hadamard basis. Accept if the measurement result is 0 and reject if not.

The swap test is illustrated as a quantum circuit in Figure 1.

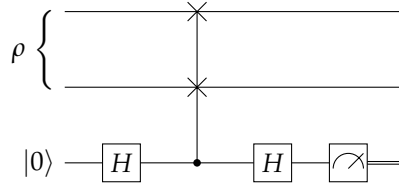


Figure 1. Quantum circuit representation of the swap test.

Let $\Pi_H : H \otimes H \rightarrow H \otimes H$ be the projector onto the symmetric subspace. Then, the swap test implements the channel $\rho \mapsto \Pi_H \rho \Pi_H \otimes |0\rangle\langle 0| + (\mathbb{I} - \Pi_H) \rho (\mathbb{I} - \Pi_H) \otimes |1\rangle\langle 1|$. It's also useful to note that, if ρ is a separable state $\rho = \rho_1 \otimes \rho_2$, then the probability of the swap test accepting is $\frac{1}{2} + \frac{1}{2} \text{Tr}[\rho_1 \rho_2]$.

3. Graph Parameters for Quantum Channels

In this section, we recall the definitions of quantum graphs and their graph parameters. Since every quantum graph can be generated by a quantum channel, we discuss how the clique and independent set graph parameters can be pulled back to the level of quantum channels. This serves to motivate the approximate version of these parameters that we introduce in Definition 6.

Definition 4. A quantum graph is an operator system $S \subseteq \mathcal{B}(H)$ for H a finite-dimensional Hilbert space, that is a vector subspace closed under the adjoint and containing $\mathbb{I}$.

As noted in the introduction, any quantum channel generates a quantum graph, sometimes referred to as its quantum confusability graph. For $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ with Kraus decomposition $\Phi(\rho) = \sum_{i=1}^n A_i \rho A_i^*$, the corresponding quantum graph is

$$S_\Phi = \text{span}_{\mathbb{C}} \{A_i^* A_j \mid 1 \leq i, j \leq n\}.$$

Proposition 1 ([24, Proposition 7.18]). *Let $S \subseteq \mathcal{B}(H)$ be a quantum graph. Then, there exists a finite-dimensional Hilbert space K and a quantum channel $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ such that $S = S_\Phi$.*

However, as for classical confusability graphs, the quantum channel corresponding to a quantum graph is not unique. Next, we recall the definitions of the independent set and clique graph parameters for a quantum graph.

Definition 5 ([2,26]). *Let $S \subseteq \mathcal{B}(H)$ be an operator system. A collection of orthogonal states $|v_1\rangle, \dots, |v_k\rangle \in H$ form a*

- *k -independent set of S if $|v_i\rangle\langle v_j|$ is orthogonal to S for all $i \neq j$;*
- *k -clique of S if $|v_i\rangle\langle v_j| \in S$ for all $i \neq j$.*

These parameters are known to naturally extend the classical definitions.

Theorem 5 ([24]). *A graph G , with corresponding quantum graph S_G , has a k -independent set (resp k -clique) if and only if S_G does.*

As for the classical case, these graph parameters can also be described in terms of a channel generating S . This will allow us to work with quantum graph parameters for quantum channels.

In the case of independent sets, Duan, Severini, and Winter establish that the independent sets for S correspond to states which are perfectly distinguishable after passing through the channel.

Lemma 2 ([2]). *Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a quantum channel and $S = S_\Phi \subseteq \mathcal{B}(H)$ be its quantum confusability graph. Then, pure states $|v_1\rangle, \dots, |v_k\rangle \in H$ are a k -independent set of S if and only if the overlap*

$$\text{Tr}[\Phi(|v_i\rangle\langle v_i|)\Phi(|v_j\rangle\langle v_j|)] = 0$$

for all $i \neq j$.

The connection between cliques for quantum graphs and channels is not as direct, since in general the quantum graph corresponding to a channel does not retain enough information about the original channel. Nevertheless, given a quantum graph S , one can associate to it a channel Φ whose behavior determines cliques in the quantum graph S .

Lemma 3. *Let $S \subseteq \mathcal{B}(H)$ be an operator system. Then, there exists a Hilbert space K , a channel $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, and a constant $c > 0$ (depending on the operating system) such that orthogonal states $|u\rangle, |v\rangle \in H$ satisfy $|u\rangle\langle v| \in S$ if and only if*

$$\text{Tr}[\Phi(|u\rangle\langle u|)\Phi(|v\rangle\langle v|)] \geq c.$$

Proof. First, we construct the requisite channel, following the construction of Paulsen [24]. Write $d = \dim H$ and $n = \dim S$. Let $X_1, \dots, X_{n-1} \in \mathcal{B}(H)$ be hermitian operators such that $\{\mathbb{I}/\sqrt{d}, X_1, \dots, X_n\}$ is an orthonormal basis of S . Then, set $H = \sum_{i=1}^{n-1} X_i \otimes (|i\rangle\langle i+1| + |i+1\rangle\langle i|)$. H is hermitian, so there exists $r > 0$ such that $r\mathbb{I} + H \geq 0$. Let $C \in \mathbb{M}_n(\mathcal{B}(H))$ be such that $C^*C = \frac{1}{nr}(r\mathbb{I} + H)$ and let $\{C_i\}_{i=1}^n$ be the $dn \times d$ block columns of C . Define $\Phi : \mathcal{B}(H) \rightarrow \mathbb{M}_n(\mathcal{B}(H))$ as $\Phi(\rho) = \sum_i C_i \rho C_i^*$. Note that Φ is a quantum channel as $\sum_i C_i^* C_i = \sum_i \frac{\mathbb{I}}{n} = \mathbb{I}$. Also, $S = S_\Phi$ since

$$C_i^* C_j = \begin{cases} \mathbb{I}/n & i = j \\ X_i/(nr) & j = i+1 \\ X_{i+1}/(nr) & j = i-1 \\ 0 & \text{else} \end{cases}$$

giving $S_\Phi = \text{span}\{C_i^* C_j\} = S$. Also, take $c = \frac{2}{(nr)^2}$.

Let $|u\rangle, |v\rangle \in H$ be orthogonal pure states. Then, the overlap

$$\begin{aligned} \text{Tr}[\Phi(|u\rangle\langle u|)\Phi(|v\rangle\langle v|)] &= \sum_{i,j} \text{Tr}\left[C_i|u\rangle\langle u|C_i^*C_j|v\rangle\langle v|C_j^*\right] = \sum_{i,j} \left|\text{Tr}\left(C_i^*C_j|v\rangle\langle u|\right)\right|^2 \\ &= \frac{2}{(nr)^2} \sum_i |\text{Tr}(X_i|v\rangle\langle u|)|^2 = c\|\text{proj}_S|u\rangle\langle v|\|_2^2, \end{aligned}$$

the length of the orthogonal projection onto S with respect to the Hilbert-Schmidt inner product. Then, if $|u\rangle\langle v| \in S$, $\|\text{proj}_S|u\rangle\langle v|\|_2^2 = 1$ and if $|u\rangle\langle v| \notin S$, $\|\text{proj}_S|u\rangle\langle v|\|_2^2 < 1$, giving that $\text{Tr}[\Phi(|u\rangle\langle u|)\Phi(|v\rangle\langle v|)] \geq c$ if and only if $|u\rangle\langle v| \in S$. $\square$

We use Lemma 3 to motivate the notion of a clique (cf. independent set) for a quantum channel Φ , as being a collection of orthogonal states with sufficiently high average (cf. low) overlap after passing through the channel.

Definition 6. Let H, K be Hilbert spaces and let $k \in \mathbb{N}$.

- A quantum channel $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ has an (α, k) -clique if there exist orthogonal states $\rho_1, \dots, \rho_k \in \mathcal{D}(H)$ such that

$$\frac{2}{k(k-1)} \sum_{1 \leq i < j \leq k} \text{Tr}[\Phi(\rho_i)\Phi(\rho_j)] \geq \alpha. \quad (5)$$

- A quantum channel $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ has an (α, k) -independent set if there exist orthogonal states $\rho_1, \dots, \rho_k \in \mathcal{D}(H)$ such that

$$\frac{2}{k(k-1)} \sum_{1 \leq i < j \leq k} \text{Tr}[\Phi(\rho_i)\Phi(\rho_j)] \leq 1 - \alpha. \quad (6)$$

4. The Complexity of the Clique Problem

4.1. The Quantum Clique Problem Is QMA(2)-Complete

In this section, we introduce and study the complexity of the promise problem of deciding the existence of an approximate independent set/clique for a quantum channel, given as a circuit.

Definition 7. Let $\mathcal{C}$ be some collection of quantum circuits, $c, s : \mathbb{N} \rightarrow [0, 1]$, and $k \in \mathbb{N}$. The quantum channel k -clique problem on $\mathcal{C}$ with completeness c and soundness s is the promise problem $\text{qClique}(k, \mathcal{C})_{c,s} = (Y, N)$ with

$$\begin{aligned} Y &= \{\text{quantum circuits } C \mid C \in \mathcal{C}, \Phi_C \text{ has a } (c(|C|), k)\text{-clique}\}, \\ N &= \{\text{quantum circuits } C \mid C \in \mathcal{C}, \Phi_C \text{ has no } (s(|C|), k)\text{-clique}\}, \end{aligned} \quad (7)$$

The quantum channel k -independent set problem on $\mathcal{C}$ with completeness c and soundness s is the promise problem $\text{qIS}(k, \mathcal{C})_{c,s} = (Y, N)$ with

$$\begin{aligned} Y &= \{\text{quantum circuits } C \mid \Phi_C \in \mathcal{C}, \Phi_C \text{ has a } (c(|C|), k)\text{-independent set}\}, \\ N &= \{\text{quantum circuits } C \mid \Phi_C \in \mathcal{C}, \Phi_C \text{ has no } (s(|C|), k)\text{-independent set}\}, \end{aligned} \quad (8)$$

Write $\text{qClique}(k)_{c,s}$ and $\text{qIS}(k)_{c,s}$ for the promise problems with respect to all circuits.

The main results of this section are to show that all of the above problems are in QMA(2) for $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap, and to show that the quantum 2-clique problem for some class of channels is in fact QMA(2)-hard. First, we recall the formal definition of QMA(2).

Definition 8. Let $c, s : \mathbb{N} \rightarrow [0, 1]$. A promise problem $Y, N \subseteq \{0, 1\}^*$ is in $\text{QMA}(2)_{c,s}$ if there exist polynomials $p, q : \mathbb{N} \rightarrow \mathbb{N}$ and a Turing machine V with one input tape and one output tape such that

- For all $x \in \{0, 1\}^*$, V halts on input x in $q(|x|)$ steps and outputs the description of a quantum circuit from $2p(|x|)$ qubits to one qubit.

- For all $x \in Y$, there exist states $\rho, \sigma \in \mathcal{D}(Q^{\otimes p(|x|)})$ such that $\langle 1 | \Phi_{V(x)}(\rho \otimes \sigma) | 1 \rangle \geq c(|x|)$.
- For all $x \in N$ and $\rho, \sigma \in \mathcal{D}(Q^{\otimes p(|x|)})$, $\langle 1 | \Phi_{V(x)}(\rho \otimes \sigma) | 1 \rangle \leq s(|x|)$.

We know that whenever $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ and have polynomial gap, then $\text{QMA}(2)_{c,s} = \text{QMA}(2)_{\frac{2}{3}, \frac{1}{3}} =: \text{QMA}(2)$, as for MA and QMA. Due to [37], we also know that $\text{QMA}(P) = \text{QMA}(2)$ for $P : \mathbb{N} \rightarrow \mathbb{N}$ any polynomial number of provers.

Lemma 4. Let $\mathcal{C}$ be any set of quantum circuits and let $c, s : \mathbb{N} \rightarrow [0, 1]$. Then, the problem $\text{qClique}(2, \mathcal{C})_{c,s} \in \text{QMA}(2)_{c',s'}$, where $c' = \frac{1}{2} + (c-s)\frac{c}{8}$ and $s' = \frac{1}{2} + (c-s)\frac{c+s}{16}$.

Thus, if $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ have polynomial gap, then $c' - s' = \frac{(c-s)^2}{16} \in \frac{1}{\text{poly}}$, giving that $\text{qClique}(2, \mathcal{C})_{c,s} \in \text{QMA}(2)$.

Proof. Suppose the verifier receives a quantum circuit C . Let $\Phi = \Phi_C$. The verifier expects to receive a proof $\rho \otimes \sigma \in \mathcal{D}(Q^{\otimes 2\text{in}(C)})$, and then effects the following procedure.

1. The verifier samples a random bit b , which is 0 with probability p (to be specified later).
2. If $b = 0$, the verifier runs the swap test on $\rho \otimes \sigma$. If it fails, she accepts and outputs 1, and if it passes, she rejects and outputs 0.
3. If $b = 1$, the verifier acts with Φ to get $\Phi(\rho) \otimes \Phi(\sigma)$. Then, she runs the swap test on this state, and outputs 1 if it passes and 0 if it fails.

Now, suppose $C \in Y$. Then, there exists a $(c(|C|), 2)$ -clique $\rho, \sigma \in \mathcal{D}(Q^{\text{in}(C)})$, so the prover provides $\rho \otimes \sigma$ to the verifier. If $b = 0$, the probability of accepting is $\frac{1}{2} - \frac{1}{2} \text{Tr}(\rho\sigma) = \frac{1}{2}$, as ρ and σ are orthogonal; if $b = 1$, since ρ and σ form a clique, the probability of accepting is $\frac{1}{2} + \frac{1}{2} \text{Tr}[\Phi(\rho)\Phi(\sigma)] \geq \frac{1}{2} + \frac{c(|C|)}{2}$. All together, the acceptance probability is at least $c'(|C|) = p\frac{1}{2} + (1-p)\left(\frac{1}{2} + \frac{c(|C|)}{2}\right) = \frac{1}{2} + (1-p)\frac{c(|C|)}{4}$. On the other hand, suppose that $C \in N$. Then, for all orthogonal $\rho, \sigma \in \mathcal{D}(Q^{\text{in}(C)})$, $\text{Tr}[\Phi(\rho)\Phi(\sigma)] \leq s(|C|)$. Without loss of generality, we may assume that the verifier receives a pure state proof $\rho \otimes \sigma = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$. Then, if $b = 0$, the probability of accepting is $\frac{1}{2} - \frac{1}{2} |\langle\psi|\phi\rangle|^2$. There exists a state $|\phi'\rangle$ orthogonal to $|\psi\rangle$ such that $|\phi\rangle = \alpha|\psi\rangle + \beta|\phi'\rangle$ with $|\alpha|^2 = |\langle\psi|\phi\rangle|^2$. Now, note that

$$\| |\phi\rangle\langle\phi| - |\phi'\rangle\langle\phi'| \|_{\text{Tr}} = \frac{1}{\sqrt{2}} \| |\phi\rangle\langle\phi| - |\phi'\rangle\langle\phi'| \|_2 = \sqrt{\frac{1}{2}(2 - 2|\beta|^2)} = |\langle\phi|\psi\rangle|. \quad (9)$$

So, if $b = 1$, the probability of accepting is

$$\begin{aligned} \frac{1}{2} + \frac{1}{2} \text{Tr}[\Phi(|\psi\rangle\langle\psi|)\Phi(|\phi\rangle\langle\phi|)] &\leq \frac{1}{2} + \frac{1}{2} \text{Tr}[\Phi(|\psi\rangle\langle\psi|)\Phi(|\phi'\rangle\langle\phi'|)] + \frac{1}{2} \| |\phi\rangle\langle\phi| - |\phi'\rangle\langle\phi'| \|_{\text{Tr}} \\ &\leq \frac{1}{2} + \frac{s(|C|)}{2} + \frac{|\langle\psi|\phi\rangle|}{2}. \end{aligned} \quad (10)$$

Thus, the total probability of accepting is at most

$$\begin{aligned} s'(|C|) &= \frac{1}{2} + (1-p)\frac{s(|C|)}{4} + \frac{1}{2} \left((1-p)|\langle\psi|\phi\rangle| - p|\langle\psi|\phi\rangle|^2 \right) \\ &\leq \frac{1}{2} + (1-p)\frac{s(|C|)}{4} + \frac{(1-p)^2}{8p}, \end{aligned} \quad (11)$$

by maximizing over $|\langle\phi|\psi\rangle| \in \mathbb{R}$. Now taking $p = 1 - \frac{c(|C|)-s(|C|)}{2}$, we get $c' = \frac{1}{2} + (c-s)\frac{c}{8}$ and $s' = \frac{1}{2} + (c-s)\frac{s}{8} + \frac{(c-s)^2}{16(2-(c-s))} \leq \frac{1}{2} + (c-s)\frac{s}{8} + \frac{(c-s)^2}{16} = \frac{1}{2} + (c-s)\frac{c+s}{16}$ $\square$

Next, we want to show that the quantum clique problem is in fact $\text{QMA}(2)$ -hard as well. In fact, we show the stronger property that a subclass of all channels achieves this hardness. First, we introduce this subclass.

