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Enumeration of inversion sequences according to the outer and inner perimeter

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Abstract

The integer sequence $\pi = \pi_1 \cdots \pi_n$ is said to be an *inversion sequence* if $0 \leq \pi_i \leq i-1$ for all i. Let $\mathcal{I}_n$ denote the set of inversion sequences of length n, represented using positive instead of non-negative integers. We consider here two new statistics defined on the bargraph representation $b(\pi)$ of an inversion sequence π which record the number of unit squares touching the boundary of $b(\pi)$ and that are either exterior or interior to $b(\pi)$. We denote these statistics on $\mathcal{I}_n$ recording the number of outer and inner perimeter squares respectively by oper and iper. In this paper, we study the distribution of oper and iper on $\mathcal{I}_n$ and also on members of $\mathcal{I}_n$ that end in a particular letter. We find explicit formulas for the maximum and minimum values of oper and iper achieved by a member of $\mathcal{I}_n$ as well as for the average value of these parameters. We make use of both algebraic and combinatorial arguments in establishing our results.

Mathematics Subject Classifications (2020). 05A15, 05A05

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1 Introduction

Let $\sigma = \sigma_1 \cdots \sigma_n$ be a permutation of $[n] = \{1, \ldots, n\}$, represented using the one-line notation. Define the sequence $\mathbf{a} = a_1 \cdots a_n$, where a_i records the number of elements of $[i-1]$ occurring to the right of the letter i in σ for $1 \leq i \leq n$. Then $\mathbf{a}$ is called the *inversion sequence* (or *inversion table*) of σ (see, e.g., [20, p. 21]). For example, $\sigma = 364215 \in S_6$ has inversion sequence $\mathbf{a} = 012204$; note that $0 \leq a_i \leq i-1$ for all i. Conversely, starting with $\mathbf{a}$, one can easily reconstruct the corresponding permutation σ. For our purposes, we will add 1 to each entry of $\mathbf{a}$ since it will be more convenient to represent the resulting sequence geometrically. The enumeration of inversion sequences satisfying various restrictions has been an ongoing object of interest in combinatorics. For example, the pattern avoidance problem on inversion sequences

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has been studied from several perspectives, initiated in the papers [14] and [7] concerning the classical avoidance of a single permutation or word pattern of length three, in analogy with the comparable problem on permutations. For extensions of this work in various directions, see, e.g., [6, 8, 9, 10, 18, 21, 22].

Recall that a bargraph is a self-avoiding random walk in the first quadrant starting at the origin and ending at $(n, 0)$ consisting of up $(0, 1)$, down $(0, -1)$ and horizontal $(1, 0)$ steps. The bargraph representation $b(\tau)$ of a sequence $\tau = \tau_1 \cdots \tau_n$ of positive integers is obtained by requiring that the number of unit squares in the i-th column of $b(\tau)$ be given by τ_i for $1 \leq i \leq n$ (i.e., the height above the x-axis of the i-th horizontal step of $b(\tau)$ equals τ_i). Many different parameters have been considered on bargraphs representing various kinds of sequences τ; see, for example, the review paper [13] and references contained therein. Let $\mathcal{I}_n$ denote the set of inversion sequences $\pi = \pi_1 \cdots \pi_n$ of length n, represented using positive integers (i.e., $1 \leq \pi_i \leq i$ for all i). Here, we wish to consider some new parameters on $\mathcal{I}_n$ that are defined geometrically in terms of $b(\pi)$. For other recent parameters considered on $\mathcal{I}_n$, see, e.g., [1, 4, 15, 16].

Given $\pi \in \mathcal{I}_n$, define the *outer (site-) perimeter* as the number of unit squares exterior to $b(\pi)$ that have at least one side which borders the boundary of $b(\pi)$ (including possibly the bottom boundary of $b(\pi)$ flush with the x-axis). We define the *inner (site-) perimeter* in the same way as the outer except that the squares in question are contained within the bargraph $b(\pi)$. Denote by $\mathrm{oper}(\pi)$ and $\mathrm{iper}(\pi)$ the outer and inner perimeter, respectively, of $\pi \in \mathcal{I}_n$. For example, if $\pi = 121345283419 \in \mathcal{I}_{12}$, then we have $\mathrm{oper}(\pi) = 51$ and $\mathrm{iper}(\pi) = 38$; see Figure 1 below, where the outer and inner perimeter squares of π are shaded or indicated by a circle. The oper statistic was originally considered on arbitrary bargraphs (which are synonymous with compositions) in [5], where it is referred to as just the *site-perimeter*, and was later studied on k-ary words [3] and finite set partitions [12] both represented geometrically as bargraphs, the latter via restricted growth sequences. The iper distribution on compositions was studied in [2] where a generating function formula was found, a result which was refined in [11]. We use here the descriptors *outer* and *inner* to distinguish further the oper and iper parameters on $\mathcal{I}_n$.

