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Article

Qubit Geometry through Holomorphic Quantization

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Abstract: We develop a wave mechanics formalism for qubit geometry using holomorphic functions and Möbius transformations, providing a geometric perspective on quantum computation. This framework extends the standard Hilbert space description, offering a natural interpretation of standard quantum gates on the Riemann sphere that is examined through their Möbius action on holomorphic wavefunction. These wavefunctions emerge via a quantization process, with the Riemann sphere serving as the classical phase space of the qubit geometry. We quantize this space using Isham's canonical group quantization with holomorphic polarization, yielding holomorphic wavefunctions and spin angular momentum operators that recover the standard $SU(2)$ algebra with interesting geometric properties. Such properties reveal how geometric transformations induce quantum logic gates on the Riemann sphere, providing a novel perspective on quantum information processing. This result provides a new direction for exploring quantum computation through Isham's canonical group quantization and its holomorphic polarization method.

Keywords: canonical group quantization; compact phase space; holomorphic polarization; qubit geometry; qubit operations

1. Introduction

Quantum information processing (QIP) represents one of the most promising frontiers in information technology, offering transformative potential in computation [1,2], secure communication [3], cryptography [4], and sensing and metrology [5]. At its core, QIP involves the manipulation of quantum states, typically represented as vectors or density matrices in a Hilbert space [6]. However, beyond their algebraic descriptions, these states inhabit rich geometric structures that profoundly influence their dynamics and applications. Geometry provides a natural framework for understanding the properties of quantum states and operations [7]. The state spaces of quantum systems are not merely mathematical abstractions; they possess symplectic structures, Riemannian metrics, and topological features that dictate the evolution and interaction of quantum states [8–10]. Tools from geometry facilitate insights into entanglement, quantum error correction, and computational algorithms, offering powerful approaches to problems in quantum information science [11].

The basic unit of information in quantum computers is based on a two-level quantum system called a qubit, which serves the same function as a classical bit in a classical computer. In this study, we will investigate the idea of wave mechanics formalism of qubit geometry, for which the latter is not new [12–19]. The main idea of our study is to reveal the formalism of qubit in the realm of traditional wave mechanics formalism of quantum theory through the process of quantization, which is a well-known and very useful technique in theoretical physics, for instance in quantum gravity [20] and quantum field theory [21]. In particular, the idea is that qubits and qubit operations are often represented elementarily as matrices, and there seems to be a disconnect between such formalism with the

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traditional wave mechanics formalism of quantum theory. We reconnect with the wave formalism through the process of quantization, and there are advantages of having various representations [22].

The starting point is to recognize the projective Hilbert space as a classical phase space [23]. For finite-dimensional Hilbert spaces, these will be complex projective spaces and in particular, for a single qubit is the Riemann sphere (or Bloch sphere). Such a structure is often discussed without referring to the quantization process [24], which we would like to discuss in this study. The idea of Riemann sphere is treated as a compact classical phase space of qubit because it has the natural properties that could produce the angular momenta through the natural polarization in our chosen quantization framework. Our relevant studies also explore the role of Möbius transformations in quantum information theory [25] for which in this study it is reinterpreted not merely as geometric symmetries but as dynamical and computational operations on qubits that produce the quantum gates in the holomorphic representation.

The quantization approach adopted is Isham's canonical group quantization (CGQ) [26], a variant of geometric quantization [27]. This approach has been applied to problems in quantum gravity [28,29], as well as to particle and string dynamics on tori [30]. It was later applied to various quantum systems, including a particle on a torus in a constant magnetic field [31], the quantum Hall effect [32], a particle in a noncommutative configuration space [33], a particle on a sphere with a magnetic monopole [34,35], and nonperturbative quantum gravity [36]. Our approach relies on a phase space geometry which requires a natural polarization where it differs from the work done in [37], who used coherent state quantization that is purely constructed from a group-theoretic point of view and also the work done in [38], who discussed the application of such coherent states in quantum information theory. The CGQ provides a global version of canonical quantization, closely aligned with the traditional foundations of quantum theory [39]. The most fundamental object in the CGQ is the symplectic form, whose symmetries are taken into consideration for starting the quantization process. It is often the case that symplectic form (being a differential form) is often insensitive to discrete symmetries.

Meanwhile, in our construction, by using the inhomogeneous coordinates $(z, \bar{z})$, a discrete symmetry naturally arises from the exchange of z and $\bar{z}$, which also addresses the question of natural polarization needed for Schrödinger-type quantization (with half of the degrees of freedom removed) (see Section 3). The idea of spin as intrinsic angular momentum remains elusive despite progress made in quantum mechanics. The current approach simply states that the simplest compact phase space carrying a natural polarization arises from a complex version of the sphere as an internal space that accommodates some physical angular momenta. It is our hope that this approach will be useful in providing a new understanding of treating a variety of fundamental problems in quantum information and computation analytically. In general, the main idea of CGQ begin with classical phase spaces modeled as symplectic manifolds endowed with a symplectic form ω, and then to identify a Lie group that preserves the global kinematical symmetries of ω, to be an "appropriate" canonical group $\mathcal{G}$. There are three types of Lie algebra associated with the phase space (a symplectic manifold) that must correspond to each other before the quantization process can be established. These are (i) the abstract algebra of the canonical group, (ii) the commutator algebra of (global) Hamiltonian vector fields, and (iii) the Poisson bracket of classical observables. An irreducible unitary representation of $\mathcal{G}$ then defines a possible quantization of the system, with inequivalent representations corresponding to different physical realizations of quantum systems.