Definition 9. A quantum channel $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is entanglement-breaking if there exists a POVM $\{M_i\}_i \subseteq \mathcal{P}(H)$ and a set of states $\{\sigma_i\}_i \subseteq \mathcal{P}(K)$ such that $\Phi(\rho) = \sum_i \text{Tr}[M_i\rho]\sigma_i$.

Write $\mathcal{C}_{\text{EB}}$ for the set of quantum circuits representing entanglement-breaking channels. Write $\mathcal{C}_{\text{Tr}}$ for the collection of circuits that are partial traces on some of the qubits.

The name *entanglement-breaking* for these maps arises since, if Φ is entanglement-breaking, then the output state of $\mathbb{I} \otimes \Phi$ is always separable. These are the only maps with the property. For more on entanglement-breaking channels, see for example [54,55].

The circuits we are interested in combine the above two restricted classes of channels by means of a direct sum. At the level of channels, we mean to use channels of the form $p\Phi_{\text{Tr}} \oplus (1-p)\Phi_{\text{EB}}$. To do so, we need a way to form the direct sum of two quantum circuits.

Definition 10. Let C_1 and C_2 be circuits with $\text{in}(C_1) = \text{in}(C_2)$ and $p \in [0, 1]$. Let P_i be the number of qubits prepared by C_i , and let $T_i = \text{in}(C_i) + P_i - \text{out}(C_i)$ be the number of qubits traced out by C_i . We define the direct sum circuit $C_1 \oplus_p C_2$ as the circuit constructed as follows:

- (i) The circuit prepares a qubit R in the state $\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$.
- (ii) The circuit prepares $\max\{P_1, P_2\} + |\text{out}(C_1) - \text{out}(C_2)|$ qubits in state $|0\rangle$.
- (iii) The circuit acts with all the unitary gates of C_1 conditioned on R . Then, it swaps, conditionally on R as well, the qubits of C_1 that would be traced out to the last positions.
- (iv) The circuit flips R with X and acts in the same way as above for C_2 conditioned on R , before flipping back with X .
- (v) The circuit traces out the last $\max\{T_1, T_2\}$ qubits.
- (vi) The circuit measures R in the computational basis.

Note that the state $\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$ cannot be generated exactly by finitely many gates from the gate set for all but countably many values of p . However, for any p , polynomially many gates (in the circuit sizes) can be used to construct an exponentially good approximation to the state, which we use to keep the circuits of reasonable length. See also Figure 2 for an example of the implementation of the above definition.

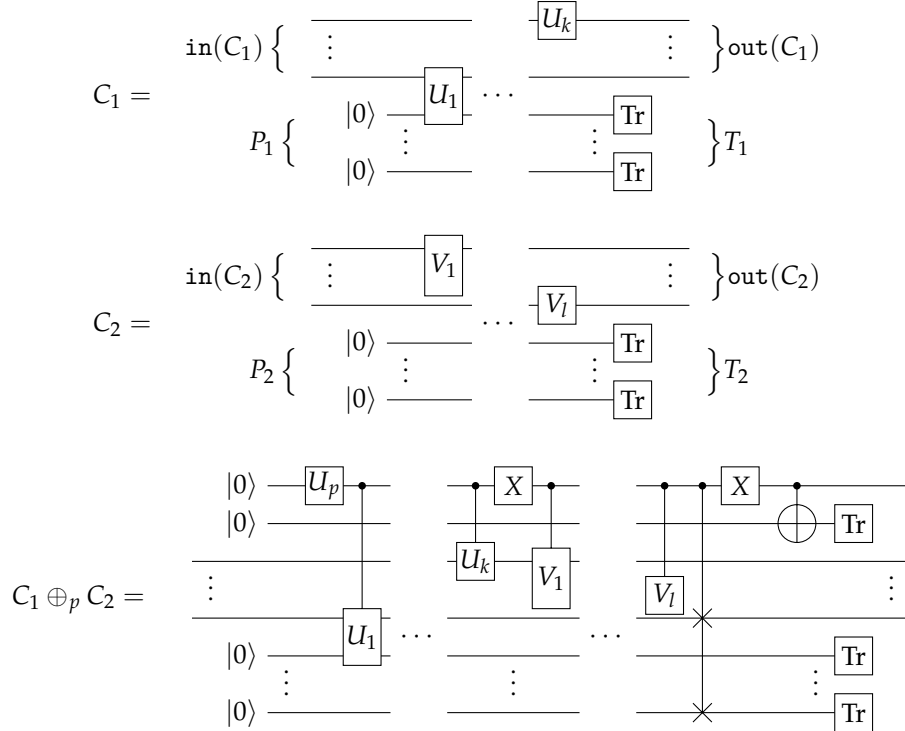


Figure 2. Representation of the construction of the circuit $C_1 \oplus_p C_2$ from the circuits C_1 and C_2 in canonical form, with $\text{out}(C_2) < \text{out}(C_1)$. U_p is a unitary that implements the map $|0\rangle \mapsto \sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$ to good approximation.

Using the natural identification, for $m > n$, of $Q^{\otimes m} \oplus Q^{\otimes n}$ with the subspace $|0\rangle \otimes Q^{\otimes m} + |1\rangle \otimes Q^{\otimes n} \otimes |0^{m-n}\rangle \subseteq Q^{\otimes(m+1)}$, we see that $\Phi_{C_1 \oplus_p C_2} = p\Phi_{C_1} \oplus (1-p)\Phi_{C_2}$. Note also that, by the above definition $|C_1 \oplus_p C_2| = O(|C_1| + |C_2|)$.

Definition 11. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be collections of quantum circuits. Write

$$\mathcal{C}_1 \oplus \mathcal{C}_2 = \{C_1 \oplus_p C_2 \mid C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2, p \in [0, 1]\}. \quad (12)$$

Theorem 6. There exist $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with constant gap such that the clique problem $qClique(2)_{c,s}(2, \mathcal{C}_{\text{Tr}} \oplus \mathcal{C}_{\text{EB}})$ is QMA(2)-hard.

To prove this theorem, we develop a technique to take a channel whose cliques have certain properties with respect to a set of states, and combine it with another channel in order to enlarge the set of states on which those properties hold. This is inspired by self-testing techniques, such as Lemma 5.5 of [40].

Lemma 5. Let H, K , and K' be Hilbert spaces, and $A, B \subseteq \mathcal{D}(H)^2$ be sets of pairs of states. Suppose there exist channels $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ and $\Psi : \mathcal{B}(H) \rightarrow \mathcal{B}(K')$ with the properties that

- For $(\rho, \sigma) \in A$, $\text{Tr}[\Phi(\rho)\Phi(\sigma)] \leq p_0$, and if $\text{Tr}[\Phi(\rho)\Phi(\sigma)] \geq p_0 - \varepsilon$ then there exist $(\rho_0, \sigma_0) \in B$ such that $\|\rho - \rho_0\|_{\text{Tr}} + \|\sigma - \sigma_0\|_{\text{Tr}} \leq C\varepsilon^\alpha$ for some constants $C, \alpha > 0$ independent of (ρ, σ) .
- For all $(\rho, \sigma) \in B$, $\text{Tr}[\Psi(\rho)\Psi(\sigma)] \leq q_0$.

Then, for all $\eta > 0$, there exists $p \in (0, 1)$ such that the channel $\Phi_p : \mathcal{B}(H) \rightarrow \mathcal{B}(K \oplus K')$ defined $\Phi_p = p\Phi \oplus (1-p)\Psi$ satisfies the property that, for all $(\rho, \sigma) \in A$,

$$\text{Tr}[\Phi_p(\rho)\Phi_p(\sigma)] \leq p^2 p_0 + (1-p)^2 q_0 + \eta. \quad (13)$$

From the proof below, it may be verified that if $\eta = 1/\text{poly}$, $C = \text{poly}$, and $\alpha < 1$ is constant, we get a probability $p, 1-p = 1/\text{poly}$. Further, the construction preserves polynomial gaps. In fact, if we have the promise that $\text{Tr}[\Psi(\rho)\Psi(\sigma)] \leq s$ for all $(\rho, \sigma) \in B$ or $\text{Tr}[\Psi(\rho)\Psi(\sigma)] \geq c$ for some $(\rho, \sigma) \in A \cap B$ with $c-s = 1/\text{poly}$, then we have that either $\text{Tr}[\Phi_p(\rho)\Phi_p(\sigma)] \leq s'$ for all $(\rho, \sigma) \in A$ or $\text{Tr}[\Phi_p(\rho)\Phi_p(\sigma)] \geq c'$ for some $(\rho, \sigma) \in A \cap B$, where $c' = p^2 p_0 + (1-p)^2 c$ and $s' = p^2 p_0 + (1-p)^2 s + \eta$. So the gap becomes $c' - s' = (1-p)^2(c-s) - \eta$. Since $(1-p)^2 = \frac{\eta^{1-\alpha}}{\text{poly}}$, we may choose $\eta = 1/\text{poly}$ small enough to preserve the polynomial gap.

Proof. Let $(\rho, \sigma) \in A$. Then, there exists some $\varepsilon \geq 0$ such that $\text{Tr}[\Phi(\rho)\Phi(\sigma)] = p_0 - \varepsilon$. By hypothesis, there exist $(\rho_0, \sigma_0) \in B$ such that $\|\rho - \rho_0\|_{\text{Tr}} + \|\sigma - \sigma_0\|_{\text{Tr}} \leq C\varepsilon^\alpha$. Also, we have by hypothesis that $\text{Tr}[\Psi(\rho_0)\Psi(\sigma_0)] \leq q_0$. Therefore,

$$\begin{aligned} \text{Tr}[\Psi(\rho)\Psi(\sigma)] &= \text{Tr}[\Psi(\rho_0)\Psi(\sigma_0)] + \text{Tr}[\Psi^\dagger(\Psi(\rho_0))(\sigma - \sigma_0)] + \text{Tr}[(\rho - \rho_0)\Psi^\dagger(\Psi(\sigma))] \\ &\leq \text{Tr}[\Psi(\rho_0)\Psi(\sigma_0)] + \|\sigma - \sigma_0\|_{\text{Tr}} + \|\rho - \rho_0\|_{\text{Tr}} \\ &\leq q_0 + C\varepsilon^\alpha. \end{aligned} \quad (14)$$

Using this bound,

$$\begin{aligned} \text{Tr}[\Phi_p(\rho)\Phi_p(\sigma)] &= p^2 \text{Tr}[\Phi(\rho)\Phi(\sigma)] + (1-p)^2 \text{Tr}[\Psi(\rho)\Psi(\sigma)] \\ &\leq p^2(p_0 - \varepsilon) + (1-p)^2(q_0 + C\varepsilon^\alpha) \\ &= p^2 p_0 + (1-p)^2 q_0 + (1-p)^2 C\varepsilon^\alpha - p^2 \varepsilon \end{aligned} \quad (15)$$

To finish the proof, it remains to show that p may be chosen so that $(1-p)^2 C\varepsilon^\alpha - p^2 \varepsilon \leq \eta$ for all $\varepsilon \geq 0$. Without loss of generality, we may assume $\alpha < 1$. Then, by extremizing the expression with respect to ε , we see that it is maximized at $\varepsilon = \left(C\alpha\left(\frac{1}{p} - 1\right)^2\right)^{1/(1-\alpha)}$, giving $(1-p)^2 C\varepsilon^\alpha - p^2 \varepsilon \leq C^{1/(1-\alpha)}\left(\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)}\right)p^2\left(\frac{1}{p} - 1\right)^{2/(1-\alpha)}$. Thus, rearranging the wanted expression $C^{1/(1-\alpha)}\left(\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)}\right)p^2\left(\frac{1}{p} - 1\right)^{2/(1-\alpha)} \leq \eta$ gives

$$\frac{1-p}{p^\alpha} \leq \left(\frac{\eta}{C^{1/(1-\alpha)}\left(\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)}\right)}\right)^{(1-\alpha)/2}. \quad (16)$$

Since $\frac{1-p}{p} \geq \frac{1-p}{p^\alpha}$, we may take $\frac{1-p}{p} = \left(\frac{\eta}{C^{1/(1-\alpha)} (\alpha^\alpha/(1-\alpha) - \alpha^{1/(1-\alpha)})} \right)^{(1-\alpha)/2}$, or

$$p = \frac{1}{1 + \left(\frac{\eta}{C^{1/(1-\alpha)} (\alpha^\alpha/(1-\alpha) - \alpha^{1/(1-\alpha)})} \right)^{(1-\alpha)/2}} \quad (17)$$

to get the result. $\square$

Lemma 6. Let H and K be Hilbert spaces and pure states $\rho = |\psi\rangle\langle\psi|, \sigma = |\phi\rangle\langle\phi| \in \mathcal{D}(H^{\otimes k} \otimes K)$. Suppose that there exist $\varepsilon_i > 0$ such that $\text{Tr}(\rho_i \sigma_i) \geq 1 - \varepsilon_i$ for all i , where ρ_i and σ_i are the marginals on the i -th copy of H . Write $\varepsilon = \frac{1}{k} \sum_i \varepsilon_i$. Then, there exist separable states $\tilde{\rho} = |\psi_1\rangle\langle\psi_1| \otimes \cdots \otimes |\psi_k\rangle\langle\psi_k| \otimes |\psi'\rangle\langle\psi'|, \tilde{\sigma} = |\phi_1\rangle\langle\phi_1| \otimes \cdots \otimes |\phi_k\rangle\langle\phi_k| \otimes |\phi'\rangle\langle\phi'| \in \mathcal{D}(H^{\otimes k} \otimes K)$ such that $\|\rho - \tilde{\rho}\|_2 \leq 2\sqrt{k\varepsilon}$, $\|\sigma - \tilde{\sigma}\|_2 \leq 2\sqrt{k\varepsilon}$, and $\|\text{Tr}_K \tilde{\rho} - \text{Tr}_K \tilde{\sigma}\|_2 \leq 6\sqrt{k\varepsilon}$.

Proof. Fix some $i = 1, \dots, k$. Then, $\text{Tr}(\rho_i \sigma_i) \geq 1 - \varepsilon_i$ implies that $\sqrt{\text{Tr} \rho_i^2} \sqrt{\text{Tr} \sigma_i^2} \geq 1 - \varepsilon_i$. In particular, since $\text{Tr} \sigma_i^2 \leq 1$, $\text{Tr} \rho_i^2 \geq (1 - \varepsilon_i)^2$. Now, let $\rho_i = \sum_j \lambda_{i,j} |\psi_j^{(i)}\rangle\langle\psi_j^{(i)}|$ be a spectral decomposition of ρ_i , with $\lambda_{i,1} \geq \lambda_{i,2} \geq \dots$. First, the above implies that $\sum_j \lambda_{i,j}^2 \geq (1 - \varepsilon_i)^2$ and the fact that ρ_i is a state means that $\sum_j \lambda_{i,j} = 1$. As such,

$$\lambda_{i,1} = \sum_j \lambda_{i,1} \lambda_{i,j} \geq \sum_j \lambda_{i,j}^2 \geq (1 - \varepsilon_i)^2. \quad (18)$$

Thus, $\langle \psi_1^{(i)} | \rho_i | \psi_1^{(i)} \rangle = \lambda_{i,1} \geq (1 - \varepsilon_i)^2$, or equivalently, $\|(\mathbb{I} \otimes \langle \psi_1^{(i)} | \otimes \mathbb{I}) |\psi\rangle\| \geq 1 - \varepsilon_i$. Now, we can expand $|\psi\rangle$ in the basis $|\psi_{j_1}^{(1)}\rangle \otimes \cdots \otimes |\psi_{j_k}^{(k)}\rangle$ of $H^{\otimes k}$ as $|\psi\rangle = \sum_{j_1, \dots, j_k} |\psi_{j_1}^{(1)}\rangle \otimes \cdots \otimes |\psi_{j_k}^{(k)}\rangle \otimes |\psi_{j_1 \dots j_k}\rangle$, where the $|\psi_{j_1 \dots j_k}\rangle \in K$ are subnormalized states such that $\sum_{j_1, \dots, j_k} \langle \psi_{j_1 \dots j_k} | \psi_{j_1 \dots j_k} \rangle = 1$. For all i , we have that

$$\sum_{j_i \neq 1} \langle \psi_{j_1 \dots j_k} | \psi_{j_1 \dots j_k} \rangle = 1 - \|(\mathbb{I} \otimes \langle \psi_1^{(i)} | \otimes \mathbb{I}) |\psi\rangle\|^2 \leq 1 - (1 - \varepsilon_i)^2 \leq 2\varepsilon_i. \quad (19)$$

Therefore, by a union bound,

$$\begin{aligned} \langle \psi_{1 \dots 1} | \psi_{1 \dots 1} \rangle &= 1 - \sum_{j_1 \neq 1 \vee \dots \vee j_k \neq 1} \langle \psi_{j_1 \dots j_k} | \psi_{j_1 \dots j_k} \rangle \geq 1 - \sum_i \sum_{j_i \neq 1} \langle \psi_{j_1 \dots j_k} | \psi_{j_1 \dots j_k} \rangle \\ &\geq 1 - 2k\varepsilon. \end{aligned} \quad (20)$$

Let $|\psi_i\rangle = |\psi_1^{(i)}\rangle$ and $|\psi'\rangle = \frac{|\psi_{1 \dots 1}\rangle}{\| |\psi_{1 \dots 1}\rangle \|}$. Then,

$$\begin{aligned} \|\rho - \tilde{\rho}\|_2^2 &= 2 - 2\|(\langle \psi_1 | \otimes \cdots \otimes \langle \psi_k | \otimes \langle \psi' |) |\psi\rangle\|^2 = 2 - 2|\langle \psi' | \psi_{1 \dots 1} \rangle|^2 \\ &= 2 - 2\langle \psi_{1 \dots 1} | \psi_{1 \dots 1} \rangle \leq 4k\varepsilon. \end{aligned} \quad (21)$$

By symmetry, we also get separable $\tilde{\sigma}$ such that $\|\sigma - \tilde{\sigma}\|_2 \leq 2\sqrt{k\varepsilon}$.