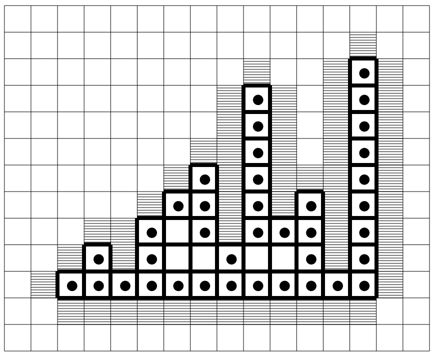

Figure 1: The outer and inner perimeter of $\pi = 121345283419 \in \mathcal{I}_{12}$

Let $\mathrm{per}(\pi)$ and $\mathrm{area}(\pi)$ denote the perimeter and area of $b(\pi)$ for $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_n$ whose distributions on $\mathcal{I}_n$ were studied in [15]. Then, by the definitions, we have $\mathrm{oper}(\pi) \leq \mathrm{per}(\pi)$ and $\mathrm{iper}(\pi) \leq \mathrm{area}(\pi)$ for all $\pi \in \mathcal{I}_n$, with equality in the first inequality only when $\pi = 11 \cdots 1$ and equality in the second only when $\min\{\pi_i, \pi_{i+1}, \pi_{i+2}\} = 1$ or 2 for all $1 \leq i \leq n-2$, where $\pi_{i+1} = 2$ if the minimum is 2. Further, by an induction on j, where j denotes the greatest column height within a member of $\mathcal{I}_n$, one can show $\mathrm{iper}(\pi) + 4 \leq \mathrm{oper}(\pi)$ for all $\pi \in \mathcal{I}_n$ and $n \geq 2$. Note that equality is achieved in the last inequality when $\pi = 12^{n-1}$ (among other sequences), and hence

$c = 4$ is best possible among all constants c in inequalities of the form $\mathrm{iper}(\pi) + c \leq \mathrm{oper}(\pi)$ for all π.

The organization of this paper is as follows. In the next section, we study the distribution of the oper statistic on $\mathcal{I}_n$ and also on members of $\mathcal{I}_n$ ending in a particular letter. To do so, we consider auxiliary generating functions for the distribution of oper on certain subsets of $\mathcal{I}_n$, namely, those obtained by considering whether the last letter of a member of $\mathcal{I}_n$ is greater than, less than or equal to its predecessor. This enables one to translate the recurrences for the distribution into a system of functional equations satisfied by the generating functions. As corollaries of this analysis, one obtains explicit formulas for the maximum, minimum and average values of oper on $\mathcal{I}_n$ as well as for the sign balance (corresponding to the case $q = -1$). Direct combinatorial proofs can then be given for these explicit formulas which do not make use of recurrences or generating functions. A comparable treatment is provided for the iper parameter on $\mathcal{I}_n$ in the third section. The final section is an appendix devoted to establishing a functional equation for a generating function related to the oper distribution on $\mathcal{I}_n$.

2 Distribution of outer perimeter

Given $n \geq 2$ with $1 \leq i \leq n-1$ and $1 \leq j \leq n$, let $\mathcal{I}_{n,i,j}$ denote the subset of $\mathcal{I}_n$ whose members end in i, j and let $\mathcal{I}_{n,j} = \cup_{i=1}^{n-1} \mathcal{I}_{n,i,j}$. Let $a(n, i, j)$ denote the distribution of the outer perimeter statistic on $\mathcal{I}_{n,i,j}$ and let $a(n, j) = \sum_{i=1}^{n-1} a(n, i, j)$ for $n \geq 2$ be the corresponding distribution on $\mathcal{I}_{n,j}$, with $a(1, 1) = q^4$. Let $a(n, i, j) = a(n, j) = 0$ in all cases where the set over which the distribution is taken is empty. Throughout, we will represent $\pi \in \mathcal{I}_n$ as a bargraph and frequently write π in place of $b(\pi)$, by a slight abuse of notation.

For example, when $n = 4$, we have $a(4, 1) = 4q^{10} + q^{11} + q^{12}$, $a(4, 2) = 5q^{11} + q^{12}$, $a(4, 3) = 3q^{12} + 3q^{13}$ and $a(4, 4) = q^{13} + 3q^{14} + 2q^{15}$; see Figure 2, which gives the outer perimeter of each member of $\mathcal{I}_4$.