The paper is organized as follows. In Section 2, we introduce the mathematical formulation of qubit geometry by identifying the sphere S^2 as a non-cotangent bundle phase space and describing its associated symplectic structure. Section 3 develops the holomorphic quantization procedure through the CGQ, emphasizing the role of holomorphic polarization in constructing wavefunctions via sections of a complex line bundle over $\mathbb{C}P^1$. In Section 4, we represent single-qubit quantum gates as Möbius transformations, examining their geometric action on holomorphic wavefunctions and demonstrating their consistency with standard quantum gate operations. Finally, Section 5 summarizes our findings and outlines future directions, including generalizations to higher-dimensional systems and potential applications in quantum information theory.

2. Mathematical Formulation and Non-cotangent Bundle Phase Space S^2

Before discussing holomorphic quantization, we provide a brief overview of the CGQ employed in our approach. The procedure begins by identifying a globally well-defined minimal set of preferred classical observables[1] for a phase space, $\mathcal{S}$ that is not necessarily a cotangent bundle. These observables generate all other classical observables and

[1] It is analogous to the position, q, and momentum, p, observables in standard quantum mechanics on the configuration space $Q = \mathbb{R}^n$.

define a set of Hamiltonian vector fields on the phase space. Given an observable $f \in C^\infty(\mathcal{S}, \mathbb{R})$, the corresponding Hamiltonian vector field ξ_f is determined by

$$\xi_f \lrcorner \omega = -df. \tag{1}$$

We assume that the commutator algebra of these vector fields is (anti)-homomorphic to the Poisson-brackets algebra of the observables. One then exponentiates these vector fields to generate the required canonical group $\mathcal{G}$. The final step is to find all inequivalent irreducible unitary representations of $\mathcal{G}$, producing inequivalent quantizations of the system. The self-adjoint generators in a unitary representation of $\mathcal{G}$ then produce the holomorphic part of the constructed vector fields.

Our new approach differs from the original idea of the CGQ scheme (for a summary, see [26]). Instead of starting with a cotangent bundle phase space endowed with a configuration space Q, we work with a non-cotangent bundle (compact) phase space, i.e., $\mathbb{C}\mathrm{P}^1$ (that is constructed from S^2), which naturally leads to discrete spectra for quantum spin systems. Furthermore, the global kinematical symmetry is analyzed by constructing the canonical group $\mathcal{G}$ in a manner that does not take the semidirect product form. In particular, the Poisson bracket algebra of preferred classical observables and commutator algebra of Hamiltonian vector fields can be constructed to be (anti-)homomorphic to the algebra of the canonical group describing kinematic symmetries of $\mathbb{C}\mathrm{P}^1$, just like the case of S^2. As a result, Mackey's induced representation techniques no longer apply directly. Instead, the induced representation is obtained from the action of $\mathcal{G}$ on the space of holomorphic wavefunctions (section of line bundle), leading to possible quantizations.

In standard quantum mechanics (QM), spin is interpreted as intrinsic angular momentum rather than arising from rotational or "spinning" motion. Meanwhile, spin-$\frac{1}{2}$ (or two-level quantum systems) carry the smallest unit of QIP. However, the classical phase space for a spin-$\frac{1}{2}$(or qubit) is not well-established. The minimal viable choice is the sphere S^2, since the simplest compact space S^1 is one-dimensional and cannot independently define a phase space. We therefore take S^2 as a compact phase space $\mathcal{S}$ that is simply connected [40], given by

$$S^2 = \{\vec{x} : \sum_{j=1}^{3} x_j x_j = 1\}. \tag{2}$$

This phase space is equipped with the natural symplectic form

$$\omega = \sin\theta d\theta \wedge d\phi, \tag{3}$$

where it is closed $d\omega = 0$, and is non-degenerate. The preferred set of observables is given by

$$x_1 = \sin\theta\cos\phi; \quad x_2 = \sin\theta\sin\phi; \quad x_3 = \cos\theta, \tag{4}$$

where $0 \le \theta \le \pi$; $0 \le \phi \le 2\pi$. Since θ and ϕ are periodic, they are not globally well-defined continuous functions on S^2. Instead, we use the observables in (4) as a minimal globally well-defined set to ensure closure of the Poisson bracket algebra.

Geometrically, S^2 can be viewed as the homogeneous space $SO(3)/SO(2)$, suggesting $SO(3)$ as the natural choice for the canonical group $\mathcal{G}$. From the fundamental equation (1), the corresponding Hamiltonian vector fields (HamVF($\mathcal{S}$)) associated with the observables (4) are

$$\xi_1 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \quad \xi_2 = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \quad \xi_3 = -\frac{\partial}{\partial\phi}. \tag{5}$$

These are precisely the standard angular momentum operators. Their commutator algebra satisfies

$$[\xi_j, \xi_k] = \varepsilon_{jkl}\xi_l, \tag{6}$$

where ε_{jkl} is the totally antisymmetric cyclic permutation that corresponds to the Lie algebra $\mathfrak{so}(3)$. The Poisson bracket algebra of the global position observables (4) is given by

$$\{x_j, x_k\} = \omega(\xi_j, \xi_k) = -\varepsilon_{jkl}x_l. \tag{7}$$

These structures reveal an (anti-)homomorphism to the abstract algebra of $SO(3)$, further justifying the choice of $SO(3)$ as the canonical group.