For the last part of the proof, note that

$$\begin{aligned} \|\text{Tr}_K \tilde{\rho} - \text{Tr}_K \tilde{\sigma}\|_2^2 &= \| |\psi_1\rangle\langle\psi_1| \otimes \cdots \otimes |\psi_k\rangle\langle\psi_k| - |\phi_1\rangle\langle\phi_1| \otimes \cdots \otimes |\phi_k\rangle\langle\phi_k| \|_2^2 \\ &= 2 - 2|\langle \psi_1 | \phi_1 \rangle|^2 \cdots |\langle \psi_k | \phi_k \rangle|^2 \\ &= 2 - 2(1 - \frac{1}{2} \| |\psi_1\rangle\langle\psi_1| - |\phi_1\rangle\langle\phi_1| \|_2^2) \cdots (1 - \frac{1}{2} \| |\psi_k\rangle\langle\psi_k| - |\phi_k\rangle\langle\phi_k| \|_2^2) \\ &\leq \sum_i \| |\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i| \|_2^2. \end{aligned} \quad (22)$$

Next, $\|\rho_i - |\psi_i\rangle\langle\psi_i|\|_2^2 = \text{Tr}(\rho_i^2) + 1 - 2\langle \psi_i | \rho_i | \psi_i \rangle \leq 2 - 2(1 - \varepsilon_i)^2 \leq 4\varepsilon_i$ and similarly for σ , so

$$\begin{aligned} \| |\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i| \|_2 &\leq \| |\psi_i\rangle\langle\psi_i| - \rho_i \|_2 + \| \rho_i - \sigma_i \|_2 + \| \sigma_i - |\phi_i\rangle\langle\phi_i| \|_2 \\ &\leq 2\sqrt{\varepsilon_i} + \sqrt{2\varepsilon_i} + 2\sqrt{\varepsilon_i} = (4 + \sqrt{2})\sqrt{\varepsilon_i}. \end{aligned} \quad (23)$$

Together, this gives $\|\text{Tr}_K \tilde{\rho} - \text{Tr}_K \tilde{\sigma}\|_2^2 \leq k(4 + \sqrt{2})^2 \varepsilon$, as wanted $\square$

Lemma 7. Let H and K be Hilbert spaces and $k \in \mathbb{N}$, and define the channel $\Phi_1 : \mathcal{B}(H^{\otimes k} \otimes K) \rightarrow \mathcal{B}(H \otimes \mathbb{C}^{[k]})$ as

$$\Phi_1(\rho) = \mathbb{E}_{i=1}^k \rho_i \otimes |i\rangle\langle i|, \quad (24)$$

where ρ_i is the marginal of ρ on the i -th copy of H and $\mathbb{E}$ is the normalized sum (expectation). Let $\rho = |\psi\rangle\langle\psi|, \sigma = |\phi\rangle\langle\phi| \in \mathcal{D}(H^{\otimes k} \otimes H')$ be orthogonal pure states. Then, $\text{Tr}[\Phi_1(\rho)\Phi_1(\sigma)] \leq \frac{1}{k}$ and if $\text{Tr}[\Phi_1(\rho)\Phi_1(\sigma)] \geq \frac{1}{k} - \varepsilon$, then there exist states $|\psi_1\rangle, \dots, |\psi_k\rangle \in H$ and orthogonal states $|\alpha\rangle, |\beta\rangle \in K$ such that

$$\|\rho - \tilde{\rho}\|_{\text{Tr}} + \|\sigma - \tilde{\sigma}\|_{\text{Tr}} \leq 10k\varepsilon^{1/4}, \quad (25)$$

where $\tilde{\rho} = |\psi_1\rangle\langle\psi_1| \otimes \dots \otimes |\psi_k\rangle\langle\psi_k| \otimes |\alpha\rangle\langle\alpha|$ and $\tilde{\sigma} = |\psi_1\rangle\langle\psi_1| \otimes \dots \otimes |\psi_k\rangle\langle\psi_k| \otimes |\beta\rangle\langle\beta|$.

Proof. First, $\text{Tr}[\Phi_1(\rho)\Phi_1(\sigma)] = \frac{1}{k} \mathbb{E}_i \text{Tr}(\rho_i \sigma_i) \leq \frac{1}{k} \mathbb{E}_i 1 = \frac{1}{k}$. Now, suppose that $\text{Tr}[\Phi_1(\rho)\Phi_1(\sigma)] \geq \frac{1}{k} - \varepsilon$. Let $\varepsilon_i \in [0, 1]$ such that $\text{Tr}(\rho_i \sigma_i) = 1 - \varepsilon_i$; we know that $k\varepsilon \geq \mathbb{E}_i \varepsilon_i$. Then, due to Lemma 6, there exist separable states $\rho' = |\psi_1\rangle\langle\psi_1| \otimes \dots \otimes |\psi_k\rangle\langle\psi_k| \otimes |\alpha\rangle\langle\alpha|, \sigma' = |\phi_1\rangle\langle\phi_1| \otimes \dots \otimes |\phi_k\rangle\langle\phi_k| \otimes |\phi'\rangle\langle\phi'| \in \mathcal{D}(H^{\otimes k} \otimes H')$ such that $\|\rho - \rho'\|_2 \leq 2k\sqrt{\varepsilon}$, $\|\sigma - \sigma'\|_2 \leq 2k\sqrt{\varepsilon}$, and $\|\text{Tr}_{H'} \rho' - \text{Tr}_{H'} \sigma'\|_2 \leq 6k\sqrt{\varepsilon}$. Since all these are pure states, this implies that $\|\rho - \rho'\|_{\text{Tr}} \leq \sqrt{2k}\sqrt{\varepsilon}$, $\|\sigma - \sigma'\|_{\text{Tr}} \leq \sqrt{2k}\sqrt{\varepsilon}$, and $\|\text{Tr}_{H'} \rho' - \text{Tr}_{H'} \sigma'\|_{\text{Tr}} \leq 3\sqrt{2k}\sqrt{\varepsilon}$. Let $\sigma'' = \text{Tr}_{H'}(\rho') \otimes |\phi'\rangle\langle\phi'|$; then

$$\|\sigma - \sigma''\|_{\text{Tr}} \leq \|\sigma - \sigma'\|_{\text{Tr}} + \|\text{Tr}_{H'} \sigma' - \text{Tr}_{H'} \rho'\|_{\text{Tr}} \leq 4\sqrt{2k}\sqrt{\varepsilon}. \quad (26)$$

Now, the inner product

$$|\langle\alpha|\phi'\rangle|^2 = \text{Tr}[\rho'\sigma''] \leq \text{Tr}[\rho\sigma] + \|\rho - \rho'\|_{\text{Tr}} + \|\sigma - \sigma''\|_{\text{Tr}} \leq 5\sqrt{2k}\sqrt{\varepsilon}. \quad (27)$$

There exists a state $|\beta\rangle \in H'$ orthogonal to $|\alpha\rangle$ such that $|\phi'\rangle = a|\alpha\rangle + b|\beta\rangle$ for some $|a|^2 \leq 5\sqrt{2k}\sqrt{\varepsilon}$. Due to this

$$\| |\phi'\rangle\langle\phi'| - |\beta\rangle\langle\beta| \|_{\text{Tr}} = \frac{1}{\sqrt{2}} \| |\phi'\rangle\langle\phi'| - |\beta\rangle\langle\beta| \|_2 = \sqrt{\frac{1}{2}(2 - 2|b|^2)} = |a| \leq \sqrt{5\sqrt{2k}\sqrt{\varepsilon}}. \quad (28)$$

Let $\tilde{\rho} = \rho'$ and $\tilde{\sigma} = \text{Tr}_{H'} \sigma'' \otimes |\beta\rangle\langle\beta|$. Then we get the wanted result

$$\begin{aligned} \|\rho - \tilde{\rho}\|_{\text{Tr}} + \|\sigma - \tilde{\sigma}\|_{\text{Tr}} &\leq \|\rho - \rho'\|_{\text{Tr}} + \|\sigma - \sigma''\|_{\text{Tr}} + \| |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| \|_{\text{Tr}} \\ &\leq 5\sqrt{2k}\sqrt{\varepsilon} + \sqrt{5\sqrt{2k}\sqrt{\varepsilon}} \leq 10k\varepsilon^{1/4}. \end{aligned} \quad (29)$$

$\square$

Lemma 8. Let H be a Hilbert space. For any channel $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(Q)$ define $\Psi : \mathcal{B}(H \otimes Q) \rightarrow \mathcal{B}(Q_{\perp})$ as

$$\Psi(\rho) = \langle 0 | (\Phi \otimes \text{Tr})(\rho) | 0 \rangle \mu_Q + \langle 1 | (\Phi \otimes \text{Tr})(\rho) | 1 \rangle | \perp \rangle \langle \perp | \quad (30)$$

Then, for any $\rho \in \mathcal{D}(H)$ and $\sigma, \sigma' \in \mathcal{D}(Q)$, if $\langle 1 | \Phi(\rho) | 1 \rangle \geq \frac{3}{4}$, then $\text{Tr}[\Psi(\rho \otimes \sigma)\Psi(\rho \otimes \sigma')] \geq \frac{19}{32}$; and if $\langle 1 | \Phi(\rho) | 1 \rangle \leq \frac{2}{3}$, then $\text{Tr}[\Psi(\rho \otimes \sigma)\Psi(\rho \otimes \sigma')] \leq \frac{1}{2}$

Proof. We can directly compute this. First, $\Psi(\rho \otimes \sigma) = \langle 0 | \Phi(\rho) | 0 \rangle \mu_Q + \langle 1 | \Phi(\rho) | 1 \rangle | \perp \rangle \langle \perp |$. Therefore,

$$\begin{aligned} \text{Tr}[\Psi(\rho \otimes \sigma)\Psi(\rho \otimes \sigma')] &= \langle 0 | \Phi(\rho) | 0 \rangle^2 \text{Tr}(\mu_Q^2) + \langle 1 | \Phi(\rho) | 1 \rangle^2 | \langle \perp | \perp \rangle |^2 \\ &= \frac{1}{2} (1 - \langle 1 | \Phi(\rho) | 1 \rangle)^2 + \langle 1 | \Phi(\rho) | 1 \rangle^2. \end{aligned} \quad (31)$$

The inequalities follow from this expression. $\square$

Proof of Theorem 6. Let $L = (Y, N) \in \text{QMA}(2)_{\frac{3}{4}, \frac{1}{4}}$, and let $x \in Y \cup N$. Then, there exists a polynomial-time computable channel $\Phi_x : \mathcal{B}(Q^{\otimes 2p(|x|)}) \rightarrow \mathcal{B}(Q)$ such that, if $x \in Y$, there exist $\rho, \sigma \in \mathcal{D}(Q^{\otimes p(|x|)})$ such that $\langle 1 | \Phi_x(\rho \otimes \sigma) | 1 \rangle \geq \frac{3}{4}$; and if $x \in N$, for all $\rho, \sigma \in \mathcal{D}(Q^{\otimes p(|x|)})$, $\langle 1 | \Phi_x(\rho \otimes \sigma) | 1 \rangle \leq \frac{1}{4}$. Construct the channel $\Psi_x : \mathcal{B}(Q^{\otimes (2p(|x|)+1)}) \rightarrow \mathcal{B}(Q_\perp)$ from Φ_x as in Lemma 8. Also, write $H = Q^{\otimes p(|x|)}$ and $K = Q$, and let $\Phi_1 : \mathcal{B}(H \otimes H \otimes K) \rightarrow \mathcal{B}(H \otimes \mathbb{C}^{[2]})$ as in Lemma 7. For $p \in (0, 1)$, consider the channel $\Phi_p = p\Phi_1 \oplus (1-p)\Psi_x$. Using the construction of Definition 10 and the fact that Φ_x admits a circuit polynomial in the size of the instance x , we see that this channel admits a circuit representation polynomial in $|x|$.

First, suppose that $x \in Y$. Then, we claim that there exist $\rho, \sigma \in \mathcal{D}(Q^{\otimes (2p(|x|)+1)})$ such that $\text{Tr}[\Phi_p(\rho)\Phi_p(\sigma)] \geq \frac{p^2}{2} + \frac{19(1-p)^2}{32}$. Since $x \in Y$, there exists a separable state on which Φ_x accepts with high probability; we may, without loss of generality, assume that this state is pure so of the form $|\psi\rangle \otimes |\phi\rangle$. Let $\rho = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| \otimes |0\rangle\langle 0|$ and $\sigma = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| \otimes |1\rangle\langle 1|$. Then, $\Phi_p(\rho) = \frac{p}{2}(|\psi\rangle\langle\psi| \otimes |1\rangle\langle 1| + |\phi\rangle\langle\phi| \otimes |2\rangle\langle 2|) \oplus (1-p)\Psi_x(\rho)$, and similarly for σ , giving that

$$\begin{aligned} \text{Tr}[\Phi_p(\rho)\Phi_p(\sigma)] &= \frac{p^2}{4}(|\langle\psi|\psi\rangle|^2 + |\langle\phi|\phi\rangle|^2) + (1-p)^2 \text{Tr}[\Psi_x(\rho)\Psi_x(\sigma)] \\ &\geq \frac{p^2}{2} + (1-p)^2 \frac{19}{32}. \end{aligned} \quad (32)$$

Now, suppose that $x \in N$. We use Lemma 5 to show that for all $\eta > 0$, there exists $p \in (0, 1)$ such that $\text{Tr}[\Phi_p(\rho)\Phi_p(\sigma)] \leq \frac{p^2}{2} + \frac{(1-p)^2}{2} + \eta$. We need that Φ_1 and Ψ_x satisfy the conditions of lemma. Let

$$A = \{(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) \mid |\psi\rangle\langle\psi|, |\phi\rangle\langle\phi| \in \mathcal{D}(Q^{\otimes (2p(|x|)+1)}), \langle\psi|\phi\rangle = 0\}, \quad (33)$$

$$\begin{aligned} B = \{(|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2| \otimes |\alpha\rangle\langle\alpha|, |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2| \otimes |\beta\rangle\langle\beta|) \\ \mid |\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2| \in \mathcal{D}(Q^{\otimes p(|x|)}), |\alpha\rangle\langle\alpha|, |\beta\rangle\langle\beta| \in \mathcal{D}(Q), \langle\alpha|\beta\rangle = 0\}. \end{aligned} \quad (34)$$

First, due to Lemma 7, for all $(\rho, \sigma) \in A$, $\text{Tr}[\Phi_1(\rho)\Phi_1(\sigma)] \leq \frac{1}{2}$, and if $\text{Tr}[\Phi_1(\rho)\Phi_1(\sigma)] \geq \frac{1}{2} - \varepsilon$, then there exists $(\tilde{\rho}, \tilde{\sigma}) \in B$ such that $\|\rho - \tilde{\rho}\|_{\text{Tr}} + \|\sigma - \tilde{\sigma}\|_{\text{Tr}} \leq 20\varepsilon^{1/4}$. Also, due to Lemma 8, we know that for all $(\tilde{\rho}, \tilde{\sigma}) \in B$, $\text{Tr}[\Psi_x(\tilde{\rho})\Psi_x(\tilde{\sigma})] \leq \frac{1}{2}$. As such, Lemma 5 gives that for all $\eta > 0$, there exists $p \in (0, 1)$ such that $\text{Tr}[\Phi_p(\rho)\Phi_p(\sigma)] \leq \frac{p^2}{2} + \frac{(1-p)^2}{2} + \eta$ for all $(\rho, \sigma) \in A$. As this bound holds for all pure states, it holds for all states. This gives that $\text{qCliques}(2)_{p^2/2+19(1-p)^2/32, p^2/2+(1-p)^2/2+\eta}$ is $\text{QMA}(2)_{c,s}$ -hard. Finally, as remarked near Lemma 5, η may be chosen so that $p^2/2 + 19(1-p)^2/32$ and $p^2/2 + (1-p)^2/2 + \eta$ have constant gap $3(1-p)^2/32 - \eta$, as the coefficient from Lemma 7 is constant in the case $k = 2$. $\square$

4.2. Reducing $\text{QMA}(k)$ to $\text{QMA}(2)$ Using Quantum Cliques

In this section, we give an alternate proof of $\text{QMA}(k) = \text{QMA}(2)$ (for k polynomial) to the original proof of Harrow and Montanaro [37]. To do this, we show that $\text{qCliques}(2)_{c,s}$ with some polynomial gap is hard for $\text{QMA}(k)$, independently of the result of [37]. The proof is similar to the case $k = 2$ that we showed in Theorem 6. This implies the wanted equality via Lemma 4, which showed that $\text{qCliques}(2)_{c,s} \in \text{QMA}(2)$ for all c, s with polynomial gap.