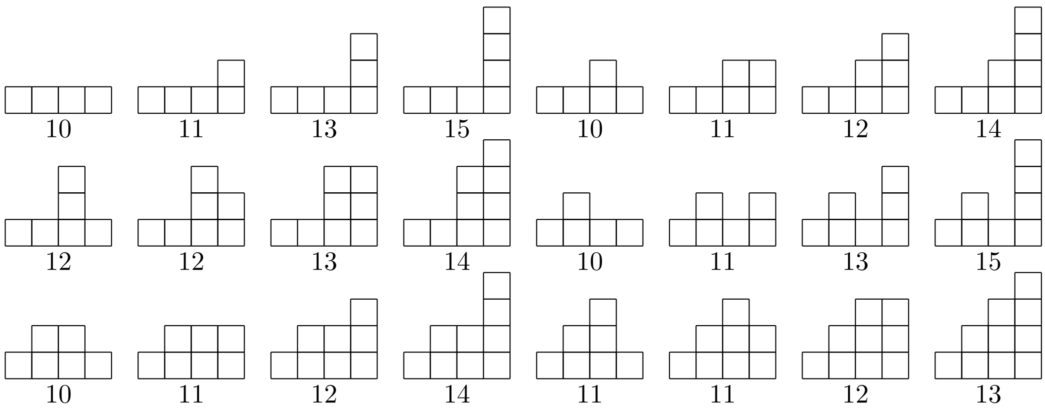

Figure 2: The outer perimeter of inversion sequences of length 4

It is possible to determine the $a(n, i, j)$, and hence the $a(n, j)$, recursively as follows.

LEMMA 1 *If $n \geq 3$, then*

$$a(n,i,j) = qa(n-1,i), \quad 1 \leq j < i \leq n-1, \tag{1}$$

$$a(n,i,i) = q^2 a(n-1,i), \quad 1 \leq i \leq n-1, \tag{2}$$

$$a(n,i,j) = \sum_{k=1}^{i} q^{2j-2i+1} a(n-1,k,i) + \sum_{k=i+1}^{j-1} a^{2j-k-i+2} a(n-1,k,i) + \sum_{k=j}^{n-2} q^{j-i+2} a(n-1,k,i), \tag{3}$$

for $1 \leq i < j \leq n$, with initial values $a(1,1) = q^4$, $a(2,1) = a(2,1,1) = q^6$ and $a(2,2) = a(2,1,2) = q^7$.

Proof. The initial conditions for $n = 1, 2$ are easily verified, so assume $n \geq 3$. Before proceeding further, we define the following terms. Let us refer to the set of j outer perimeter squares directly to the right of the n-th column of $\sigma \in \mathcal{I}_{n,j}$ as the set of *right lateral squares* of σ, denoted by $\text{rlat}(\sigma)$. Further, we refer to a square lying directly above or below some column of σ as a *high* or *low boundary* square of σ, respectively. Suppose $\pi \in \mathcal{I}_{n,i,j}$ is obtained from $\rho \in \mathcal{I}_{n-1,i}$ by appending a column of size j to ρ. We consider cases based on the relative sizes of i and j, first assuming $i > j$. Note in this case that appending a column of size j to ρ has the effect of replacing the j lowest squares of $\text{rlat}(\rho)$ by those of $\text{rlat}(\pi)$, with the $i - j$ top squares in $\text{rlat}(\rho)$ still part of the outer perimeter of π. Taking into account the low boundary square in the last column of π, we get $\text{oper}(\pi) = \text{oper}(\rho) + 1$ for all π and ρ, which implies (1). If $i = j$, then $\text{rlat}(\rho)$ is completely replaced by $\text{rlat}(\pi)$, with only the high and low boundary squares in the last column of π accounting for the difference in oper parameter values, which implies (2).

To prove (3), assume $i < j$ and we consider cases based on the penultimate letter k of ρ, where $1 \leq k \leq n-2$. If $1 \leq k \leq i$, then appending a column of size j to ρ results in their being $j - i - 1$ new outer perimeter squares in column $n - 1$ lying directly above the high boundary square in that column. Further, taking into account the high and low boundary squares of π in column n and the fact that π has $j - i$ more right lateral squares than ρ as $j > i$, we get $\text{oper}(\pi) - \text{oper}(\rho) = 2j - 2i + 1$ for all π and ρ, regardless of the value of $k \in [i]$. Considering all possible k then yields a contribution of $\sum_{k=1}^{i} q^{2j-2i+1} a(n-1,k,i)$ towards $a(n,i,j)$ in this case. If $i + 1 \leq k \leq j - 1$, then the difference $\text{oper}(\pi) - \text{oper}(\rho)$ can be attributed to outer perimeter squares of the following three types: (i) those that arise from the top $j - k$ squares in column n of π, each of which contributes two new outer perimeter squares (to its left and right), (ii) the ones that arise from the $k - i$ squares of π lying directly below those in (i), each of which contributes a single new square (to its right), and (iii) the high and low boundary squares in the final column of π. Combining these cases gives

$$\text{oper}(\pi) - \text{oper}(\rho) = 2(j-k) + k - i + 2 = 2j - k - i + 2,$$

for each π and ρ, and considering all k yields a contribution of $\sum_{k=i+1}^{j-1} a^{2j-k-i+2} a(n-1,k,i)$. Finally, if $j \leq k \leq n-2$, then the difference in oper parameter values comes about from the top $j - i$ right lateral squares of π, together with the high and low boundary squares in the final column. This gives $\text{oper}(\pi) - \text{oper}(\rho) = j - i + 2$ for all π and ρ in this case. Considering all possible k then yields the third summation on the right side of (3) and completes the proof. $\square$