An alternate route is to obtain the transitive action of the group $\mathcal{G}$ on the phase space $(\mathcal{S}, \omega)$ given by

$$(g, \vec{x}) \longmapsto \ell_g(\vec{x}) = \vec{x}', \tag{8}$$

where $g := \exp(-iJ) \in SO(3)$ is constructed from the exponential map of Lie algebra $J \in \mathfrak{so}(3)$ as one-parameter subgroups (OPS) elements, and $\vec{x} \in S^2$. These transformations generate the same HamVF($\mathcal{S}$) (5) through their integral curves on phase space.

If the quantum wavefunction were not required to depend on only half of the phase space coordinates, then $SO(3)$ might serve as a suitable canonical group for quantizing S^2. However, a fundamental requirement is the existence of a natural polarization of the phase space [27]. The sphere S^2 lacks such a polarization due to its nonvanishing Euler class [41]. Thus, the need for a compact phase space $\mathbb{C}\mathrm{P}^1$ to impose the holomorphic polarization in CGQ.

3. Holomorphic Quantization

It is well-known that S^2 is homeomorphic to $\mathbb{C}\mathrm{P}^1$ via stereographic projection [42] as illustrated in Figure 1. This projection maps a point $\vec{x} = (x_1, x_2, x_3)$ on S^2 to a point $z = x + iy$ on the extended complex plane $\hat{C} := \mathbb{C} \cup \{\infty\}$. Specifically, the line passing through the north pole $N = (0, 0, 1)$ and the point $\vec{x}$ on S^2 intersects $\hat{C}$ at $z = \cot(\frac{\theta}{2})e^{i\phi} = \infty$. This construction explicitly identifies S^2 with the Riemann sphere $\mathbb{C}\mathrm{P}^1$, a natural setting for holomorphic quantization [27]. The symplectic form on the resultant phase space $\tilde{\mathcal{S}} = \mathbb{C}\mathrm{P}^1$ is

$$\Omega = \frac{2i dz \wedge d\bar{z}}{(1 + z\bar{z})^2}, \tag{9}$$

and the complex canonical observables are given by

$$u_1 = \frac{z + \bar{z}}{(z\bar{z} + 1)}; \quad u_2 = -i\frac{z - \bar{z}}{(z\bar{z} + 1)}; \quad u_3 = \frac{z\bar{z} - 1}{(z\bar{z} + 1)}. \tag{10}$$

Unlike S^2, the Riemann sphere admits a natural polarization between z and $\bar{z}$, facilitating holomorphic quantization. The symplectic form Ω encodes an inherent extra hidden discrete symmetry under complex conjugation, $z \longmapsto \bar{z}$. Incorporating this symmetry leads to the natural choice of the canonical group as the double cover $SU(2)$ of the $SO(3)$ symmetry of S^2.

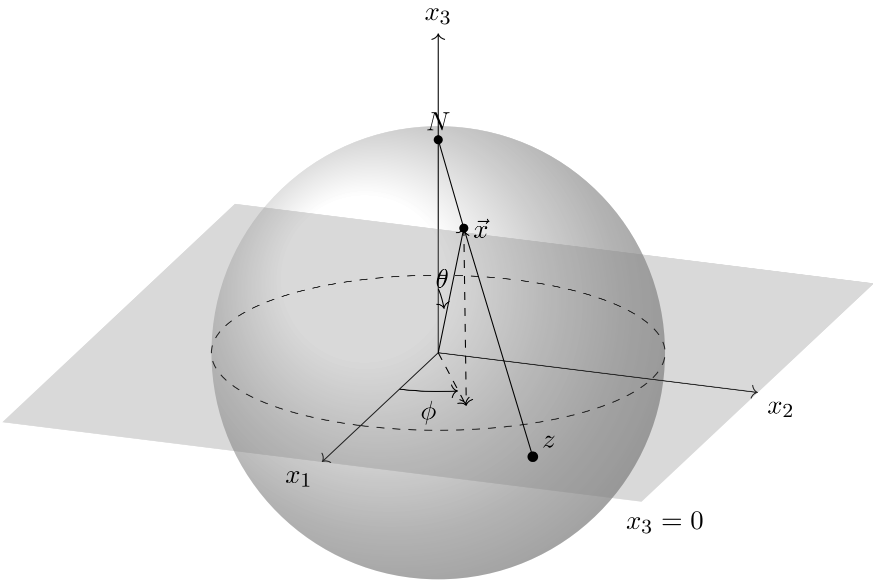

Figure 1. Stereographic projection from the north pole onto the plane $x_3 = 0$.

Furthermore, $\mathbb{C}\mathrm{P}^1$ is linked to $\mathbb{C}$ through homogeneous coordinates (z_1, z_2), where the inhomogeneous coordinate is given by $z = \frac{z_1}{z_2} \in U \subset \mathbb{C}\mathrm{P}^1$. This identification aligns with the standard representation of qubits in

$\mathbb{C}^2$. The stereographic projection from the north and south poles naturally defines two coordinate charts on $SU(2)$, corresponding to the inhomogeneous coordinates z and $\frac{1}{z} \equiv \overline{z}$, satisfying $\overline{z}z = 1$. Note that this two-chart structure underlies the transition functions essential for a consistent geometric quantization framework, and the same holds true for our approach here.