Theorem 7. For all $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap and $k : \mathbb{N} \rightarrow \mathbb{N}$ polynomial, there exist $c', s' : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap such that $\text{qCliques}(2)_{c', s'}$ is $\text{QMA}(k)_{c, s}$ -hard.

Proof. The proof proceeds analogously to 6. Let $L = (Y, N) \in \text{QMA}(k)_{c, s}$. By increasing the number of proofs, it is possible to amplify the probability gap. In particular, there exists a polynomial $k' : \mathbb{N} \rightarrow \mathbb{N}$ such that $L \in \text{QMA}(k')_{\frac{3}{4}, \frac{1}{4}}$. Let $x \in Y \cup N$ be an instance of L . Now, let $\Phi_x : \mathcal{B}(Q^{\otimes k'p(|x|)}) \rightarrow \mathcal{B}(Q)$ be the channel of the $\text{QMA}(k')_{\frac{3}{4}, \frac{1}{4}}$ verifier for L , from which we can follow the construction of Lemma 8 to get $\Psi_x : \mathcal{B}(Q^{\otimes k'p(|x|)+1}) \rightarrow \mathcal{B}(Q_\perp)$. Also, let $H = Q^{\otimes p(|x|)}$, $K = Q$, and $\Phi_1 : \mathcal{B}(H^{\otimes k'} \otimes K) \rightarrow \mathcal{B}(H \otimes \mathbb{C}^{[k']})$ be the channel from Lemma 7 with $k = k'$. Let $\Phi_p = p\Phi_1 \oplus (1-p)\Psi_x$; we consider the cliques of this channel. Note that, using the construction of Definition 10, it is direct to see that this channel admits a circuit representation polynomial in the size of the instance x .

First, suppose $x \in Y$. Then, there exist some separable state $\rho_0 = |\psi_1\rangle\langle\psi_1| \otimes \cdots \otimes |\psi_k\rangle\langle\psi_k|$ such that $\langle 1 | \Phi_x(\rho_0) | 1 \rangle \geq \frac{3}{4}$. As such, we have that with $\rho = \rho_0 \otimes |0\rangle\langle 0|$ and $\sigma = \rho_0 \otimes |1\rangle\langle 1|$, $\text{Tr}[\Phi_p(\rho)\Phi_p(\sigma)] \geq \frac{p^2}{k'} + (1-p)^2 \frac{19}{32}$, giving that (ρ, σ) forms a $(\frac{p^2}{k'} + (1-p)^2 \frac{19}{32}, 2)$ -clique. On the other hand, if $x \in N$, then, we

know that for all separable $\rho_0 \in (Q^{\otimes p(|x|)})^{\otimes k'}$, $\langle 1 | \Phi_x(\rho_0) | 1 \rangle \leq \frac{1}{4}$. As such, we know by Lemma 5 that for all $\eta > 0$, there exists p such that Φ_p has no $\left(\frac{p^2}{k'} + \frac{(1-p)^2}{2} + \eta, 2\right)$ -clique, as in Theorem 6. This gives a probability gap $(1-p)^2 \frac{3}{32} - \eta$, which can be chosen to be inverse polynomial, as the coefficient from Lemma 7, $10k'$, is polynomial. $\square$

Corollary 1. *For all $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap and $k : \mathbb{N} \rightarrow \mathbb{N}$ polynomial, there exist $c', s' : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap such that $\text{QMA}(k)_{c,s} \subseteq \text{QMA}(2)_{c',s'}$.*

The corollary follows immediately by concatenating the results of Theorem 7 and Lemma 4.

4.3. What about Quantum Independent Sets?

The quantum independent set problem, defined in Definitions 6 and 7, seems very similar to the quantum clique problem. However, its exact complexity remains vexing. Nevertheless, we are able to show that, as for the clique problem, the independent set problem is contained in $\text{QMA}(2)$.

Lemma 9. *Let $c, s : \mathbb{N} \rightarrow [0, 1]$ and $k : \mathbb{N} \rightarrow \mathbb{N}$. Then, $\text{qIS}(k)_{c,s} \in \text{QMA}(k)_{c',s'}$ for $c' = \frac{1+\frac{c-s}{2}c}{2+c-s}$ and $s' = \frac{1+\frac{c-s}{2}c+s}{2+c-s}$.*

This implies immediately that for k polynomial and $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap $\text{qIS}(k)_{c,s} \in \text{QMA}(2)$, since the gap $c' - s' = \frac{(c-s)^2}{4(2+c-s)} \geq \frac{(c-s)^2}{16}$ remains polynomial. The proof proceeds in the same way as the proof of Lemma 4.

Proof. Suppose the verifier receives an instance $C \in Y \cup N$ and a $\text{QMA}(k)$ proof $\rho_1 \otimes \dots \otimes \rho_k$. Let $\Phi = \Phi_C$. Then, the verifier effects the following procedure.

1. The verifier samples a uniformly random pair of distinct $i, j \in [k]$ and a random bit b , which is 0 with probability p .
2. If $b = 0$, the verifier runs the swap test on $\rho_i \otimes \rho_j$ and outputs 1 if and only if it fails.
3. If $b = 1$, the verifier computes $\Phi(\rho_i) \otimes \Phi(\rho_j)$, and again runs the swap test, outputting 1 if it fails.

Suppose $C \in Y$, then there exists a $(c(|C|), k)$ -independent $\rho_1, \dots, \rho_k$ set of Φ . So, the prover can send $\rho_1 \otimes \dots \otimes \rho_k$. If $b = 0$, the probability of accepting is $\mathbb{E}_{i,j} \frac{1}{2} - \frac{1}{2} \text{Tr}(\rho_i \rho_j) = \frac{1}{2}$; if $b = 1$, the probability of accepting is $\mathbb{E}_{i,j} \frac{1}{2} - \frac{1}{2} \text{Tr}(\rho_i \rho_j) \geq \frac{1}{2} - \frac{1-c(|C|)}{2} = \frac{1}{2}c(|C|)$. Thus, the probability of accepting is $c'(|C|) = \frac{p+(1-p)c(|C|)}{2}$.

Now, suppose $C \in N$. We may suppose that each $\rho_i = |\psi_i\rangle\langle\psi_i|$ is pure. If $b = 0$, then the probability of accepting is $\frac{1}{2} - \frac{1}{2} \mathbb{E}_{i \neq j} |\langle\psi_i|\psi_j\rangle|^2$; if $b = 1$, the probability of accepting is $\frac{1}{2} - \frac{1}{2} \mathbb{E}_{i \neq j} \text{Tr}[\Phi(\rho_i)\Phi(\rho_j)]$. Let Ψ be the matrix whose columns are $|\psi_i\rangle$, and let $\Psi = U\Sigma V^\dagger$ be its singular value decomposition. The matrix with orthonormal columns Ψ' nearest to Ψ in Frobenius norm is $\Psi' = UV^\dagger$ —this is the solution to the orthogonal Procrustes problem [56], see also Lemma A5 for more. The norm distance between these matrices is

$$\|\Psi - \Psi'\|_2^2 = \|\Sigma - \mathbb{I}\|_2^2 = \sum_i (\sigma_i - 1)^2 \leq \sum_i (\sigma_i^2 - 1)^2 = \|\Psi^\dagger \Psi - \mathbb{I}\|_2^2 = \sum_{i \neq j} |\langle\psi_i|\psi_j\rangle|^2. \quad (35)$$

Let $|\psi'_i\rangle$ be the columns of Ψ' . Then, as these vectors are orthonormal, the fact that there is no independent set implies $\mathbb{E}_{i \neq j} \text{Tr}[\Phi(|\psi'_i\rangle\langle\psi'_i|)\Phi(|\psi'_j\rangle\langle\psi'_j|)] \geq 1 - s(|C|)$. As such, we can upper bound the probability of accepting if $b = 1$ by

$$\begin{aligned} \frac{1}{2} - \frac{1}{2} \mathbb{E}_{i \neq j} \text{Tr}[\Phi(\rho_i)\Phi(\rho_j)] &\leq \frac{1}{2} - \frac{1}{2} \mathbb{E}_{i \neq j} \text{Tr}[\Phi(|\psi'_i\rangle\langle\psi'_i|)\Phi(|\psi'_j\rangle\langle\psi'_j|)] + \mathbb{E}_i \| |\psi_i\rangle\langle\psi_i| - |\psi'_i\rangle\langle\psi'_i| \|_{\text{Tr}} \\ &\leq \frac{1}{2} - \frac{1-s(|C|)}{2} + \mathbb{E}_i \| |\psi_i\rangle - |\psi'_i\rangle \| \\ &\leq \frac{s(|C|)}{2} + \sqrt{\mathbb{E}_i \| |\psi_i\rangle - |\psi'_i\rangle \|^2} = \frac{s(|C|)}{2} + \|\Psi - \Psi'\|_2, \end{aligned} \quad (36)$$

using Jensen's inequality. Thus, putting the two cases together, the probability of accepting is

$$\begin{aligned}
s'(|C|) &\leq p \left(\frac{1}{2} - \frac{1}{2} \|\Psi - \Psi'\|_2^2 \right) + (1-p) \left(\frac{s(|C|)}{2} + \|\Psi - \Psi'\|_2 \right) \\
&= \frac{p + (1-p)s(|C|)}{2} + (1-p) \|\Psi - \Psi'\|_2 - \frac{p}{2} \|\Psi - \Psi'\|_2^2 \\
&\leq \frac{p + (1-p)s(|C|)}{2} + \frac{(1-p)^2}{2p},
\end{aligned} \tag{37}$$

by maximizing over $\|\Psi - \Psi'\|_2 \in \mathbb{R}$. Taking $p = \frac{2}{2+c-s}$ gives the wanted values of c' and s' . $\square$

5. Specifying to QMA

Many NP-complete computational problems quantize naturally to QMA-complete. We have seen, however, that the clique problem quantizes rather to a QMA(2)-complete problem. This situation begs the question of whether there is some subclass of circuits on which the clique problem is sufficiently weakened to be QMA-complete. In this section we show that these circuits are those representing (a large subclass of) the entanglement-breaking channels, consisting of a measurement followed by state preparation. This contrasts well with the result of the previous section, wherein we showed that the clique problem for direct sums of the entanglement-breaking channels with partial traces end up being QMA(2)-complete. First, we recall the definition of the class QMA.

Definition 12. Let $c, s : \mathbb{N} \rightarrow (0, 1)$. A promise problem $Y, N \subseteq \{0, 1\}^*$ is in $\text{QMA}_{c,s}$ if there exist polynomials $p, q : \mathbb{N} \rightarrow \mathbb{N}$ and a Turing machine V with one input tape and one output tape such that

- For all $x \in \{0, 1\}^*$, V halts on input x in $q(|x|)$ steps and outputs the description of a quantum circuit from $p(|x|)$ qubits to one qubit.
- For all $x \in Y$, there exists a state $\rho \in \mathcal{D}(Q^{\otimes p(|x|)})$ such that $\langle 1 | \Phi_{V(x)}(\rho) | 1 \rangle \geq c(|x|)$.
- For all $x \in N$ and $\rho \in \mathcal{D}(Q^{\otimes p(|x|)})$, $\langle 1 | \Phi_{V(x)}(\rho) | 1 \rangle \leq s(|x|)$.

As long as $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap, we have that $\text{QMA}_{c,s} = \text{QMA}_{\frac{2}{3}, \frac{1}{3}}$, so as previously we take this to be the class QMA.

Definition 13. Write $\mathcal{C}_{\text{EB}'} \subseteq \mathcal{C}_{\text{EB}}$ for those circuits whose output is independent of at least one of the input qubits, i.e., there exists a qubit R such that $\Phi_C(\rho) = \Phi_C(\text{Tr}_R(\rho) \otimes \sigma_R)$ for some $\sigma \in \mathcal{D}(Q)$.

We aim to first show that $\text{qClique}(2, \mathcal{C}_{\text{EB}'}) \in \text{QMA}$. Note that we are only showing inclusion of a subclass of the entanglement-breaking channels in QMA. Why not all of them? For general channels, our proof of inclusion of the clique problem in QMA(2) has the verifier run the swap test to verify orthogonality of the clique. However, in QMA, the verifier does not have this power—in fact, Harrow and Montanaro show that there is no LOCC swap test [37]. As such, we need some other way that the verifier can guarantee that the provided clique is orthogonal. Leaving the result independent of one of the qubits allows the orthogonality to be provided by that qubit. In fact, the prover need not even provide orthogonal states to the verifier for a yes instance: the verifier checks on those qubits that are measured, and for the remaining one, she may be certain that there are extensions of the given states to that qubit that are both orthogonal and a clique.

We use a result of Brandão [43] to prove the inclusion. First, we need to define a useful subclass of QMA(2). A two-answer POVM $\{M, \mathbb{I} - M\}$ is a Bell measurement if there exist some $m, n \in \mathbb{N}$, $p_{ij} \in [0, 1]$, and POVMs $\{M_i\}_{i=1}^m, \{N_j\}_{j=1}^n$ such that

$$M = \sum_{i=1}^m \sum_{j=1}^n p_{ij} M_i \otimes N_j. \tag{38}$$

Let $\text{QMA}^{\text{Bell}}(2) \subseteq \text{QMA}(2)$ be the subset of problems where the verifier may make a Bell measurement in order to decide whether to accept or reject.

Theorem 8 (Special case of Theorem 6.5.2 in [43]). $\text{QMA}^{\text{Bell}}(2) = \text{QMA}$.

Now, we use this to show the inclusion.

Proposition 2. *Let $c, s : \mathbb{N} \rightarrow [0, 1]$. Then, $\text{qClique}(2, \mathcal{C}_{\text{EB}'})_{c,s} \subseteq \text{QMA}^{\text{Bell}}(2)^{\frac{1+c}{2}, \frac{1+s}{2}}$.*

Using the result of Brandão, we get as a corollary that $\text{qClique}(2, \mathcal{C}_{\text{EB}'})_{c,s} \in \text{QMA}$ when $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap.

Proof. Note that if Φ is a channel whose output is independent of one of its qubits Q_i , then if there exists a pair of states ρ, σ such that $\text{Tr}[\Phi(\rho)\Phi(\sigma)] \geq \alpha$, then there exists a pair of orthogonal states $\rho' = \text{Tr}_{Q_i}(\rho) \otimes |0\rangle\langle 0|$, $\sigma' = \text{Tr}_{Q_i}(\sigma) \otimes |1\rangle\langle 1|$ such that $\text{Tr}[\Phi(\rho')\Phi(\sigma')] = \text{Tr}[\Phi(\rho)\Phi(\sigma)] \geq \alpha$. As such, the verifier need not verify that the two states in the clique are orthogonal. Thus, given some circuit C and a purported proof $\rho \otimes \sigma$, the verifier simply computes $\Phi(\rho) \otimes \Phi(\sigma)$, performs the swap test, and outputs 1 if and only if the swap test accepts. If C is a yes instance, then the prover can provide a proof $\rho \otimes \sigma$ such that $\text{Tr}[\Phi(\rho)\Phi(\sigma)] \geq c(|C|)$, in which case the prover accepts with probability $\frac{1+c(|C|)}{2}$. Otherwise, if C is a no instance, then for all states ρ, σ , $\text{Tr}[\Phi(\rho)\Phi(\sigma)] \leq s(|C|)$, by the remark at the beginning of the proof. As such, the swap test accepts with probability $\frac{1+s(|C|)}{2}$.