Let $a(n) = \sum_{i=1}^{n-1} \sum_{j=1}^{n} a(n,i,j) = \sum_{j=1}^{n} a(n,j)$ for $n \geq 2$, with $a(1) = q^4$. Then $a(n)$ gives the distribution on all of $\mathcal{I}_n$ for the outer perimeter statistic. For example, from the formulas above for $a(4,j)$, we have

$$a(4) = \sum_{j=1}^{4} a(4,j) = 4q^{10} + 6q^{11} + 5q^{12} + 4q^{13} + 3q^{14} + 2q^{15}.$$

Note that $a(n) = n!$ for all $n \geq 1$ when $q = 1$. We wish to determine the (ordinary) generating function for $a(n)$. In order to do so, we consider the following three auxiliary generating functions:

$$AE(x,v) = \sum_{n\geq 2}\sum_{i=1}^{n-1} a(n,i,i)x^n v^{i-1},$$
$$AN(x,v,u) = \sum_{n\geq 3}\sum_{j=1}^{n-2}\sum_{i=j+1}^{n-1} a(n,i,j)x^n v^{i-1}u^{i-j-1},$$
$$AP(x,v,u) = \sum_{n\geq 2}\sum_{j=2}^{n}\sum_{i=1}^{j-1} a(n,i,j)x^n v^{i-1}u^{j-i},$$

which are introduced so that one can translate the recurrences in Lemma 1. Note that the distribution for oper on $\mathcal{I}_n$ for $n \geq 1$ then has generating function given by $q^4x + AE(x,1) + AN(x,1,1) + AP(x,1,1)$.

We have the following functional equation satisfied $AE(x,v)$ for general q and v, which is derived in the appendix.

Theorem 1 *The generating function $AE(x,v)$ satisfies*

$$\begin{aligned}
&\frac{(1-v)(1-q^2v)+q(1-q)(q^2v^2-q^2v+qv+1)x+q^5vx^2}{1-v}AE(x,v)\\
&=\frac{qx(q^6vx-q^2+1)}{q^2-1}AE(q^2vx,\frac{1}{q^2})+\frac{qx(q^4(1-q)v^2x+(1-q^2v)(1-qv))}{(1-qv)(1-v)}AE(x,1)\\
&\quad+\frac{q^6vx^2}{1-q}AE(q^2vx,\frac{1}{q})-\frac{q^6vx^2}{1-q^2}AE(q^2vx,1)+\frac{q^5vx^2}{1-qv}AE(x,qv)+q^6x^2(1-q^2v). \qquad (4)
\end{aligned}$$

It is also shown in the appendix that $AN(x,v,u)$ and $AP(x,v,u)$ may both be expressed in terms of $AE(x,v)$ (see Theorem 7 below). Hence, finding a formula for the generating function of the oper distribution on $\mathcal{I}_n$ for $n \geq 1$ is equivalent to finding $AE(x,v)$ when $v = 1$. Though it does not seem likely that one can solve for $AE(x,v)$ or $AE(x,1)$ in (4) explicitly for general q, it is possible nonetheless to deduce the following further properties of the outer perimeter distribution on bargraphs of inversion sequences.

2.1 The case $q = -1$

Using (4), one can show

$$AE(x,v)\mid_{q=-1}=\frac{x^2(1-x-vx)}{1-2x}.$$

Thus, by Theorem 7, we have

$$AN(x,v,u)\mid_{q=-1}=\frac{vx^3}{1-2x} \text{ and } AP(x,v,u)\mid_{q=-1}=-\frac{ux^2(2ux^2-ux-vx-x+1)}{(1-2x)(1-2ux)}.$$

Taking $u = v = 1$ shows that $AE(x,1)+AN(x,1,1)+AP(x,1,1)$ at $q = -1$ is zero and thus yields the following sign-balance result for the oper statistic on $\mathcal{I}_n$.

Corollary 1 *For all $n \geq 2$, the number of inversion sequences of length n with odd outer perimeter is the same as the number with even outer perimeter.*

One can also explain bijectively the prior result.