By treating the inhomogeneous coordinates as equivalence classes, i.e., $(z_1, z_2)^T \equiv (z, 1)^T$, one observes that the $SU(2)$-action $\ell_{\tilde{g}}$ on $\mathbb{C}\mathrm{P}^1$ is of Möbius transformations

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} \alpha z + \beta \\ -\overline{\beta} z + \overline{\alpha} \end{pmatrix} \equiv \begin{pmatrix} \frac{\alpha z + \beta}{-\overline{\beta} z + \overline{\alpha}} \\ 1 \end{pmatrix}; \ \alpha, \beta \in \mathbb{C}, \tag{11}$$

where $\alpha\overline{\alpha} + \beta\overline{\beta} = 1$ and $-\overline{\beta}z + \overline{\alpha} \neq 0$, integrating a nontrivial phase factors. The Hamiltonian vector fields of $\tilde{S}$ (HamVF($\tilde{S}$)) can then be constructed through the similar method as S^2 to obtain

$$\tilde{\xi}_1 = \frac{i}{2}(1 - z^2)\frac{\partial}{\partial z} - \frac{i}{2}(1 - \bar{z}^2)\frac{\partial}{\partial \bar{z}}, \quad \tilde{\xi}_2 = \frac{1}{2}(1 + z^2)\frac{\partial}{\partial z} + \frac{1}{2}(1 + \bar{z}^2)\frac{\partial}{\partial \bar{z}}, \quad \tilde{\xi}_3 = iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial \bar{z}}. \tag{12}$$

Alternatively, (12) can also be constructed by substituting the OPS elements $\tilde{g} := \exp(-\frac{i\theta\sigma}{2})$ (where $\sigma = \{\sigma_j\}_{j=1}^3$ are elements of Lie algebra $\mathfrak{su}(2)$) into (11). The commutator algebra of (12) is equivalent to (6) and corresponds to the Lie algebra $\mathfrak{su}(2)$, meanwhile the Poisson bracket algebra of observables (10) is equivalent to (7). In fact, at the face value, locally their algebraic structures are isomorphic.

In holomorphic quantization, choosing the natural holomorphic polarization allows one to construct inequivalent irreducible unitary representations of $SU(2)$ by its action on a geometric space of states: holomorphic wavefunctions interpreted as sections of a line bundle over the classical phase space. We consider first the nontrivial Hopf bundle,

$$U(1) \simeq S^1 \longrightarrow S^3 \longrightarrow S^2 \simeq \mathbb{C}\mathrm{P}^1, \tag{13}$$

as our principal $U(1)$-bundle (with the first Chern number $\frac{i}{2\pi}\int_{\mathbb{C}\mathrm{P}^1} \Omega = 2\pi$), then we define an associated vector bundle[2] over $\mathbb{C}\mathrm{P}^1$,

$$\mathbb{C} \longrightarrow \mathcal{L} := S^3 \times_{U(1)} \mathbb{C} \longrightarrow \mathbb{C}\mathrm{P}^1, \tag{14}$$

where for all $\lambda \in U(1)$ there is an equivalence relation $(w\lambda, \mathcal{U}(\lambda^{-1})v) \sim (w = (w_1, w_2), v)$ that gives an equivalence class $[w, v]$ to be the element of $\mathcal{L}$ (for more details refer [26]). The space of local sections σ of the local trivialization $\pi^{-1}(U) \simeq U \times \mathbb{C}$ of $\mathcal{L}$ is given by

$$\Gamma_{\text{hol}}(U \times \mathbb{C}) \simeq \{\psi : U \subset \mathbb{C}\mathrm{P}^1 \longrightarrow \mathbb{C} \big| \psi(z) = \mathcal{U}(\lambda^{-1})\psi(z)\}, \tag{15}$$

will be the representation space for $\tilde{\mathcal{G}}$. Since we have the Möbius action in (11), this action has a nontrivial lift $\ell_{\tilde{g}}^{\uparrow}$ to the $\mathcal{L}$, as shown in the following commutative diagram

$$\begin{array}{ccccc} \mathbb{C} & \longrightarrow & \mathcal{L} & \xrightarrow{\ \ell_{\tilde{g}}^{\uparrow}\ } & \mathcal{L} \\ & & \downarrow{\scriptstyle \pi_{\mathbb{C}}} & & \downarrow{\scriptstyle \pi_{\mathbb{C}}} \\ & & \mathbb{C}\mathrm{P}^1 & \xrightarrow{\ \ell_{\tilde{g}}\ } & \mathbb{C}\mathrm{P}^1 \end{array}$$

such that $\ell_{\tilde{g}} \cdot \pi_{\mathbb{C}} = \pi_{\mathbb{C}} \cdot \ell_{\tilde{g}}^{\uparrow}$. Here, the lift $\ell_{\tilde{g}}^{\uparrow}$-action give rise to an action of the $U(1)$ structure group of $\mathcal{L}$ on the fibers v in terms of complex variables,

$$\ell_{\tilde{g}}^{\uparrow}\Big([w, v]\Big) \longmapsto \Big[(w_1', w_2'), \Big(\frac{\beta w + \overline{\alpha}}{|\beta w + \overline{\alpha}|}\Big) \cdot v\Big)\Big], \tag{16}$$

where $w_1' := \alpha w_1 + \beta w_2, w_2' := -\overline{\beta} w_1 + \overline{\alpha} w_2 \in S^3; \tilde{g} \in SU(2)$. In the following argument, the structure group of the fibers v will appear as the phase factor of the wavefunctions in the representation operators.