It remains to show that the measurement that the verifier makes is a Bell measurement. The measurement operator for outcome 1 is $M = (\Phi^\dagger \otimes \Phi^\dagger)(\Pi_{\text{sym}})$ where Π_{sym} is the projector onto the symmetric subspace. Writing $\Phi(\rho) = \sum_i \text{Tr}[M_i \rho] \sigma_i$, as it is entanglement-breaking, we have that $\Phi^\dagger(A) = \sum_i \text{Tr}[A \sigma_i] M_i$, so

$$M = \sum_{i,j} \text{Tr}[\Pi_{\text{sym}}(\sigma_i \otimes \sigma_j)] M_i \otimes M_j, \quad (39)$$

which is manifestly a Bell measurement operator. $\square$

Theorem 9. *There exist $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap such that $\text{qClique}(2, \mathcal{C}_{\text{EB}'})_{c,s}$ is QMA-hard.*

Proof. We show this by reducing the local Hamiltonian problem to the clique problem. Let $H = \sum_{i=1}^t H_i \in \mathcal{B}(\mathcal{Q}^{\otimes n})$ be an instance of the k -local Hamiltonian problem with completeness α and soundness β . By construction [12], we may assume that the Hamiltonian is normalized in the sense that $0 \leq H_i \leq \mathbb{I}$. By gap amplification, we may further assume that $\beta > 4\alpha$ with polynomial gap. Let $c = 1 - 2\alpha/t$ and $s = 1 - \beta/(2t)$. This gives c and s with polynomial gap.

We construct the channel $\Phi : \mathcal{B}(\mathcal{Q}^{\otimes n+1}) \rightarrow \mathcal{B}(\mathcal{Q}_\perp)$ as

$$\begin{aligned} \Phi(\rho) &= \text{Tr}[(H/t \otimes \mathbb{I})\rho] \mu_Q + \text{Tr}[(\mathbb{I} - H/t) \otimes \mathbb{I}]\rho | \perp \rangle \langle \perp | \\ &= \frac{1}{t} \sum_{i=1}^t \text{Tr}[(H_i \otimes \mathbb{I})\rho] \mu_Q + \text{Tr}[(\mathbb{I} - H_i) \otimes \mathbb{I}]\rho | \perp \rangle \langle \perp |. \end{aligned} \quad (40)$$

Since this channel consists of randomly sampling $i \in \{1, \dots, t\}$ uniformly at random, measuring according to the POVM $\{H_i, \mathbb{I} - H_i\}$, and then preparing a state of fixed size, it can be described by a circuit C_H of size polynomial in the size of the description of H .

Now, suppose H is a yes instance. Then, there exists a state $\langle \psi | H | \psi \rangle \leq \alpha$. So, taking $\rho = |\psi\rangle\langle \psi| \otimes |0\rangle\langle 0|$, $\sigma = |\psi\rangle\langle \psi| \otimes |1\rangle\langle 1|$, we get that

$$\begin{aligned} \text{Tr}[\Phi(\rho)\Phi(\sigma)] &= \frac{1}{2} (\langle \psi | H | \psi \rangle / t)^2 + (1 - \langle \psi | H | \psi \rangle / t)^2 \\ &\geq 1 - 2\langle \psi | H | \psi \rangle / t \geq 1 - 2\alpha/t = c. \end{aligned} \quad (41)$$

On the other hand, if H is a no instance, then for all states $\rho \in \mathcal{B}(\mathcal{Q}^{\otimes n})$, $\text{Tr}[H\rho] \geq \beta$. Then, for any state $\rho \in \mathcal{B}(\mathcal{Q}^{\otimes n+1})$, we have that $\text{Tr}[(H \otimes \mathbb{I})\rho] \geq \beta$, giving

$$\begin{aligned} \text{Tr}[\Phi(\rho)^2] &= \frac{1}{2} (\text{Tr}[(H \otimes \mathbb{I})\rho] / t)^2 + (1 - \text{Tr}[(H \otimes \mathbb{I})\rho] / t)^2 \\ &\leq 1 - \frac{1}{2} \text{Tr}[(H \otimes \mathbb{I})\rho] / t \leq 1 - \beta/2t = s. \end{aligned} \quad (42)$$

Thus, for any pair of orthogonal states ρ, σ , the triangle inequality gives that $\text{Tr}[\Phi(\rho)\Phi(\sigma)] \leq \sqrt{\text{Tr}[\Phi(\rho)^2] \text{Tr}[\Phi(\sigma)^2]} \leq s$. $\square$

We finish this section by noting what happens to the quantum 2-independent set problem on this reduced set of circuits. It turns out it is also a QMA-complete problem in an almost identical way to the clique problem, so we only provide a sketch of this here. It is an easy exercise to verify that Proposition 2 may be slightly modified to show that $\text{qIS}(2, \mathcal{C}_{\text{EB}'}^s)_{c,s} \subseteq \text{QMA}^{\text{Bell}}(2)_{\frac{\epsilon}{2}, \frac{\epsilon}{2}}$, and hence the quantum 2-independent set problem for this subset of entanglement-breaking channels is in QMA. On the other hand, by a construction similar to Theorem 9 and to a construction in Theorem 3.1 of [15], we also get that the problem is QMA-hard. As such, this leads to the following theorem.

Theorem 10. *There exist $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap such that $\text{qIS}(2, \mathcal{C}_{\text{EB}'}^s)_{c,s}$ is QMA-complete.*

6. Classical Collision and Clique Problems

In this section, we formally introduce the classical versions of the clique and collision problems and study their computational complexity.

6.1. The Deterministic Case

The *collision problem* for functions is of fundamental interest to complexity theory and cryptography. It asks whether, given some function $f : X \rightarrow Y$, is there a pair of elements in the domain $x, y \in X$ with the same image $f(x) = f(y)$ —a collision.

Collisions also have a natural interpretation in the language of channel capacity. The function f can be seen as a deterministic (i.e., not noisy) channel from the input set X to the output set Y . As in the case for a noisy channel, this induces a *confusability graph* on the set X : x and y are connected by an edge if and only if $f(x) = f(y)$. That is, the confusability graph indicates the collisions of the function. The graphs that this procedure induces are, in themselves, not particularly interesting—they are simply disjoint unions of complete graphs on the subsets of the input set that collide—but they allow to interpret collisions in a graph-theoretic way, as cliques. With this in mind, we define the following graph-theoretic notions for functions.

Definition 14. *Let X, Y be sets and $k \in \mathbb{N}$.*

- *A function $f : X \rightarrow Y$ has a k -clique (or k -collision) if there exist distinct $x_1, \dots, x_k \in X$ such that $f(x_1) = \dots = f(x_k)$.*
- *A function $f : X \rightarrow Y$ has a k -independent set if there exist distinct $x_1, \dots, x_k \in X$ such that $f(x_i) \neq f(x_j)$ for all $i \neq j$.*

To study the complexity of finding cliques and independent sets, we fix a presentation for the functions we consider. The most basic way to present a function is as a table of values. Though this is the most general, it is highly inefficient: a function that may be easy to compute is described with the same complexity as a function that is more difficult, and the length of the function description is exponential in the size of the input set. Two natural and equivalent options from the point of view of computational complexity are to present a function as a Turing machine or as a circuit diagram. We will use the circuit description, as this will generalize more naturally to quantum channels, represented as quantum circuits. With this choice of convention, we can formally present the clique and independent set problems in the deterministic case.

Definition 15. *Let $k \in \mathbb{N}$. The language for the k -clique problem is*

$$\text{Clique}(k) = \left\{ \text{circuits } C \mid f_C : \{0, 1\}^{\text{in}(C)} \rightarrow \{0, 1\}^{\text{out}(C)} \text{ has a } k\text{-clique} \right\}. \quad (43)$$

The language for the k -independent set problem is

$$\text{IS}(k) = \left\{ \text{circuits } C \mid f_C : \{0, 1\}^{\text{in}(C)} \rightarrow \{0, 1\}^{\text{out}(C)} \text{ has a } k\text{-independent set} \right\}. \quad (44)$$

It is well known that the collision problem for general functions is NP-complete; we will give simple constructions that provide this completeness in the context of our graph-theoretic language. First, we recall the definition of NP.

Definition 16. A language $L \subseteq \{0,1\}^*$ is in NP if there exist polynomials $p, q : \mathbb{N} \rightarrow \mathbb{N}$ and a Turing machine V with two input tapes and one output tape such that

- For all $x, y \in \{0,1\}^*$, V halts on input (x, y) in $q(|x|)$ steps.
- For all $x \in L$, there exists $y \in \{0,1\}^{p(|x|)}$ such that $V(x, y) = 1$.
- For all $x \notin L$ and for all $y \in \{0,1\}^*$, $V(x, y) = 0$.

To finish this section, we give proofs of the NP-completeness of both the 2-clique and the 2-independent set problems. Note that the proofs of inclusion of the 2-clique and 2-independent set problems in NP generalize directly to the case of k -cliques and k -independent sets (for k polynomial in the instance size), showing that these *a priori* harder problems are also NP-complete. In Appendix A, we also provide direct reductions to the 2-clique and independent set problems.

Proposition 3. *Clique(2) is NP-complete.*

Proof. First, it is direct to see that the problem $\text{Clique}(2) \in \text{NP}$. In fact, given a circuit C , the verifier expects to receive a proof of the form (x, y) for $x, y \in \{0,1\}^{\text{in}(C)}$, computes $f_C(x)$ and $f_C(y)$, and accepts if and only if they are equal. If C is a yes instance, then the prover can simply provide a collision, causing the verifier to accept; otherwise, there is no clique for the prover to provide, so the verifier never accepts.

Next, we need to see that $\text{Clique}(2)$ is NP-hard. Let $L \in \text{NP}$, let $V : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$ be its verifier Turing machine, q be the halting time, and p be the proof length. Then, for each $x \in \{0,1\}^*$, let $f_x : \{0,1\}^{p(|x|)} \times \{0,1\} \rightarrow \{0,1\}^{p(|x|)} \times \{0,1\}$ be the function $f_x(y, b) = (y, b \vee V(x, y))$. Since $V(x, y)$ halts in $q(|x|)$ steps, f_x is computable in time polynomial in $|x|$, and hence has a circuit description C_x that is polynomial size in $|x|$. This provides an instance of the collision problem. Now, if $x \in L$, there exists y such that $V(x, y) = 1$, so $f_x(y, 0) = (y, 1) = f_x(y, 1)$, so $C_x \in \text{Clique}(2)$; on the other hand, if $x \notin L$, then for all y , $V(x, y) = 0$, giving $f_x(y, b) = (y, b)$ is injective, so $C_x \notin \text{Clique}(2)$. As such, we have the wanted Karp reduction. $\square$

Proposition 4. *IS(2) is NP-complete.*

Proof. First, we need that $\text{IS}(2) \in \text{NP}$. As for the clique problem, the proof for a circuit C can be taken to be the description of the independent set (x, y) . In that case, the verifier simply needs to compute $f_C(x)$ and $f_C(y)$ and check that they are different, something that it can do efficiently in the length of the circuit.

Conversely, let $L \in \text{NP}$ be some language and let V be a verifier for L . Now for $x \in \{0,1\}^*$, consider the function $f_x : \{0,1\}^{p(|x|)} \rightarrow \{0,1\}^2$ defined as $f_x(y) = (y_1 \wedge V(x, y), \neg y_1 \wedge V(x, y))$, where y_1 is the first bit of y . If $x \notin L$, then $V(x, y) = 0$ for all $y \in \{0,1\}^{p(|x|)}$, so $f_x(y) = (0, 0)$ giving that there is no independent set. On the other hand, if there is y such that $V(x, y) = 1$, $f_x(y) = (y_1, \neg y_1) \neq (0, 0)$. Then, for any $z \in \{0,1\}^{p(|x|)}$ such that $z_1 \neq y_1$, $f_x(y) \neq f_x(z)$. So, f_x has a 2-independent set if and only if $x \in L$. Finally, it is direct to see that, since the circuit of $y \mapsto V(x, y)$ is efficiently describable in $|x|$, the circuit of f_x is also. $\square$

6.2. The Probabilistic Case

The results of the previous section generalize easily to probabilistic functions. This is the setting where cliques and independent sets of functions were originally considered, since probabilistic functions model noisy channels [1]. First, we formally introduce probabilistic functions.

Definition 17. A probabilistic function $f : X \rightarrow Y$ is a function that to each $x \in X$ associates a random variable $f(x)$ on Y .

Intuitively, cliques of a noisy channel describe inputs that are likely to become confused, and independent sets describe inputs that are unlikely to. We can think of this in the context of zero-error capacity, wherein cliques are those inputs that are always confused, and independent sets are those that are never confused. However, from a complexity-theoretic point of view, this is too restrictive, since we are looking to extract a property of a probabilistic object that should hold exactly. To remedy this, we define cliques/independent sets with respect to an additional parameter that describes the probability of confusion.

Definition 18. Let X, Y be sets, $k \in \mathbb{N}$, and $\alpha \in [0, 1]$.

- A probabilistic function $f : X \rightarrow Y$ has an (α, k) -clique if there exist distinct $x_1, \dots, x_k \in X$ such that for all $i \neq j$,

$$\frac{2}{k(k-1)} \sum_{1 \leq i < j \leq k} \Pr[f(x_i) = f(x_j)] \geq \alpha, \quad (45)$$

where the probability is with respect to independent evaluations of the function, that is $\Pr[f(x_i) = f(x_j)] = \sum_{y \in Y} \Pr[y = f(x_i)] \Pr[y = f(x_j)]$.

- A probabilistic function $f : X \rightarrow Y$ has an (α, k) -independent set if there exist distinct $x_1, \dots, x_k \in X$ such that for all $i \neq j$,

$$\frac{2}{k(k-1)} \sum_{1 \leq i < j \leq k} \Pr[f(x_i) = f(x_j)] \leq 1 - \alpha. \quad (46)$$

Similarly to the deterministic case, we describe functions by circuits. Here, however, to allow probabilistic functions, we will consider probabilistic circuits, where there are some bits of random input.

Definition 19.

- A probabilistic circuit C^m with n inputs is a circuit C on $n + m$ inputs such that on each evaluation of the circuit, the final m wires are given uniformly random bits as input.
- A probabilistic Turing machine M^p with k input tapes is a Turing machine M with $k + 1$ input tapes along with a function $p : \mathbb{N} \rightarrow \mathbb{N}$. On input x , $M^p(x) = M(x, r)$ where $r \in \{0, 1\}^{p(|x|)}$ is sampled uniformly at random.

Next, we can introduce the complexity problem for probabilistic cliques and independent sets. Rather than a language, we present it as a promise problem, which provides a probability gap between the yes and no instances.

Definition 20. Let $k \in \mathbb{N}$ and let $c, s : \mathbb{N} \rightarrow (0, 1)$. The probabilistic k -clique problem with completeness c and soundness s is the promise problem $\text{pClique}(k)_{c,s} = (Y, N)$ with

$$\begin{aligned} Y &= \{\text{probabilistic circuits } C^m \mid f_{C^m} \text{ has a } (c(|C^m|), k)\text{-clique}\}, \\ N &= \{\text{probabilistic circuits } C^m \mid f_{C^m} \text{ has no } (s(|C^m|), k)\text{-clique}\}. \end{aligned} \quad (47)$$

The probabilistic k -independent set problem with completeness c and soundness s is the promise problem $\text{pIS}(k)_{c,s} = (Y, N)$ with

$$\begin{aligned} Y &= \{\text{probabilistic circuits } C^m \mid f_{C^m} \text{ has a } (c(|C^m|), k)\text{-independent set}\}, \\ N &= \{\text{probabilistic circuits } C^m \mid f_{C^m} \text{ has no } (s(|C^m|), k)\text{-independent set}\}. \end{aligned} \quad (48)$$

Analogously to the deterministic case, the probabilistic clique and independent set problems are complete for the probabilistic analogue of NP, which is MA. First, we provide a definition of this class.

Definition 21. Let $c, s : \mathbb{N} \rightarrow (0, 1)$. A promise problem $Y, N \subseteq \{0, 1\}^*$ is in $\text{MA}_{c,s}$ if there exist polynomials $p, q, l : \mathbb{N} \rightarrow \mathbb{N}$ and a probabilistic Turing machine V^l with two input tapes and one output tape such that

- For all $x, y \in \{0, 1\}^*$, V^l halts on input (x, y) in $q(|x|)$ steps.
- For all $x \in Y$, there exists $y \in \{0, 1\}^{p(|x|)}$ such that $\Pr[V^l(x, y) = 1] \geq c(|x|)$.
- For all $x \in N$ and for all $y \in \{0, 1\}^*$, $\Pr[V^l(x, y) = 1] \leq s(|x|)$.

Note that, as long as there exists $N \in \mathbb{N}$, a polynomial $P : \mathbb{N} \rightarrow \mathbb{N}$, and a function $E : \mathbb{N} \rightarrow \mathbb{N}$ no larger than exponential such that $c(n) - s(n) \geq \frac{1}{P(n)}$ and $s(n), 1 - c(n) \geq \frac{1}{E(n)}$ for all $n \geq N$, $\text{MA}_{c,s} = \text{MA}_{\frac{2}{3}, \frac{1}{3}}$, so we write this class simply MA.

Next, we show completeness of the 2-clique and independent set problems. As in the deterministic case, we refer to Appendix A for the reduction from k -cliques and independent sets to this case.