Bijective proof of Corollary 1: Let $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_n$. We define an involution on $\mathcal{I}_n$ that reverses the parity of oper(π) for all π. First suppose π contains at least one entry ≥ 3 and let ℓ be the smallest index such that $\pi_\ell \geq 3$. Then $\ell \geq 3$ and $\pi_{\ell-1} \in \{1,2\}$. Consider switching $\pi_{\ell-1}$ to the other option, which changes the value of oper(π) by one, and hence reverses the oper parity. This follows from $\pi_{\ell-2} \in \{1,2\}$ and the fact that all squares lying above the $(\ell-1)$-st column of π up to height π_ℓ belong to the outer perimeter as they border the ℓ-th column of π on its left. On the other hand, if no such ℓ exists, then π is binary (on $\{1,2\}$) and switching π_n to the other option is seen to reverse the parity. Combining the two operations above then yields the desired involution on $\mathcal{I}_n$. □

Remarks: The preceding involution ϕ can be used to establish the formulas given above for $AN = AN(x,v,u)$ and $AE = AE(x,v)$ evaluated at $q=-1$, when restricted to the subsets of $\mathcal{I}_n$ enumerated by AN or AE. For the first formula, note that the restriction of ϕ to members of $\mathcal{I}_n$ enumerated by AN is not defined on the binary members of $\mathcal{I}_{n,2,1}$. For changing the last entry to 2 as described would result in a member of $\mathcal{I}_{n,2,2}$, which is enumerated by AE, and not AN. Further, it is seen that the restriction of ϕ is defined on all other members of $\mathcal{I}_n$ enumerated by AN, as in this case there would exist a smallest index ℓ such that $\pi_\ell \geq 3$ with $\ell \leq n-1$. Thus, the sum of the (signed) weights of the survivors of the involution is given by $2^{n-3}v$ for $n \geq 3$ and the formula for AN at $q=-1$ follows. On the other hand, the restriction of ϕ is not defined on the binary members of $\mathcal{I}_n$ enumerated by AE, which must end in either 11 or 22. Then the survivors of the involution in this case have weight given by $2^{n-3}(1-v)$ if $n \geq 3$ and 1 if $n=2$, which implies the formula for AE. A comparable argument though it involves more cases can also be given for the formula above for $AP(x,v,u)$ at $q=-1$.

2.2 Maximum and minimum outer perimeter

Let $d(p)$ denote the degree of a polynomial $p = p(q)$ in the indeterminate q. In Table 1 below are given $d(a(n,i,j))$ for all i and j, where $2 \leq n \leq 5$.

n	i	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
2	1	6	7			
3	1	8	9	11		
	2	8	9	10		
4	1	10	11	13	15	
	2	10	11	12	14	
	3	12	12	13	14	
5	1	14	15	16	18	20
	2	13	14	15	17	19
	3	14	14	15	16	18
	4	16	16	16	17	18

Table 1: Degrees of the polynomials $a(n,i,j)$ for $2 \leq n \leq 5$

By Lemma 1 and induction on $n = 3m+k$ where $m \geq 2$ and $k = 0,1,2$, one can prove the following general formula for $d(a(n,i,j))$.

Theorem 2 *For all $m \geq 2$,*

$$d(a(3m,i,j)) = \begin{cases} 3m^2+m+4-i+j, & 1 \leq i \leq j \leq 2m-1, \\ 3m^2-m+4-i+2j, & 1 \leq i \leq 2m-1,\, 2m \leq j \leq 3m, \\ 3m^2-3m+5+2j, & 2m \leq i \leq j-1,\, 2m \leq j \leq 3m, \\ 3m^2-3m+6+2i, & 2m \leq i = j \leq 3m-1, \\ 3m^2+m+3, & 1 \leq j \leq 2m-2,\, j+1 \leq i \leq 2m-1, \\ 3m^2-3m+5+2i, & 1 \leq j \leq 2m-1,\, 2m \leq i \leq 3m-1, \\ 3m^2-3m+5+2i, & 2m \leq j \leq 3m,\, j+1 \leq i \leq 3m-1, \end{cases}$$

$$d(a(3m+1,i,j)) = \begin{cases} 3m^2+3m+5-i+j, & 1 \leq i \leq j \leq 2m, \\ 3m^2+m+4-i+2j, & 1 \leq i \leq 2m,\, 2m+1 \leq j \leq 3m+1, \\ 3m^2-m+4+2j, & 2m+1 \leq i \leq j-1,\, 2m+1 \leq j \leq 3m+1, \\ 3m^2-m+5+2i, & 2m \leq i = j \leq 3m, \\ 3m^2+3m+4, & 1 \leq j \leq 2m-1,\, j+1 \leq i \leq 2m, \\ 3m^2-m+4+2i, & 1 \leq j \leq 2m-1,\, 2m \leq i \leq 3m, \\ 3m^2-m+4+2i, & 2m \leq j \leq 3m+1,\, j+1 \leq i \leq 3m, \end{cases}$$