[2]The UIR of $U(1)$ is indexed by its character i.e., $\chi(\lambda) = \lambda^n; n \in \mathbb{Z}$ and such an integer n topologically is related to the first Chern number.

Thus, through the $\ell^{\uparrow}_{\tilde{g}}$-action on the space of local sections (15), one has the following representation of $\tilde{\mathcal{G}}$

$$(\mathcal{U}_{\tilde{g}}\psi)(z) = (\beta z + \overline{\alpha})\psi\Big(\frac{\alpha z - \overline{\beta}}{\beta z + \overline{\alpha}}\Big), \tag{17}$$

where $\psi(z) = \sum_{j=-l}^{l} c_j z^{l+j}$ (c_j are complex coefficients) is the holomorphic wavefunction (see Figure 2), and (17) is the simplest form of homogenized polynomial. By substituting $\tilde{g} := \exp(-\frac{i\theta\sigma}{2})$ into (17) one gets the following representation of $\tilde{\mathcal{G}}$,

$$(\mathcal{U}_{\tilde{g}_1}\psi)(z) = (i\sin\frac{\theta_1}{2}z + \cos\frac{\theta_1}{2})\psi\Big(\frac{\cos\frac{\theta_1}{2}z + i\sin\frac{\theta_1}{2}}{i\sin\frac{\theta_1}{2}z + \cos\frac{\theta_1}{2}}\Big), \tag{18}$$

$$(\mathcal{U}_{\tilde{g}_2}\psi)(z) = (\sin\frac{\theta_2}{2}z + \cos\frac{\theta_2}{2})\psi\Big(\frac{\cos\frac{\theta_2}{2}z - \sin\frac{\theta_2}{2}}{\sin\frac{\theta_2}{2}z + \cos\frac{\theta_2}{2}}\Big), \tag{19}$$

$$(\mathcal{U}_{\tilde{g}_3}\psi)(z) = e^{-\frac{i\theta_3}{2}}\psi(e^{i\theta_3}z), \tag{20}$$

and a set of "spin angular momentum" operators is constructed as follows:

$$\hat{S}_1 = \frac{z}{2} + \frac{1}{2}(1 - z^2)\frac{\partial}{\partial z}, \hat{S}_2 = -\frac{iz}{2} + \frac{i}{2}(1 + z^2)\frac{\partial}{\partial z}, \hat{S}_3 = -\frac{1}{2} + z\frac{\partial}{\partial z}. \tag{21}$$

It is the holomorphic part (after polarization) of the constructed vector fields (12) for the system appended with a connection-type term. Moreover, its Casimir operator is $\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 = \frac{3}{4}\hat{\mathbb{I}}$, and the eigenfunctions of $\hat{S}_3$ are $c_{-\frac{1}{2}}$ and $c_{\frac{1}{2}}z$. Eq. (21) obeys the commutation relation

$$[\hat{S}_j, \hat{S}_k] = i\varepsilon_{jkl}\hat{S}_l. \tag{22}$$

Furthermore, one can generalize the representation by noting that the Hopf bundle (13) is nontrivial (first Chern number $2\pi n; n \in \mathbb{Z}$), hence the representation (17) becomes

$$(\mathcal{U}^l_{\tilde{g}}\psi)(z) = (\beta z + \overline{\alpha})^{2l}\psi\Big(\frac{\alpha z - \overline{\beta}}{\beta z + \overline{\alpha}}\Big), \tag{23}$$

and the general holomorphic part of the constructed vector fields after polarization also serves in part as general spin angular momentum operators for the system associated with a general connection-type term l are

$$\hat{S}_1 = lz + \frac{1}{2}(1 - z^2)\frac{\partial}{\partial z}, \quad \hat{S}_2 = -ilz + \frac{i}{2}(1 + z^2)\frac{\partial}{\partial z}, \quad \hat{S}_3 = -l + z\frac{\partial}{\partial z}, \tag{24}$$

obeying a similar commutation relation as (22). Thereafter, one could classify the wavefunctions as in standard $SU(2)$ eigenfunctions with the help of a Casimir operator $l(l+1)$. Throughout (24), the ladder operators are obtained

$$\hat{S}_+ = 2lz - z^2\frac{\partial}{\partial z}, \quad \hat{S}_- = \frac{\partial}{\partial z}; \tag{25}$$

satisfying

$$[\hat{S}_+, \hat{S}_-] = 2\hat{S}_3, \quad [\hat{S}_3, \hat{S}_\pm] = \pm\hat{S}_\pm. \tag{26}$$