Proposition 5. *There exist $c, s : \mathbb{N} \rightarrow (0, 1)$ with constant gap such that $\text{pClique}(2)_{c,s}$ is MA-complete.*

Proof. First, we see that $\text{pClique}(2)_{c,s} \in \text{MA}_{c,s}$ for all c, s . An instance is simply a probabilistic circuit C^m , and the verifier expects a proof that is a $(c, 2)$ -clique (x, y) . Then, the verifier computes samples of $f_{C^m}(x)$ and $f_{C^m}(y)$, and accepts if and only if they are equal. Then, if $C^m \in Y$, $\Pr[f_{C^m}(x) = f_{C^m}(y)] \geq c(|C^m|)$, so $\Pr[V^l(C^m, (x, y)) = 1] \geq c(|C^m|)$. Conversely, if $C^m \in N$, $\Pr[f_{C^m}(x) = f_{C^m}(y)] \leq s(|C^m|)$, so $\Pr[V^l(C^m, (x, y)) = 1] \leq s(|C^m|)$, completing the proof of inclusion.

Let $L = (Y, N) \in \text{MA}_{\frac{2}{3}, \frac{1}{3}}$. Then, there exists a probabilistic polynomial time Turing machine V^l such that for all $x \in Y$, there exists y such that $\Pr[V^l(x, y) = 1] \geq \frac{2}{3}$; and if $x \in N$, then for all y , $\Pr[V^l(x, y) = 1] \leq \frac{1}{3}$. For each x , let C_x^l be a probabilistic circuit describing the probabilistic function $f_{C_x^l}(y, b) = (y, V^l(x, y) \vee b)$. If $x \in Y$, then $(y, 0), (y, 1)$ forms a $(2/3, 2)$ -clique, as

$$\Pr[f_{C_x^l}(y, 0) = f_{C_x^l}(y, 1)] = \Pr[V^l(x, y) = 1] = \frac{2}{3}. \quad (49)$$

Conversely, if $x \in N$, then suppose there exists a $(s, 2)$ -clique $(y, b), (y', b')$ for $s \geq 1/3$. First, since the probability of acceptance is greater than 0, we must have $y = y'$. Also, since $(y, b) \neq (y, b')$, we can take without loss of generality $b = 0$ and $b' = 1$. Therefore,

$$\frac{1}{3} < s \leq \Pr[f_{C_x^l}(y, 0) = f_{C_x^l}(y, 1)] = \Pr[V^l(x, y) = 1], \quad (50)$$

which is a contradiction. This provides that $\text{qClique}_{\frac{2}{3}, \frac{1}{3}}(2)$ is MA-hard. $\square$

Proposition 6. *There exist $c, s : \mathbb{N} \rightarrow (0, 1)$ with constant gap such that $\text{pIS}_{c,s}(2)$ is a MA-complete.*

Proof. It is, as before, direct to see that $\text{pIS}_{c,s}(2) \in \text{MA}_{c,s}$ for all c, s with polynomial gap. Given a circuit C^m as instance and a proof (x, y) , the verifier simply computes samples of $f_{C^m}(x)$ and $f_{C^m}(y)$ and accepts iff they are not equal. If (x, y) is a $(c, 2)$ -independent set of f_{C^m} , then $\Pr[V^l(C^m, (x, y)) = 1] = \Pr[f_{C^m}(x) \neq f_{C^m}(y)] \geq c$; and conversely, if f_{C^m} has no $(s, 2)$ -independent set, $\Pr[V^l(C^m, (x, y)) = 1] \leq s$.

Now, let $L = (Y, N) \in \text{MA}_{\frac{2}{3}, \frac{1}{3}}$ with verification circuit V_l , and define the probabilistic circuit C_x^l with function $f_{C_x^l}(y) = (y_1 \wedge V^l(x, y), \neg y_1 \wedge V^l(x, y))$. If $x \in Y$, then there exists y such that $\Pr[V^l(x, y) = 1] \geq \frac{2}{3}$, giving that $\Pr[f_{C_x^l}(y) = (y_1, \neg y_1)] \geq \frac{2}{3}$. Taking any z such that $z_1 \neq y_1$, the possible values of $f_{C_x^l}(z)$ are $(0, 0)$ and $(z_1, \neg z_1)$, so $f_{C_x^l}(z) \neq (y_1, \neg y_1)$. As such, $\Pr[f_{C_x^l}(y) \neq f_{C_x^l}(z)] \geq \Pr[f_{C_x^l}(y) = (y_1, \neg y_1)] \Pr[f_{C_x^l}(z) \neq (y_1, \neg y_1)] \geq \frac{2}{3}$. On the other hand, if $x \in N$, for all y , $\Pr[V^l(x, y) = 1] \leq \frac{1}{3}$, so $\Pr[f_{C_x^l}(y) = (0, 0)] \geq \frac{2}{3}$. Thus, for all y, y' , $\Pr[f_{C_x^l}(y) = f_{C_x^l}(y')] \geq \Pr[f_{C_x^l}(y) = (0, 0)] \Pr[f_{C_x^l}(y') = (0, 0)] \geq \frac{4}{9}$. Thus, $\text{pIS}_{\frac{2}{3}, \frac{1}{9}}$ is MA-hard. $\square$

Author Contributions

E.C. and A.M. contributed to all work in this manuscript. All authors have read and agreed to the published version of the manuscript.

Funding

This research was funded by the Natural Sciences and Engineering Research Council of Canada (NSERC)(ALLRP-578455-2022).

Conflicts of Interest Statement

The authors declare no conflicts of interest.

Data Availability Statement

There is no supporting data required or used for these results.

Appendix A. Reductions from k -cliques to 2-cliques

In this appendix, we provide reductions for deterministic, probabilistic, and quantum k -clique and independent set problems to the corresponding 2-clique/independent set problem. Matching the completeness results, we are able

to show the reduction for both the clique and independent set problems, but for the quantum case, we show only the reduction in the case of cliques. For conciseness, we work at the level of channels—it is direct that the results below imply complexity-theoretic results at the level of circuits.

Lemma A1. *Let $f : X \rightarrow Y$ be a function. Define $g : X^k \times \mathbb{Z}_2 \rightarrow X^k \times \mathbb{Z}_2$ as*

$$g(x_1, \dots, x_k, b) = \begin{cases} (x_1, \dots, x_k, 1) & \text{if } f(x_1) = \dots = f(x_k) \text{ and } x_1, \dots, x_k \text{ all distinct} \\ (x_1, \dots, x_k, b) & \text{else} \end{cases}. \quad (\text{A1})$$

Then, g has a 2-clique if and only if f has a k -clique.

It is direct to note that the mapping from the circuit of f to the circuit of g is efficient, that is polynomial in k and the size of the description of f .

Proof. Suppose f has a k -clique. Then, there exist distinct $x_1, \dots, x_k \in X$ such that $f(x_1) = \dots = f(x_k)$. By definition of g , $g(x_1, \dots, x_k, 0) = (x_1, \dots, x_k, 1) = g(x_1, \dots, x_k, 1)$, so g has a 2-clique. Conversely, suppose f does not have a k -clique, i.e., the pre-image of every element in Y has cardinality at most $k - 1$. Then, g is identity—in particular injective—so it does not have a 2-clique. $\square$

Lemma A2. *Let $f : X \rightarrow Y$ be a function. Define $g : X^k \rightarrow \mathbb{Z}_2$ as*

$$g(x_1, \dots, x_k) = \begin{cases} 1 & \text{if } x_1, \dots, x_k \text{ all distinct and } f(x_1), \dots, f(x_k) \text{ all distinct as well} \\ 0 & \text{else.} \end{cases} \quad (\text{A2})$$

Then, g has a 2-independent set if and only if f has a 2-independent set.

As for the previous lemma, it is direct that the construction of g is efficient.

Proof. Suppose f has a k -independent set. Then, there exist $x_1, \dots, x_k \in X$ distinct such that $f(x_1), \dots, f(x_k)$ are all distinct. As such $g(x_1, \dots, x_k) = 1$. But since $g(x_1, \dots, x_1) = 0$, we have a 2-independent set $((x_1, \dots, x_k), (x_1, \dots, x_1))$. Conversely, suppose that f does not have a k -independent set. Then, for all $x_1, \dots, x_k \in X$ there exist i, j such that $f(x_i) = f(x_j)$. As such, we always have $g(x_1, \dots, x_k) = 0$, giving that there is no 2-independent set. $\square$

Next, we extend this to the probabilistic case.

Lemma A3. *Let $f : X \rightarrow Y$ be a probabilistic function, and let $S = \{(i, j) \mid 1 \leq i < j \leq k\}$. Define $g : X^k \times \mathbb{Z}_2 \rightarrow X^k \times \mathbb{Z}_2 \times S$ to be the probabilistic function where $g(x_1, \dots, x_k, b)$ is the random variable $(x_1, \dots, x_k, b \vee \delta_{f(x_i), f(x_j)}, b \vee d, (i, j))$ for $d = 1$ if and only if $x_1, \dots, x_k$ are all distinct and (i, j) sampled uniformly at random from S . Then, g has an $(2\alpha/k(k-1), 2)$ -clique if and only if f has a (α, k) -clique, with $\alpha > 0$.*

As in the deterministic case, this gives a reduction from the k -clique problem $\text{pClique}(k)_{c,s}$ to the 2-clique problem $\text{pClique}(2)_{2c/k(k-1), 2s/k(k-1)}$, which preserves the polynomial completeness-soundness gap if and only if k is polynomial in the circuit size. This however is a reasonable restriction on k , as a proof for a clique of size larger than polynomial cannot even be read in polynomial time.

Proof. Suppose f has an (α, k) -clique. Then, there exist distinct elements $x_1, \dots, x_k \in X$ such that $\frac{2}{k(k-1)} \sum_{(i,j) \in S} \Pr[f(x_i) = f(x_j)] \geq \alpha$. Then,

$$\begin{aligned} \Pr[g(x_1, \dots, x_k, 0) = g(x_1, \dots, x_k, 1)] &= \sum_{(i,j), (i',j') \in S} \Pr[(i, j) = (i', j') \wedge \delta_{f(x_i), f(x_j)} = 1] \\ &= \frac{2}{k(k-1)} \sum_{(i,j) \leftarrow S} \Pr[f(x_i) = f(x_j)] \geq \frac{2\alpha}{k(k-1)}. \end{aligned} \quad (\text{A3})$$

Conversely, suppose that g has a $(2\alpha/k(k-1), 2)$ -clique. Then, there exist $x_1, \dots, x_k, x'_1, \dots, x'_k \in X$ and $b, b' \in \mathbb{Z}_2$ such that $\Pr[g(x_1, \dots, x_k, b) = g(x'_1, \dots, x'_k, b')] \geq 2\alpha/k(k-1)$. If $x_i \neq x'_i$ for some i , then this probability is 0, which is a contradiction. On the other hand, if $b = b'$, then the two elements of the clique are not distinct. As such, the clique must be $(x_1, \dots, x_k, 0), (x_1, \dots, x_k, 1)$. Further, if $x_i = x_j$ for any $i \neq j$, then $d = 0$, so $g(x_1, \dots, x_k, 0) \neq g(x_1, \dots, x_k, 1)$, giving that the $x_1, \dots, x_k$ are all distinct. By the calculation from the previous direction, this gives that

$$\alpha \leq \frac{k(k-1)}{2} \Pr[g(x_1, \dots, x_k, b) = g(x'_1, \dots, x'_k, b')] = \frac{2}{k(k-1)} \sum_{(i,j) \in S} \Pr[f(x_i) = f(x_j)], \quad (\text{A4})$$

so $x_1, \dots, x_k$ is an (α, k) -clique of f . $\square$

Lemma A4. Let $k \geq 3$. Let $f : X \rightarrow Y$ be a probabilistic function, and let $S = \{(i, j) \mid 1 \leq i < j \leq k\}$. Define $g : X^k \rightarrow \mathbb{Z}_2 \times S$ to be the probabilistic function where $g(x_1, \dots, x_k)$ is the random variable $(d_{i,j}, (i, j))$ for $d = 1$ if and only if $x_1, \dots, x_k$ are all distinct and $f(x_i) \neq f(x_j)$ with (i, j) sampled uniformly at random from S . Then, g has an $(1 - 2(1 - \alpha)/k(k-1), 2)$ -independent set if and only if f has a (α, k) -independent set, with $\alpha > 0$.

Proof. Suppose f has an (α, k) -independent set. Then, there exist distinct $x_1, \dots, x_k \in X$ such that $\frac{2}{k(k-1)} \sum_{(i,j)} \Pr[f(x_i) = f(x_j)] \leq 1 - \alpha$. Then, I claim $(x_1, \dots, x_k)$ and $(x_1, \dots, x_1)$ form a $(2\alpha/k(k-1), 2)$ -independent set of g . In fact,

$$\Pr[g(x_1, \dots, x_k) = g(x_1, \dots, x_1)] = \frac{2}{k(k-1)} \sum_{(i,j) \in S} \Pr[f(x_i) = f(x_j)] \leq \frac{2}{k(k-1)} (1 - \alpha). \quad (\text{A5})$$

Conversely, suppose that g has a $(1 - 2(1 - \alpha)/k(k-1), 2)$ -independent set. Then, there exist $(x_1, \dots, x_k)$ and $(x'_1, \dots, x'_k)$ such that $\Pr[g(x_1, \dots, x_k) = g(x'_1, \dots, x'_k)] \leq \frac{2(1-\alpha)}{k(k-1)}$. First, this probability is $\Pr[g(x_1, \dots, x_k) = g(x'_1, \dots, x'_k)] = \Pr[g(x_1, \dots, x_k) = 0] \Pr[g(x'_1, \dots, x'_k) = 0] + (1 - \Pr[g(x_1, \dots, x_k) = 0]) (1 - \Pr[g(x'_1, \dots, x'_k) = 0])$. If $x_1, \dots, x_k$ are all distinct, we have

$$\Pr[g(x_1, \dots, x_k) = 0] = \frac{2}{k(k-1)} \sum_{(i,j) \in S} \Pr[f(x_i) = f(x_j)], \quad (\text{A6})$$

and otherwise $\Pr[g(x_1, \dots, x_k) = 0] = 1$. The same holds in the same way for $\Pr[g(x'_1, \dots, x'_k) = 0]$

Next, consider the function $p(x, y) = xy + (1-x)(1-y)$ on $x, y \in [0, 1]$. We have that, if $x \leq \frac{1}{2}$ or $y \leq \frac{1}{2}$, $p(x, y) \geq \min\{x, y\}$. Further, if $p(x, y) \leq \frac{1}{2}$, then either $x \leq \frac{1}{2}$ or $y \leq \frac{1}{2}$. The above implies that $p(\Pr[g(x_1, \dots, x_k) = 0], \Pr[g(x'_1, \dots, x'_k) = 0]) \leq \frac{2(1-\alpha)}{k(k-1)} \leq \frac{2}{k(k-1)} \leq \frac{1}{2}$, so we may without loss of generality assume $\Pr[g(x_1, \dots, x_k) = 0] \leq \frac{1}{2}, \Pr[g(x'_1, \dots, x'_k) = 0] \leq \frac{1}{2}$. This implies that $\sum_{(i,j) \in S} \Pr[f(x_i) = f(x_j)] \leq \frac{k(k-1)}{2} \Pr[g(x_1, \dots, x_k) = g(x'_1, \dots, x'_k)] \leq (1 - \alpha)$, giving that $(x_1, \dots, x_k)$ is an (α, k) -independent set of f . $\square$

Finally, we provide the reduction between the quantum clique problems. To do so, we make use of the techniques introduced in Section 4. First, we need to add an additional channel to our toolkit, that will allow us to guarantee orthogonality of a purported clique.

Lemma A5. Let H, K be Hilbert spaces and let $\Pi_H : H \otimes H \rightarrow H \otimes H$ be the projector onto the symmetric subspace. Define the set $S = \{(i, j) \mid 1 \leq i < j \leq k\}$ and the quantum channel $\Phi_2 : \mathcal{B}(H^{\otimes k} \otimes K) \rightarrow \mathcal{B}(Q' \otimes \mathbb{C}^S)$ as

$$\Phi_2(\rho) = \mathbb{E}_{(i,j) \in S} \left(\text{Tr}[\Pi_H \rho_{i,j}] \mu_Q + \text{Tr}[(\mathbb{I} - \Pi_H) \rho_{i,j}] |\perp\rangle\langle\perp| \right) \otimes |(i, j)\rangle\langle(i, j)|, \quad (\text{A7})$$

where $\rho_{i,j}$ is the marginal on the i -th and j -th subregisters. Then, for any pure separable states of the form $\rho = |\psi_1\rangle\langle\psi_1| \otimes \cdots \otimes |\psi_k\rangle\langle\psi_k| \otimes |\alpha\rangle\langle\alpha|$, $\sigma = |\psi_1\rangle\langle\psi_1| \otimes \cdots \otimes |\psi_k\rangle\langle\psi_k| \otimes |\beta\rangle\langle\beta| \in \mathcal{D}(H^{\otimes k} \otimes K)$, $\text{Tr}[\Phi_2(\rho)\Phi_2(\sigma)] \leq \frac{1}{2k(k-1)}$ and if $\text{Tr}[\Phi_2(\rho)\Phi_2(\sigma)] \geq \frac{1}{2k(k-1)} - \varepsilon$, there exist orthogonal states $|\psi'_1\rangle, \dots, |\psi'_k\rangle \in H$ such that

$$\|\rho - \tilde{\rho}\|_{\text{Tr}} + \|\sigma - \tilde{\sigma}\|_{\text{Tr}} \leq 4k\sqrt{k(k-1)}\sqrt{\varepsilon}, \quad (\text{A8})$$

where $\tilde{\rho} = |\psi'_1\rangle\langle\psi'_1| \otimes \cdots \otimes |\psi'_k\rangle\langle\psi'_k| \otimes |\alpha\rangle\langle\alpha|$ and $\tilde{\sigma} = |\psi'_1\rangle\langle\psi'_1| \otimes \cdots \otimes |\psi'_k\rangle\langle\psi'_k| \otimes |\beta\rangle\langle\beta|$.