and

$$d(a(3m+2,i,j)) = \begin{cases} 3m^2+5m+6-i+j, & 1 \leq i \leq j \leq 2m, \\ 3m^2+3m+5-i+2j, & 1 \leq i \leq 2m,\, 2m+1 \leq j \leq 3m+2, \\ 3m^2+m+4+2j, & 2m+1 \leq i \leq j-1,\, 2m+1 \leq j \leq 3m+2, \\ 3m^2+m+5+2i, & 2m+1 \leq i = j \leq 3m+1, \\ 3m^2+5m+5, & 1 \leq j \leq 2m-1,\, j+1 \leq i \leq 2m, \\ 3m^2+m+4+2i, & 1 \leq j \leq 2m,\, 2m+1 \leq i \leq 3m+1, \\ 3m^2+m+4+2i, & 2m \leq j \leq 3m+2,\, j+1 \leq i \leq 3m+1. \end{cases}$$

Note that the result above is seen also to hold for $n = 5$, which could serve as the basis of an inductive argument. When $1 \leq i \leq 2m-1$, the preceding formula may be written more compactly for all $m \geq 2$ and $k = 0, 1, 2$ as

$$d(a(3m+k,i,j)) = \begin{cases} 3m^2+(2k+1)m+3+k, & 1 \leq j \leq i-1, \\ 3m^2+(2k+1)m+4+k-i+j, & i \leq j \leq 2m, \\ 3m^2+(2k-1)m+4+k-i+2j-\delta_{k\in\{1,2\}}, & 2m+1 \leq j \leq 3m+k, \end{cases}$$

where $\delta_X = 1$ or 0 depending on the truth or falsity of the statement X.

By considering the largest value of $d(a(n,i,j))$ over all i and j for a fixed n in Theorem 2, we obtain the following result for $d(a(n))$.

Corollary 2 *Let $d_m^{(k)}$ denote the maximum outer perimeter of a member of $\mathcal{I}_{3m+k}$. Then we have $d_m^{(k)} = 3m^2 + (2k+5)m + \binom{k+2}{2} + 2$ for $k = 0, 1, 2$ and all $m \geq 1$.*

One can provide a direct explanation of this result as follows.

Combinatorial proof of Corollary 2: Let $e_m^{(k)}$ denote the number of members of $\mathcal{I}_{3m+k}$ such that $\mathrm{oper}(\pi) = d_m^{(k)}$. We will establish the formula for $d_m^{(k)}$ in Corollary 2 and at the same time show further that

$$e_m^{(0)} = e_m^{(1)} = 3 \text{ and } e_m^{(2)} = 1, \quad m \geq 2,$$

with $e_1^{(0)} = e_1^{(2)} = 1$ and $e_1^{(1)} = 2$. By a *maximal* member of $\mathcal{I}_n$, we mean one where the maximum oper value, which we denote by d_n, is achieved. Let e_n denote the number of maximal members of $\mathcal{I}_n$. One may verify $e_3 = 1$, $e_4 = 2$, $e_5 = 1$ and $e_6 = e_7 = 3$, with the corresponding sets of maximal sequences given by $\{113\}$, $\{1114, 1214\}$, $\{11315\}$, $\{111416, 113116, 121416\}$ and $\{1131517, 1214117, 1114117\}$, respectively, which implies $d_3 = 11$, $d_4 = 15$, $d_5 = 20$, $d_6 = 25$ and $d_7 = 31$.

We proceed to establish the formulas for d_n and e_n by induction on n, assuming $n \geq 8$. Henceforth, let $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_n$. If $\pi_n < n$, then we may replace π_n with n to obtain a member of $\mathcal{I}_n$ with (strictly) larger outer perimeter. Further, if $\pi_n = n$ and $\pi_{n-1} > 1$, then replacing π_{n-1} with 1 is seen to increase the outer perimeter. Applying Lemma 2 below, repeatedly if necessary, it follows that if π is maximal, then we must have $\pi_i \in \{1, i\}$ for all $i \in [n]$, with $\pi_{n-1} = 1$ and $\pi_n = n$. Clearly, the sequences $11 \cdots 1n$ and $121 \cdots 1n$ cannot be maximal as $n \geq 8$, so consider the largest index $s \in [3, n-2]$ such that $\pi_s = s$. If $s \leq n-4$, then such π cannot be maximal, as replacing $\pi_{n-2} = 1$ with $n-2$ is seen to increase the oper value, whence $s \in \{n-2, n-3\}$. It follows that the maximal members of $\mathcal{I}_n$ must belong to the subset $\{\pi \in \mathcal{I}_n : \pi = \tau 1n \text{ or } \rho 11n\}$, where τ and ρ denote maximal members of $\mathcal{I}_{n-2}$ and $\mathcal{I}_{n-3}$, respectively. Then we have $\text{oper}(\pi) = d_{n-2} + n + 4$ if $\pi = \tau 1n$ and $\text{oper}(\pi) = d_{n-3} + 2n + 2$ if $\pi = \rho 11n$, where d_{n-2} and d_{n-3} are given by the appropriate formula for $d_m^{(k)}$ above based on the value of n mod 3, by the induction hypothesis.