In each case, the eigenfunction of $\hat{S}_3$ is a monomial z^{l+j} of $\psi(z) = \sum_{j=-l}^{l} c_j z^{l+j}$, and can use standard methods [43] to generate representations of $SU(2)$ that will be discussed shortly after this. Let us write the monomial $z^{l+j} \equiv \psi^l_j$ (that will be used interchangeably after this), thus the action of the $\hat{S}$'s on the monomial ψ^l_j produces $(l \mp j)\psi^l_{j\pm1}$, therefore

$$\hat{S}_\pm\psi^l_j = \sqrt{(l \pm j + 1)(l \mp j)}\psi^l_{j\pm1}, \quad \hat{S}_3\psi^l_j = j\psi^l_j, \tag{27}$$

where ψ_j^l is the canonical basis, which treats the raising and lowering operators in a manifestly similar way to the standard angular operators in QM. Therefore, from all the above constructions it is straightforward to deduce that the eigenvalue j is generally,

$$j \in \{-l, -l+1, ..., l-1, l\};\ l := \frac{n}{2}; n \in \mathbb{Z}.$$

To complete the quantization, we introduce a Hermitian vector bundle (which is an inner product as in standard QM). Choose

$$\psi_j^l = \frac{z^{l+j}}{\sqrt{(l+j)!(l-j)!}}; \quad -l \leq j \leq l, \tag{28}$$

as an orthonormal basis, thus for any holomorphic wavefunctions in the Hilbert space, the inner product can be written as the holomorphic integral

$$\frac{i}{2\pi} \sum_{j=-l}^{l} \int_{\mathbb{C}\mathrm{P}^1} \bar{c_j} c_j \frac{(l+j)!(l-j)!}{2l!} \Omega_{(n)}, \tag{29}$$

where $\Omega_{(n)} = \frac{2in\, dz \wedge d\bar{z}}{(1+z\bar{z})^2}$ is the integration over $\mathbb{C}\mathrm{P}^1$ as the Hilbert space measure.

From (28), we shall find matrix entries

$$\mathcal{U}_{kj}^l(\tilde{g}) = \langle e_k, \mathcal{U}_{\tilde{g}}^l e_j \rangle, \tag{30}$$

where e_j is a basis that is equivalent to ψ_j^l. Notice that the inner product (29) can be expressed by means of the differential operators

$$D_k^l = \sum_{k=-l}^{l} (l-k)! c_k \left(\frac{d}{dz}\right)^{l+k} \bigg|_{z=0}; \quad -l \leq k \leq l \tag{31}$$

associated with the monomial z^{l+j}. The inner product is given by

$$\left\langle D_k^l, \psi_j^l \right\rangle \Big|_{z=0},$$

and by using this new form of inner product, the kj-th matrix element (the explicit derivation can be referred to [43]) is given in terms of Euler angles as a product of an exponential function, trigonometric terms, and Jacobi polynomial,

$$\begin{aligned} \mathcal{U}_{kj}^l(\theta_3, \theta_2, \theta_3') = \frac{(-1)^{l+k}}{2^l} \sqrt{\frac{(l+k)!}{(l-k)!(l-j)!(l+j)!}} e^{i(j\theta_3 + k\theta_3')} \\ \times (\sin\frac{\theta_2}{2})^{j-k} (\cos\frac{\theta_2}{2})^{-(k+j)} P_{l+k}^{(j-k;-(k+j))}(\cos\theta_2). \end{aligned} \tag{32}$$

where $P_{l+k}^{(j-k;-(k+j))}(\cos\theta_2)$ is equivalent to the Rodrigues' formula [44]. Therefore, equation (32) represents the unitary operator in terms of numerical functions compared to (23). However, it is interesting to discuss the wavefunctions of qubits operations through the latter.

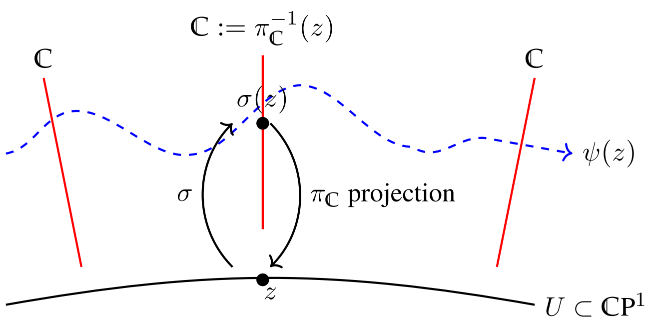

Figure 2. Holomorphic wavefunction from local section of $\mathcal{L}$.

4. Single-Qubit Gates in the Holomorphic Formalism

Next, we determine the representation corresponding to the single-qubit unitary operation using (18)–(20). As an example, the construction of Hadamard gate H by taking $\tilde{g}_1, \tilde{g}_2$ from OPS elements $\{\tilde{g}_1, \tilde{g}_2, \tilde{g}_3\}$ of $\tilde{g} := \exp(-i\frac{\theta\sigma}{2}) \in SU(2)$ yields

$$\tilde{g}_H = \begin{pmatrix} \cos(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) + i\sin(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) & i\cos(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) + \sin(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) \\ -\sin(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) + i\cos(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) & -i\sin(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) + \cos(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) \end{pmatrix}. \tag{33}$$