Proof. First, it is direct to see that

$$\text{Tr}[\Phi_2(\rho)\Phi_2(\sigma)] = \frac{1}{2k(k-1)} \mathbb{E}_{(i,j)} \left[\frac{1}{2} \left| \langle\psi_i|\psi_j\rangle \right|^4 + \left(1 - \left| \langle\psi_i|\psi_j\rangle \right|^2 \right)^2 \right] \leq \frac{1}{2k(k-1)}. \quad (\text{A9})$$

Now, suppose $\text{Tr}[\Phi_2(\rho)\Phi_2(\sigma)] \geq \frac{1}{2k(k-1)} - \varepsilon$. Let $\varepsilon_{ij} = \left| \langle\psi_i|\psi_j\rangle \right|^2$. Then,

$$\mathbb{E}_{(i,j)} \varepsilon_{ij} \leq 2 \mathbb{E}_{(i,j)} 1 - (1 - \varepsilon_{ij})^2 - \frac{1}{2} \varepsilon_{ij}^2 \leq 4k(k-1)\varepsilon. \quad (\text{A10})$$

Let $\Psi = \begin{bmatrix} |\psi_1\rangle & \cdots & |\psi_k\rangle \end{bmatrix}$ be a $\dim H \times k$ matrix. For any matrix $\Psi' = \begin{bmatrix} |\psi'_1\rangle & \cdots & |\psi'_k\rangle \end{bmatrix}$, the Frobenius norm

$$\begin{aligned} \|\Psi - \Psi'\|_2^2 &= \sum_i \|\psi_i - \psi'_i\|^2 \geq \frac{1}{2} \sum_i \|\psi_i - \psi'_i\|_2^2 \\ &\geq \frac{1}{2k} \left(\sum_i \|\psi_i - \psi'_i\|_2 \right)^2. \end{aligned} \quad (\text{A11})$$

So, our goal becomes to find an isometry Ψ' that minimizes this norm, since this is the class of matrices whose columns are orthonormal—this question is called the *orthogonal Procrustes problem*. Let $\Psi = U\Sigma V^\dagger$ be the singular-value decomposition, where $\Sigma \in \mathbb{M}_{k \times k}(\mathbb{C})$ is a diagonal matrix with elements $\sigma_i \geq 0$ and $U \in \mathbb{M}_{\dim H \times k}$, $V \in \mathbb{M}_{k \times k}(\mathbb{C})$ are isometries: the solution of the problem is $\Psi' = UV^\dagger$. In fact, the unitary that minimizes $\|\Psi - \Psi'\|_2$ is the one that maximizes $\text{Tr}[\Psi^\dagger \Psi'] = \sum_i \sigma_i (U^\dagger \Psi' V)_{i,i}$, which by unitarity is maximized when the diagonal elements of $U^\dagger \Psi' V$ are all 1, giving $U^\dagger \Psi' V = \mathbb{I}$. With this candidate, it suffices to compute the norm. Note that $(\Psi^\dagger \Psi)_{ij} = \langle\psi_i|\psi_j\rangle$, so

$$\|\Psi^\dagger \Psi - \mathbb{I}\|_2^2 = \sum_{i \neq j} |\langle\psi_i|\psi_j\rangle|^2 \leq 4k^2(k-1)^2\varepsilon. \quad (\text{A12})$$

On the other hand, $\|\Psi^\dagger \Psi - \mathbb{I}\|_2^2 = \|\Sigma^2 - \mathbb{I}\|_2^2 = \sum_i (\sigma_i^2 - 1)^2$. This provides that

$$\|\Psi - \Psi'\|_2^2 = \|U\Sigma V^\dagger - UV^\dagger\|_2^2 = \sum_i (\sigma_i - 1)^2 \leq \sum_i (\sigma_i^2 - 1)^2 \leq 4k^2(k-1)^2\varepsilon. \quad (\text{A13})$$

Therefore, we have orthonormal vectors $\{|\psi'_1\rangle, \dots, |\psi'_k\rangle\}$ such that $\sum_i \|\psi_i - \psi'_i\|_2 \leq 2\sqrt{2}k^{3/2}(k-1)\sqrt{\varepsilon}$. Taking $\tilde{\rho}, \tilde{\sigma}$ as in the statement, we have immediately

$$\begin{aligned} \|\rho - \tilde{\rho}\|_{\text{Tr}} + \|\sigma - \tilde{\sigma}\|_{\text{Tr}} &= \sqrt{2} \|\psi_1 - \psi'_1\|_2 + \cdots + \|\psi_k - \psi'_k\|_2 \\ &\leq \sqrt{2} \sum_i \|\psi_i - \psi'_i\|_2 \leq 4k\sqrt{k(k-1)}\sqrt{\varepsilon}. \end{aligned} \quad (\text{A14})$$

□

Lemma A6. Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a quantum channel and let $p_1, p_2, p_3 > 0$ such that $p_1 + p_2 + p_3 = 1$. Let $S = \{(i, j) \mid 1 \leq i < j \leq k\}$ as in the previous lemma. Let Π_H and Π_K be the projectors onto the symmetric subspaces of $H \otimes H$ and $K \otimes K$, respectively. Let $\Phi_1 : \mathcal{B}(H^{\otimes k} \otimes Q) \rightarrow \mathcal{B}(H \otimes \mathbb{C}^{[k]})$ be the channel from Lemma 7

and $\Phi_2 : \mathcal{B}(H^{\otimes k} \otimes Q) \rightarrow \mathcal{B}(Q' \otimes \mathbb{C}^S)$ be the channel from Lemma A5. Define the channel $\Phi_3 : \mathcal{B}(H^{\otimes k} \otimes Q) \rightarrow \mathcal{B}(Q' \otimes \mathbb{C}^S)$ as

$$\Phi_3(\rho) = \mathbb{E}_{(i,j) \in S} \left(\text{Tr} \left(\Pi_K(\Phi \otimes \Phi)(\rho_{i,j}) \right) |\perp\rangle\langle\perp| + \text{Tr} \left((\mathbb{I} - \Pi_K)(\Phi \otimes \Phi)(\rho_{i,j}) \right) \mu_Q \right) \otimes |(i,j)\rangle\langle(i,j)|. \quad (\text{A15})$$

Define $\Phi' = p_1\Phi_1 \oplus p_2\Phi_2 \oplus p_3\Phi_3$. Suppose Φ has an (α, k) -clique. Then, Φ' has an $(\alpha', 2)$ -clique, with $\alpha' = \frac{p_1^2}{k} + \frac{p_2^2 + p_3^2(3/2 + \alpha)}{2k(k-1)}$.

Proof. Let $\rho_1, \dots, \rho_k$ be an (α, k) -clique of Φ . We may, without loss of generality, suppose that $\rho_i = |\psi_i\rangle\langle\psi_i|$ are all pure, as convex combinations cannot increase the overlap. Now, I claim that $\sigma_0 = \rho_1 \otimes \dots \otimes \rho_k \otimes |0\rangle\langle 0|$ and $\sigma_1 = \rho_1 \otimes \dots \otimes \rho_k \otimes |1\rangle\langle 1|$ forms a $(\alpha', 2)$ -clique of Φ' . First, σ_0 and σ_1 are orthogonal. Next, due to 7, $\text{Tr}[\Phi_1(\sigma_0)\Phi_1(\sigma_1)] = \frac{1}{k}$ and due Lemma A5, $\text{Tr}[\Phi_2(\sigma_0)\Phi_2(\sigma_1)] = \frac{1}{2k(k-1)}$. Finally,

$$\Phi_3(\sigma_y) = \mathbb{E}_{(i,j)} \left(\left(\frac{1}{2} + \frac{1}{2} \text{Tr}(\Phi(\rho_i)\Phi(\rho_j)) \right) |\perp\rangle\langle\perp| + \left(\frac{1}{2} - \frac{1}{2} \text{Tr}(\Phi(\rho_i)\Phi(\rho_j)) \right) \mu_Q \right) \otimes |(i,j)\rangle\langle(i,j)| \quad (\text{A16})$$

Putting this together,

$$\begin{aligned} \text{Tr}(\Phi'(\sigma_0)\Phi'(\sigma_1)) &= p_1^2 \text{Tr}(\Phi_1(\sigma_0)\Phi_1(\sigma_1)) + p_2^2 \text{Tr}(\Phi_2(\sigma_0)\Phi_2(\sigma_1)) + p_3^2 \text{Tr}(\Phi_3(\sigma_0)\Phi_3(\sigma_1)) \\ &= \frac{p_1^2}{k} + \frac{p_2^2}{2k(k-1)} + \frac{2p_3^2}{k(k-1)} \mathbb{E}_{(i,j)} \left(\frac{1}{2} + \frac{1}{2} \text{Tr}(\Phi(\rho_i)\Phi(\rho_j)) \right)^2 + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \text{Tr}(\Phi(\rho_i)\Phi(\rho_j)) \right)^2 \end{aligned} \quad (\text{A17})$$

Since the function $(\frac{1}{2} + \frac{x}{2})^2 + \frac{1}{2}(\frac{1}{2} - \frac{x}{2})^2 \geq \frac{3}{8} + \frac{x}{4}$ we have that $\mathbb{E}_{(i,j)} \left(\frac{1}{2} + \frac{1}{2} \text{Tr}(\Phi(\rho_i)\Phi(\rho_j)) \right)^2 + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \text{Tr}(\Phi(\rho_i)\Phi(\rho_j)) \right)^2 \geq \frac{3}{8} + \frac{\alpha}{4}$, by the clique property. Thus, we have $\text{Tr}(\Phi'(\sigma_0)\Phi'(\sigma_1)) = \frac{p_1^2}{k} + \frac{p_2^2}{2k(k-1)} + \frac{p_3^2(3/2 + \alpha)}{2k(k-1)} = \alpha'$, as wanted. $\square$

Theorem A1. Let Φ and Φ' be as in Lemma A6. Then, if Φ has no (β, k) -clique, then, for all η , there are p_1, p_2, p_3 such that Φ' has no $(\beta' + \eta, 2)$ -clique, with $\beta' = \frac{p_1^2}{k} + \frac{p_2^2 + p_3^2(3/2 + (5/2)\beta)}{2k(k-1)}$.

Together with the previous lemma, this gives a reduction from $\text{qClique}(k)_{c,s}$ to $\text{qClique}(2)_{c',s'}$, where as long as $c \geq \frac{5}{2}s$ with polynomial gap, c' and s' have polynomial gap.

Proof. The proof proceeds by two successive applications of Lemma 5. For the first application, let $A = \{(|\psi_1\rangle\langle\psi_1| \otimes \dots \otimes |\psi_k\rangle\langle\psi_k| \otimes |\alpha\rangle\langle\alpha|, |\psi_1\rangle\langle\psi_1| \otimes \dots \otimes |\psi_k\rangle\langle\psi_k| \otimes |\beta\rangle\langle\beta|) \mid |\psi_1\rangle, \dots, |\psi_k\rangle \in H \text{ and } |\alpha\rangle, |\beta\rangle \in H' \text{ orthonormal}\}$, and $B = \{(\rho, \sigma) \in A \mid |\psi_1\rangle, \dots, |\psi_k\rangle \text{ orthonormal}\}$. Then, as Φ has no (β, k) -clique, for all $(\rho, \sigma) \in B$, $\text{Tr}[\Phi_3(\rho)\Phi_3(\sigma)] \leq \frac{3/2 + (5/2)\beta}{2k(k-1)}$, due to the fact that $(\frac{1}{2} + \frac{x}{2})^2 + \frac{1}{2}(\frac{1}{2} - \frac{x}{2})^2 \leq \frac{3}{8} + \frac{5}{8}x$ for $x \in [0, 1]$. Then, by using Lemma A5 there exists $p \in (0, 1)$ such that $p\Phi_2 \oplus (1-p)\Phi_3$ has no $(\frac{p^2 + (1-p)^2(3/2 + (5/2)\beta)}{2k(k-1)} + \eta_1, 2)$ -clique. Now for the second application, we take B to be A from above and $A = \{(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|)\}$. Then, using Lemma 7, there exists $q \in (0, 1)$ such that $q\Phi_1 \oplus (1-q)p\Phi_2 \oplus (1-q)(1-p)\Phi_3$ has no $(\frac{q^2}{k} + \frac{((1-q)p)^2 + ((1-p))^2(3/2 + (5/2)\beta)}{2k(k-1)} + (1-q)^2\eta_1 + \eta_2, 2)$ -clique. Take $\eta = (1-q)^2\eta_1 + \eta_2 \leq \eta_1 + \eta_2$, which can be made arbitrarily small, and $p_1 = q$, $p_2 = (1-q)p$, $p_3 = (1-q)(1-p)$, which completes the proof. $\square$

Appendix B. Alternate proof of QMA(2)-hardness

In this appendix, we give an alternate proof that the quantum clique problem is QMA(2)-hard. The proof is conceptually simpler than the proof in Section 4, but provides no additional constraints on the structure of the channels.

Theorem A2. There exist $c, s : \mathbb{N} \rightarrow (0, 1)_{\text{exp}}$ with polynomial gap such that $\text{qClique}(2)_{c,s}$ is QMA(2)-complete.

Proof. Let $L \in \text{QMA}(2)$. Based on the construction of [37], we can make some assumptions about the structure of the verification circuit. First, we may assume that the witness state is pure, and that it consists of two copies of the same state $|\psi\rangle \otimes |\psi\rangle \in (\mathbb{C}^2)^{\otimes 2m}$. Second, we can assume that, for any classical input x , the verification circuit takes the form of some unitary V_x acting on the space of the witness state and some polynomial-sized workspace $W = (\mathbb{C}^2)^{\otimes k}$, and that the verification algorithm accepts or rejects by making a measurement of the first qubit of the output of the unitary. That is, if $x \in L$, then there exists $|\psi\rangle \in (\mathbb{C}^2)^{\otimes m}$ such that $\|(\langle 1| \otimes \mathbb{I})V_x|\psi\rangle|\psi\rangle|0^k\rangle\|^2 \geq c$; and if $x \notin L$, then for all $|\psi\rangle, |\phi\rangle \in (\mathbb{C}^2)^{\otimes m}$, $\|(\langle 1| \otimes \mathbb{I})V_x|\psi\rangle|\phi\rangle|0^k\rangle\|^2 \leq s$.

Now, we want to construct a channel Φ_x that has a 2-clique if and only if $x \in L$. Write $H = (\mathbb{C}^2)^{\otimes m}$, $W = (\mathbb{C}^2)^{\otimes k}$, $Y = \mathbb{C}^2$, and $G = (\mathbb{C}^2)^{\otimes 2m+k-1}$, so that $V_x : H \otimes H \otimes W \rightarrow Y \otimes G$. Run the channel $\Phi_x : \mathcal{B}(G \otimes \mathbb{C}^2) \rightarrow \mathcal{B}(H)$ on input ρ as follows:

1. Trace out the last qubit of ρ to get ρ_G .
2. Act with $V_x^\dagger$ on $|1\rangle\langle 1| \otimes \rho_G$.
3. Measure W in the computational basis. If the measurement result is 0^k , do nothing to the remaining state; otherwise replace it with a maximally mixed state.
4. Trace out the first copy of H .

This construction is illustrated in Figure A1

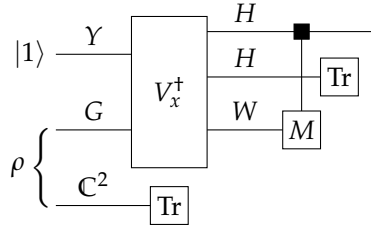


Figure A1. Construction of the channel from a QMA(2)-language verification circuit.