One may verify for all cases of n mod 3 that $d_{n-3}+n-2 > d_{n-2}$ for $n \geq 8$, i.e., $d_n+n+1 > d_{n+1}$ for $n \geq 5$, and hence only members of $\mathcal{I}_n$ of the form $\pi = \rho 11n$ are maximal when $n \geq 8$. Further, we get $d_m^{(k)} = d_{m-1}^{(k)} + 2(3m + k + 1)$ and $e_m^{(k)} = e_{m-1}^{(k)}$ if $3m + k \geq 8$, which implies by induction the $d_m^{(k)}$ and $e_m^{(k)}$ formulas for $k = 0, 1, 2$ and all $m \geq 1$. Note that if $n = 6$ or 7, then we get $d_{n-3} + n - 2 = d_{n-2}$, which accounts for the formulas in these cases. □

Lemma 2 *Let $n \geq 5$ and $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_n$ ending in $1n$. Suppose that there exists at least one index $i \in [3, n-2]$ such that $\pi_i \notin \{1, i\}$ and let t be the largest such index. If $t \geq 4$, then oper(π) can be increased by replacing π_t with 1 or t (possibly both). The same applies if $t = 3$, unless π starts with 12215, in which case one can replace 22 with 13 to increase oper(π).*

Proof. Given $i \in [n]$ and $1 \leq \ell \leq i$, let $\pi^{(i,\ell)}$ denote the member of $\mathcal{I}_n$ obtained from π by replacing π_i with ℓ. Let $\Pi^{(i)}$ denote the sequence of oper statistic values $(\text{oper}(\pi^{(i,\ell)}))_{\ell=1}^{i}$, starting with $\ell = 1$. We will use the terms *descent*, *level* and *ascent* in the usual way to indicate the relative sizes of a pair of adjacent entries in the sequence $\Pi^{(i)}$. First suppose $\pi_{t+1} = t+1$. Then one may verify that the sequence $\Pi^{(t)}$ is weakly decreasing and starts with a descent, which implies oper(π) is increased by replacing $1 < \pi_t < t$ with 1. Now suppose $\pi_{t+1} = 1$, $\pi_{t+2} = t+2$. Then it can be shown that $\Pi^{(t)}$ in this case starts with zero or more descents, followed by one or more levels, followed by zero or more ascents, where there must be at least one descent or ascent. Indeed, if $t \geq 4$, then $\Pi^{(t)}$ must contain at least one ascent if $\pi_{t-1} = 1$ or 2 and at least one descent if $\pi_{t-1} \geq 3$. In either case, we have $\text{oper}(\pi) < \max\{\text{oper}(\pi^{(t,1)}), \text{oper}(\pi^{(t,t)})\}$. If $t = 3$, then π starts either as 11215 or 12215, and one can make the replacements 2 by 3 or 22 or 13, respectively, to increase oper(π). Finally, if $\pi_{t+1} = \pi_{t+2} = 1$, then $\Pi^{(t)}$ starts with zero or one descents, followed by zero or more levels, followed by one or more ascents, where there are at least two ascents if $\Pi^{(t)}$ starts with a descent. Thus, replacing π_t with t always increases the outer perimeter in cases where $\pi_{t+1} = \pi_{t+2} = 1$, which completes the proof. □

Determining the greatest value of $d(a(n, i, j))$ amongst all i where n and j are fixed in Theorem 2 yields the following formula for $d(a(n, j))$.

Corollary 3 *Let $n \geq 4$ and $1 \leq j \leq n$. Then we have*

$$d(a(n,j)) = \begin{cases} 3m^2+3+\max\{2j-m,3m\}, & \text{if } n=3m; \\ 3m^2+2m+4+\max\{2j-m-1,3m\}, & \text{if } n=3m+1; \\ 3m^2+4m+6+\max\{2j-m-2,3m\}, & \text{if } n=3m+2. \end{cases}$$

One may extend the combinatorial argument given for Corollary 2 to explain the formula for $d(a(n,j))$ as well.