By substituting (33) into (17) and taking $\theta_2 = \frac{\pi}{2}, \theta_1 = \pi$, we obtain the representation of H on $\mathbb{C}\mathrm{P}^1$

$$(\mathcal{U}_{\tilde{g}_H}\psi)(z) = (z-1)\psi\left(\frac{z+1}{z-1}\right) \simeq \psi\left(\frac{z+1}{z-1}\right), \tag{34}$$

that is $z \longrightarrow \frac{z+1}{z-1}$. Then setting $\theta = \frac{\pi}{4}, \phi = \{0, \pi\}$ gives fixed points $\{1 \pm \sqrt{2}\}$. In the standard understanding, these transformations act on the basis states $|0\rangle$ and $|1\rangle$, but here it is represented by the stereographic coordinates z: $|0\rangle$ corresponds to $z = \infty$ and $|1\rangle$ corresponds to $z = 0$. Explicitly, for (34) if $z = \infty$, we obtain

$$\psi\left(\frac{z+1}{z-1}\right) = \psi\left(\frac{\infty+1}{\infty-1}\right) = 1, \tag{35}$$

corresponding to $|0\rangle \longmapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, and if $z = 0$, we obtain

$$\psi\left(\frac{z+1}{z-1}\right) = \psi\left(\frac{0+1}{0-1}\right) = -1, \tag{36}$$

which corresponding to $|1\rangle \longmapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}}$. Moreover, the fixed points align with the eigenstates of the corresponding standard Hadamard logic gate, i.e., recall

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

to find its eigenstates, one solves the eigenvalue equation $H|\psi\rangle = \lambda|\psi\rangle$ and yields two normalized eigenstates corresponding to eigenvalues $\lambda = \pm 1$:

$$|\psi_+\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}, \quad |\psi_-\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}. \tag{37}$$

These eigenstates lie on irrational points of the Riemann sphere and are fixed under Möbius transformations associated with H.

Furthermore, the wavefunction of the phase shift gate that shifts the phase of the $|0\rangle$ state relative to the $|1\rangle$ state, as in a common single-qubit formalism that modifies the phase of the quantum state, can be constructed as follows. Recall (20) and set $\theta_3 = \pi$ thus it yields the S-gate wavefunction

$$(\mathcal{U}_{\tilde{g}_3}\psi)(z) = \psi(iz), \tag{38}$$

where the $z \mapsto \psi(iz)$ fixing $x_3 = 0$ that is a π radian rotation around z-axis on $\mathbb{C}\mathrm{P}^1$. This is corresponding to the common S-gate on the Bloch sphere. And if we set $\theta_3 = \frac{\pi}{2}$ thus it yields the wavefunction of T-gate as follows:

$$(\mathcal{U}_{\tilde{g}_3}\psi)(z) = \frac{1}{\sqrt{2}}\psi(z + iz), \tag{39}$$

that is $\frac{\pi}{4}$ radian rotation on z-axis fixing $z = 0$. Geometrically, this is equivalent to tracing a horizontal fiber (or circle) of the total space over $\mathbb{C}\mathrm{P}^1$ along the z-axis by θ_3.

In conclusion, the S- and T-gates only fix 0, acting as pure (fiber) phase shifts. The Hadamard gate is the most nontrivial, fixing two intermediate points, and the Pauli gates are simple reflections, with clear fixed points at geometrically well-defined locations. All of these results confirm that Möbius transformations on $\mathbb{C}\mathrm{P}^1$ correctly encode single-qubit gates, and the representation remains valid within the holomorphic wavefunction framework constructed via sections of the line bundle over $\mathbb{C}P^1$. Table 1 summarizes the correspondence between the common gates and our gates' holomorphic wavefunction, and Figure 3 shows the fixed points of quantum gate wavefunctions on $\mathbb{C}P^1$.

These results provide a novel perspective on qubit geometry using the wave mechanics formalism derived from Isham's canonical group quantization on a non-cotangent bundle phase space. The Möbius action on holomorphic wavefunctions naturally encodes single-qubit operations, yielding representations that align with standard quantum gates while offering a geometric interpretation of quantum transformations. This framework confirms the validity of standard quantum gates in a holomorphic setting and highlights the role of complex analytic structures in quantum computation.

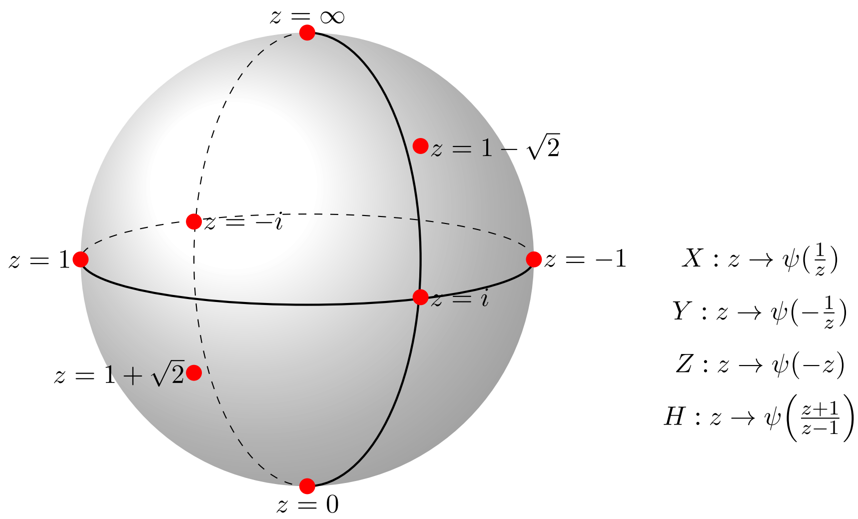

Figure 3. Fixed points of quantum gate wavefunction on $\mathbb{C}P^1$

Table 1. Geometric interpretation of quantum gates via their holomorphic wavefunctions, including their action on the Riemann sphere, fixed points, and induced transformations.