First, consider the case that $x \in L$. Then, there exists $|g\rangle \in G$ such that $\left| \langle 1| \langle g| V_x |\psi\rangle |\psi\rangle |0^k\rangle \right|^2 \geq c$. We claim the wanted clique is $\rho = |g\rangle\langle g| \otimes |0\rangle\langle 0|$, $\sigma = |g\rangle\langle g| \otimes |1\rangle\langle 1|$. For either of those two states, the state after step 2 is $V_x^\dagger |1\rangle |g\rangle = \alpha |\psi\rangle |\psi\rangle |0^k\rangle + |\perp\rangle$, where $|\alpha|^2 \geq c$. Were $\alpha = 1$, the output would be $|\psi\rangle$, in which case $\text{Tr}(\Phi(\rho_0)\sigma(\sigma_0)) = 1$. In reality, $\|V_x^\dagger |1\rangle |g\rangle - |\psi\rangle |\psi\rangle |1\rangle\| = \|\perp\| = \sqrt{|1-\alpha|^2 + 1 - |\alpha|^2} = \sqrt{2 - 2\text{Re } \alpha} \leq \sqrt{2 - 2c}$. Thus,

$$\begin{aligned} \text{Tr}(\Phi(\rho)\Phi(\sigma)) &\geq \text{Tr}(\Phi(\rho_0)\Pi(\sigma_0)) - \|\Phi(\rho) - \Phi(\rho_0)\|_{\text{Tr}} - \|\Phi(\sigma) - \Phi(\sigma_0)\|_{\text{Tr}} \\ &\geq 1 - 2\sqrt{2 - 2c} \geq 4(c - \frac{7}{8}) \end{aligned} \quad (\text{A18})$$

Now, consider the case that $x \notin L$. In this case, for all $|\psi\rangle, |\phi\rangle \in H$, and $|g\rangle \in G$, $\left| \langle 1| \langle g| V_x |\psi\rangle |\phi\rangle |0^k\rangle \right|^2 \leq s$. Since every mixed state is a convex combination of pure states, if we show that for all pure states $|\psi\rangle, |\phi\rangle \in G \otimes \mathbb{C}^2$, $\text{Tr}(\Phi(|\psi\rangle\langle\psi|)\Phi(|\phi\rangle\langle\phi|))$ is upper bounded by some constant, the same bound must hold for all states. Further, we may, by the same argument, work only with states that remain pure after the partial trace is taken in step 1—these are the separable states $|\psi\rangle = |\psi_G\rangle \otimes |\psi_0\rangle$, $|\phi\rangle = |\phi_G\rangle \otimes |\phi_0\rangle$. With this assumption, after step 2, $|\psi\rangle$ becomes $V_x^\dagger |1\rangle |\psi_G\rangle = \alpha |\psi_{HH}\rangle |0^k\rangle + \beta |\perp\rangle$ for some states $|\psi_{HH}\rangle \in H \otimes H$ and $|\perp\rangle \in H \otimes H \otimes W$, where $|\alpha|^2 + |\beta|^2 = 1$. Letting μ_{HH} be the maximally mixed state on $H \otimes H$, the state after step 3 is the $|\alpha|^2 |\psi_{HH}\rangle\langle\psi_{HH}| + |\beta|^2 \mu_{HH}$. Expanding $|\psi_{HH}\rangle = \sum_i \sqrt{p_i} |\psi_i^1\rangle |\psi_i^2\rangle$ via the Schmidt decomposition, the final state after step 4 is

$$\Phi_x(|\psi\rangle\langle\psi|) = \sum_i |\alpha|^2 p_i |\psi_i^2\rangle\langle\psi_i^2| + |\beta|^2 \mu_H = \sum_i \left(|\alpha|^2 p_i + \frac{|\beta|^2}{2^m} \right) |\psi_i^2\rangle\langle\psi_i^2|. \quad (\text{A19})$$

Note also that $|\alpha|^2 p_i = \left| \langle 1 | \langle \psi | V_x | \psi_i^1 \rangle | \psi_i^2 \rangle | 0^k \rangle \right|^2 \leq s$ for all i . In the same way, $\Phi_x(|\phi\rangle\langle\phi|) = \sum_i \left(|\alpha'|^2 q_i + \frac{|\beta'|^2}{2^m} \right) |\phi_i^2\rangle\langle\phi_i^2|$ for some $|\alpha'|^2 q_i \leq s$. Finally, we get that, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \text{Tr}(\Phi(|\psi\rangle\langle\psi|)\Phi(|\phi\rangle\langle\phi|)) &\leq \sqrt{\sum_i \left(|\alpha|^2 p_i + \frac{|\beta|^2}{2^m} \right)^2 \sum_j \left(|\alpha'|^2 q_j + \frac{|\beta'|^2}{2^m} \right)^2} \\ &\leq \sqrt{\left(s + \frac{|\beta|^2}{2^m} \right) \sum_i \left(|\alpha|^2 p_i + \frac{|\beta|^2}{2^m} \right) \left(s + \frac{|\beta'|^2}{2^m} \right) \sum_j \left(|\alpha'|^2 q_j + \frac{|\beta'|^2}{2^m} \right)} \quad (\text{A20}) \\ &\leq s + \frac{1}{2^m}. \end{aligned}$$

□

References

1. C. Shannon (1956). “The zero error capacity of a noisy channel.” *IRE Transactions on Information Theory*, 2: 3, 8–19, DOI: 10.1109/TIT.1956.1056798.
2. R. Duan, S. Severini, and A. Winter (2013). “Zero-error communication via quantum channels, noncommutative graphs, and a quantum Lovász number.” *IEEE Transactions on Information Theory*, 59: 2, 1164–1174. DOI: 10.1109/tit.2012.2221677.
3. M. Brannan, K. Eifler, C. Voigt, and M. Weber (2022). “Quantum Cuntz-Krieger algebras.” *Transactions of the American Mathematical*, 9. DOI: <https://doi.org/10.1090/btran/88>.
4. A. Connes and W. D. van Suijlekom (2022). “Tolerance relations and operator systems.” *Acta Scientiarum Mathematicarum*, 88. DOI: <https://doi.org/10.1007/s44146-022-00012-3>.
5. A. Chirvasitu and M. Wasilewski (2022). “Random quantum graphs.” *Transactions of the American Mathematical Society*. DOI: 10.1090/tran/8584.
6. P. Ganesan (2023). “Spectral bounds for the quantum chromatic number of quantum graphs.” *Linear Algebra and its Applications*, 674, 351–376. DOI: <https://doi.org/10.1016/j.laa.2023.06.007>.
7. M. Kennedy, T. Kolomatski, and D. Spivak (2020). “An infinite quantum Ramsey theorem.” *Journal of Operator Theory*, 84. DOI: 10.7900/jot.2018dec11.2259.
8. J. Matsuda (2022). Classification of quantum graphs on $M2$ and their quantum automorphism groups. *Journal of Mathematical Physics*, 63, 9. DOI: 10.1063/5.0081059.
9. D. Stahlke (2016). “Quantum zero-error source-channel coding and non-commutative graph theory.” *IEEE Transactions on Information Theory*, 62: 1, 554–577. DOI: 10.1109/tit.2015.2496377.
10. I. G. Todorov and L. Turowska (2020). “Quantum no-signalling correlations and non-local games.” arXiv: 2009.07016.
11. M. Sipser (2012). *Introduction to the Theory of Computation*, 3rd ed. Cengage Learning.
12. A. Y. Kitaev, A. Shen, and M. N. Vyalyi (2002). *Classical and Quantum Computation*. American Mathematical Society.
13. A. Broadbent and A. B. Grilo (2022). “QMA-hardness of consistency of local density matrices with applications to quantum zero-knowledge.” *SIAM Journal on Computing*, 51: 4, 1400–1450. DOI: 10.1137/21m140729x.
14. S. Bravyi (2006). “Efficient algorithm for a quantum analogue of 2-SAT.” arXiv: quant-ph/0602108.
15. S. Beigi and P. W. Shor (2008). “On the complexity of computing zero-error and holevo capacity of quantum channels.” arXiv: 0709.2090.
16. S. Aaronson, S. Beigi, A. Drucker, B. Fefferman, and P. Shor (2009). “The power of unentanglement.” *Theory of Computing*, 5: 1, 1–42. DOI: 10.4086/toc.2009.v005a001.
17. H. Kobayashi, K. Matsumoto, and T. Yamakami (2003). “Quantum Merlin-Arthur proof systems: Are multiple Merlins more helpful to Arthur?” In *Algorithms and Computation*, pp. 189–198, Berlin, Heidelberg. Springer Berlin Heidelberg. DOI: 10.1007/978-3-540-24587-2_21.
18. A. Chailloux and O. Sattath (2012). “The complexity of the separable Hamiltonian problem.” In *2012 IEEE 27th Conference on Computational Complexity*. IEEE. DOI: 10.1109/cc.2012.42.
19. G. Gutoski, P. Hayden, K. Milner, and M. M. Wilde (2015). “Quantum interactive proofs and the complexity of separability testing.” *Theory of Computing*, 11: 1, 59–103. DOI: 10.4086/toc.2015.v011a003.
20. J. Erdos, A. Katavolos, and V. Shulman (1998). “Rank one subspaces of bimodules over maximal abelian selfadjoint algebras.” *Journal of Functional Analysis*, 157: 2, 554–587. DOI: <https://doi.org/10.1006/jfan.1998.3274>.

21. N. Weaver (2010). “Quantum relations.” *Memoirs of the American Mathematical Society*, 215. DOI: 10.1090/S0065-9266-2011-00637-4.
22. N. Weaver (2021). “Quantum graphs as quantum relations.” *The Journal of Geometric Analysis*, 31: 9, 9090–9112. DOI: 10.1007/s12220-020-00578-w.
23. V. I. Paulsen (2002). *Completely bounded maps and operator algebras*. Cambridge studies in advanced mathematics; 78. Cambridge: Cambridge University Press.
24. V. Paulsen (2016). “Online course notes: Entanglement and non-locality,” University of Waterloo. Available at www.math.uwaterloo.ca/~vpaulsen/EntanglementAndNonlocality_LectureNotes_7.pdf.
25. M. Daws (2022). “Quantum graphs: different perspectives, homomorphisms and quantum automorphisms.” arXiv: 2203.08716.
26. G. Boreland, I. Todorov, and A. Winter (2021). “Sandwich theorems and capacity bounds for non-commutative graphs.” *Journal of Combinatorial Theory, Series A*, 177, 105302. DOI: 10.1016/j.jcta.2020.105302.
27. S.-J. Kim and A. Mehta (2019). “Chromatic numbers, Sabidussi’s theorem and Hedetniemi’s conjecture for non-commutative graphs.” *Linear Algebra and its Applications*, 582, 291–309. DOI: <https://doi.org/10.1016/j.laa.2019.08.002>.
28. C. M. Ortiz and V. I. Paulsen (2016). “Quantum graph homomorphisms via operator systems.” *Linear Algebra and its Applications*, 497, 23–43. DOI: <https://doi.org/10.1016/j.laa.2016.02.019>.
29. J. W. Helton, K. P. Meyer, V. I. Paulsen, and M. Satriano (2019). “Algebras, synchronous games, and chromatic numbers of graphs.” *New York Journal of Mathematics*, 25, 328–361. Available at nyjm.albany.edu/j/2019/25-16.html.
30. L. Mančinska and D. E. Roberson (2016). “Quantum homomorphisms.” *Journal of Combinatorial Theory, Series B*, 118, 228–267. DOI: <https://doi.org/10.1016/j.jctb.2015.12.009>.
31. M. Brannan, P. Ganesan, and S. J. Harris (2022). “The quantum-to-classical graph homomorphism game.” *Journal of Mathematical Physics*, 63: 11, 112204. DOI: 10.1063/5.0072288.
32. M. Coudron and W. Slofstra (2019). “Complexity lower bounds for computing the approximately-commuting operator value of non-local games to high precision.” In A. Shpilka (ed.), *34th Computational Complexity Conference (CCC 2019)*, volume 137 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pp. 25:1–25:20, Dagstuhl, Germany. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. DOI: 10.4230/LIPIcs.CCC.2019.25.
33. Z. Ji, A. Natarajan, T. Vidick, J. Wright, and H. Yuen (2021). “MIP* = RE.” *Communications of the ACM*, 64: 11, 131–138. DOI: 10.1145/3485628.
34. H. Mousavi, S. S. Nezhadi, and H. Yuen (2022). “Nonlocal games, compression theorems, and the arithmetical hierarchy.” In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*. ACM. DOI: 10.1145/3519935.3519949.
35. S. J. Harris (2023). “Universality of graph homomorphism games and the quantum coloring problem.” arXiv: 2305.18116.
36. H. Galperin and A. Wigderson (1983). “Succinct representations of graphs.” *Information and Control*, 56: 3, 183–198, DOI: [https://doi.org/10.1016/S0019-9958\(83\)80004-7](https://doi.org/10.1016/S0019-9958(83)80004-7).
37. A. W. Harrow and A. Montanaro (2013). “Testing product states, quantum Merlin-Arthur games and tensor optimization.” *Journal of the ACM*, 60: 1, 1–43. DOI: 10.1145/2432622.2432625.
38. D. Aharonov and A. B. Grilo (2019). “Stoquastic pcg vs. randomness.” In *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*, pp. 1000–1023. IEEE. DOI: 10.1109/FOCS.2019.00065.
39. S. Bravyi and B. Terhal (2010). “Complexity of stoquastic frustration-free Hamiltonians.” *SIAM Journal on Computing*, 39: 4, 1462–1485. DOI: 10.1137/08072689X.
40. A. Broadbent, A. Mehta, and Y. Zhao (2024). “Quantum delegation with an off-the-shelf device.” In *19th Conference on the Theory of Quantum Computation, Communication and Cryptography*, DOI: 10.4230/LIPIcs.TQC.2024.12.
41. A. B. Grilo (2019). “A simple protocol for verifiable delegation of quantum computation in one round.” In *46th International Colloquium on Automata, Languages, and Programming—ICALP 2019*, pp. 28. DOI: 10.4230/LIPIcs.ICALP.2019.28.
42. R. Bassirian, B. Fefferman, and K. Marwaha (2023). “Quantum Merlin-Arthur and proofs without relative phase.” arXiv: 2306.13247.
43. F. G. d. S. L. Brandão (2008). *Entanglement Theory and the Quantum Simulation of Many-Body Physics*. PhD thesis, University of London. arXiv: 0810.0026.

44. L. Lovász (1973). “Coverings and colorings of hypergraphs.” In *Proceedings of the 4th Southeastern Conference on Combinatorics, Graph Theory, and Computing*. Online: www.cs.elte.hu/~lovasz/scans/covercolor.pdf.
45. M. Ben-Or, O. Goldreich, S. Goldwasser, J. Håstad, J. Kilian, S. Micali, and P. Rogaway (1988). “Everything provable is provable in zero-knowledge.” In *Advances in Cryptology—CRYPTO 1988*, pp. 37–56. DOI: 10.1007/0-387-34799-2_4.
46. O. Goldreich, S. Micali, and A. Wigderson (1991). “Proofs that yield nothing but their validity or all languages in NP have zero-knowledge proof systems.” *Journal of the ACM*, 38: 3, 690–728. DOI: 10.1145/116825.116852.
47. A. Chiesa, M. Forbes, T. Gur, and N. Spooner (2018). “Spatial isolation implies zero knowledge even in a quantum world.” In *59th Annual Symposium on Foundations of Computer Science—FOCS 2018*, pp. 755–765. DOI: 10.1109/FOCS.2018.00077.
48. A. Grilo, W. Slofstra, and H. Yuen (2019). “Perfect zero knowledge for quantum multiprover interactive proofs.” In *Proceedings of FOCS*, pp. 611–635. DOI: 10.1109/FOCS.2019.00044.
49. T. Vidick and T. Zhang (2020). “Classical zero-knowledge arguments for quantum computations.” *Quantum*, 4, 266. DOI: 10.22331/q-2020-05-14-266.
50. F. Ramsey (1930). “On a problem of formal logic.” *Proceedings of the London Mathematical Society*, 30, 264–286. DOI: 10.1112/plms/s2-30.1.264.
51. S. A. Burr (1990). *On the Computational Complexity of Ramsey—Type Problems*, pp. 46–52. Springer Berlin Heidelberg, Berlin, Heidelberg. DOI: 10.1007/978-3-642-72905-8_5.
52. N. Weaver. (2017). “A ‘quantum’ Ramsey theorem for operator systems.” *Proceedings of the American Mathematical Society*, 145, 4595–4605. DOI: 10.1090/proc/13606.
53. H. Buhrman, R. Cleve, J. Watrous, and R. de Wolf (2001). “Quantum fingerprinting.” *Physical Review Letters*, 87, 167902. DOI: 10.1103/PhysRevLett.87.167902.
54. M. Horodecki, P. W. Shor, and M. B. Ruskai (2003). “Entanglement breaking channels.” *Reviews in Mathematical Physics*, 15: 6, 629–641. DOI: 10.1142/s0129055x03001709.
55. J. Watrous (2018). *The Theory of Quantum Information*, 1st ed. Cambridge University Press.
56. P. H. Schönemann (1966). “A generalized solution of the orthogonal Procrustes problem.” *Psychometrika*, 31: 1, 1–10. DOI: 10.1007/BF02289451.