Common Gate	**Symbol**	**Gates' Wavefunctions**	**Fixed Points**	**Corresponding Eigenstates**
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	I	$\psi(z)$	Every point on $\mathbb{C}P^1$	Every point on $\mathbb{C}P^1$
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	X	$\psi(\frac{1}{z})$	$\{\pm 1\}$	$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$
$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$	Y	$\psi(-\frac{1}{z})$	$\{\pm i\}$	$\left\{\begin{pmatrix} 0 \\ i \end{pmatrix}, \begin{pmatrix} -i \\ 0 \end{pmatrix}\right\}$
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Z	$-\psi(z)$	$\{\infty, 0\}$	$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$
$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$	H	$\psi(\frac{z+1}{z-1})$	$\{1 \pm \sqrt{2}\}$	$\left\{\begin{pmatrix} 1 \pm \sqrt{2} \\ 1 \end{pmatrix}\right\}$
$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$	S	$\psi(iz)$	$\{0\}$	$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix}\right\}$
$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$	T	$\frac{1}{\sqrt{2}}\psi(z + iz)$	$\{0\}$	$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^{i\pi/4} \end{pmatrix}\right\}$
$\begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$	$R_X(\theta)$	$\psi\left(\frac{\cos\frac{\theta_1}{2}z + i\sin\frac{\theta_1}{2}}{i\sin\frac{\theta_1}{2}z + \cos\frac{\theta_1}{2}}\right)$	$\{\pm 1\}$	Similar as X-gate
$\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$	$R_Y(\theta)$	$\psi\left(\frac{\cos\frac{\theta_2}{2}z - \sin\frac{\theta_2}{2}}{\sin\frac{\theta_2}{2}z + \cos\frac{\theta_2}{2}}\right)$	$\{\pm i\}$	Similar as Y-gate
$\begin{bmatrix} e^{\frac{-i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{bmatrix}$	$R_Z(\theta)$	$\psi(e^{i\theta_3}z)$	$\{\infty, 0\}$	Similar as Z-gate

5. Conclusion and Further Outlook

In this work, we have introduced a wave mechanics formalism for qubit geometry based on holomorphic functions and Mobius transformations. By treating the Riemann sphere $\mathbb{C}\mathrm{P}^1$ as a non-cotangent bundle phase space, we applied holomorphic quantization to construct a natural representation of quantum states. This approach led to a formulation of spin angular momentum operators that reproduce the standard $SU(2)$ algebra while providing new geometric insights into the evolution of quantum state. We demonstrated how the standard single-qubit gates act as Mobius transformations on holomorphic wavefunctions fundamentally. This interpretation offers a novel geometric perspective on quantum computation, shedding light on the implications of geometric properties of Möbius transformations on quantum gates and their corresponding eigenstates. The results can be translated to just how quantum theory benefits from multiple equivalent formulations, that is operator, path integral, or phase-space, and quantum information can similarly benefit from holomorphic and geometric reformulations that reveal structural insights invisible in standard matrix representations.

For future outlook, firstly, on the quantum gates' wavefunctions, one could also raise other geometric properties of these transformations, e.g., exact 3-transitivity and invariance of lines and circles on the Riemann sphere and find out what implications they have on the logic gates. Secondly, the current work is an initial step in quantizing the more general case of $\mathbb{C}\mathrm{P}^n$, which poses an even more difficult quantization in a more general structure. One could generalize the technique for qubit to qudit by using the generalized Hopf fibration,

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}\mathrm{P}^n,$$

via parametrization of S^n to find separable coordinates for $\mathbb{C}\mathrm{P}^n$. Here, we have lost the spherical character of its base space in general to take advantage, hence it is necessary to utilize the higher dimensional sphere S^{2n+1} to find its separable coordinates for characterizing the separable coordinates on $\mathbb{C}\mathrm{P}^n$ that comes from Cartan subalgebra of $SU(n+1)$ and the rest from Casimir operators of different $SU(2)$ subalgebras [45]. Thirdly, one knows that qubits can also combine to form higher-dimensional qudits. Mathematically, this translates into the problem of how the multiple $\mathbb{C}\mathrm{P}^1$ are combined to form the higher-dimensional complex projective spaces. Formally, it is given by

$$SP^n(\mathbb{C}\mathrm{P}^1) = \mathbb{C}\mathrm{P}^1/S_n$$

where SP^n stands for the symmetric product of $\mathbb{C}\mathrm{P}^1$ and S_n is the symmetric group [46,47]. As such, it is interesting to further explore that the symplectic form $\sum_{j=0}^{n} dz_j \wedge d\overline{z}_j$ of $\mathbb{C}\mathrm{P}^n$ is invariant under any change of z_j and $\overline{z}_j$ that should be reflected in the results of its quantization which physically could be used to describe quantum entanglement.

Author Contributions

A.H.A.S. conceptual, methodology, formal analysis, investigation, writing original draft. N.M.S. investigation, validation, funding acquisition, review, and editing. U.A.H validation, reviewed drafts of the paper. H.Z. conceptual, review, and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest Statement

All authors declare no conflicts of interest.

Data Availability Statement

All data generated or analyzed for this work are included in this published article.

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