Combinatorial proof of Corollary 3: The formula can be verified directly for $n=4$ and $n=5$, so we may assume $n \geq 6$. Let $\Pi^{(i)}$ be as in the proof of Lemma 2. Then it can be shown that no descents or levels can occur anywhere to the right of the first ascent (if it exists) of $\Pi^{(i)}$ for $3 \leq i \leq n-1$. Thus, when determining $d(a(n,j))$, i.e., the maximum outer perimeter of a member of $\mathcal{I}_{n,j}$, one may restrict attention to $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_{n,j}$ such that $\pi_i \in \{1,i\}$ for $1 \leq i \leq n-1$. Note that if $\pi_{n-3}=\pi_{n-2}=\pi_{n-1}=1$, then replacing $\pi_{n-2}=1$ with $n-2$ is seen to increase the oper value. Therefore, in finding $d(a(n,j))$, we need only consider π having one of the following three forms: (i) $\pi=\alpha(n-1)j$, (ii) $\pi=\beta(n-2)1j$ or (iii) $\pi=\gamma(n-3)11j$.

Let d_n denote the maximum outer perimeter achieved by a member of $\mathcal{I}_n$. Recall that maximal members of $\mathcal{I}_n$ necessarily end in n for all $n \geq 1$. Let $1 \leq j \leq n-2$. Then the maximum oper value achieved by a member of $\mathcal{I}_{n,j}$ of the form (i), (ii) or (iii) above is given by (a) $d_{n-1}+1$, (b) $d_{n-2}+j+2$ or (c) $d_{n-3}+2j+2$, respectively. If $j=n-1$, then one gets for these maximum values $d_{n-1}+2$, $d_{n-2}+n+2$ and $d_{n-3}+2n$ instead. If $j=n$, then clearly the maximum is given by d_n. We then need to determine the largest of (a), (b) and (c) for each j. To do so, we consider cases on n mod 3 and make use of the formula from Corollary 2.

If $n=3m$, where $m \geq 2$, then we need to compare (a) $d_{3(m-1)+2}+1=3m^2+3m+3$, (b) $d_{3(m-1)+1}+j+2=3m^2+m+j+3$ and (c) $d_{3(m-1)}+2j+2=3m^2-m+2j+3$. Note that if $1 \leq j \leq 2m$, then (a)$\geq$(b)$\geq$(c), whereas if $2m+1 \leq j \leq 3m-2$, then (c)>(b)>(a). Hence, we have

$$d(a(3m,j)) = \begin{cases} 3m^2+3m+3, & 1 \leq j \leq 2m, \\ 3m^2-m+2j+3, & 2m+1 \leq j \leq 3m-2. \end{cases}$$

Note that this may be written as a single formula as $d(a(3m,j)) = 3m^2+3+\max\{2j-m,3m\}$ for $1 \leq j \leq 3m-2$. If $j=3m-1$, then we must compare $3m^2+3m+4$, $3m^2+4m+3$ and $3m^2+5m+1$, with the last of these quantities being the greatest as $m \geq 2$. If $j=3m$, then the maximum possible oper value is $d_{3m}=3m^2+5m+3$. Thus, the preceding formula for $d(a(3m,j))$ where $1 \leq j \leq 3m-2$ is seen to hold also for $j=3m-1,3m$. This completes the proof of the formula for $d(a(n,j))$ when $n=3m$. The arguments for the $n=3m+1$ and $n=3m+2$ cases are similar, which we leave to the reader. □

Remarks: It is possible to extend the combinatorial proof of Corollary 3 and obtain the formulas for $d(a(n,i,j))$ given in Theorem 2 for any i and j, though several cases are required based on the modular class of n mod 3 and the relative sizes of i and j. Note that when $i \geq j$, there are the following simple relations between $d(a(n,i,j))$ and $d(a(n,j))$:

$$d(a(n,i,j)) = d(a(n-1,i))+1,\ i>j, \text{ and } d(a(n,i,i)) = d(a(n-1,i))+2,$$

which are easily realized directly.

We now consider members of $\mathcal{I}_n$ for which the outer perimeter is a minimum. Inspection of the terms of $a(n)$ for the first several values of n suggests that the coefficient corresponding to the smallest term q^{2n+2} is given by M_{n-1}, the $(n-1)$-st Motzkin number (see A001006 in [19]). Though we do not have a complete analytic proof of this result, a simple combinatorial explanation can be given.

Proposition 1 *There are M_{n-1} members of $\mathcal{I}_n$ for $n \geq 1$ for which the minimum outer perimeter of $2n+2$ is achieved. Moreover, the number of members of $\mathcal{I}_{n,j}$ where $1 \leq j \leq n$ for which the minimum outer perimeter value of $2n+j+1$ on $\mathcal{I}_{n,j}$ is achieved corresponds to the array $A064189[n-1, j-1]$.*

Proof. One may verify that in order for $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_n$ to achieve the minimum possible oper value of $2n+2$, it is necessary and sufficient for the π_i to satisfy $|\pi_{i+1} - \pi_i| \leq 1$ for $1 \leq i \leq n-1$, with $\pi_n = 1$. Upon putting an up step $u = (1,1)$, a down step $d = (1,-1)$ or a horizontal step $h = (1,0)$ according to if $\pi_{i+1} - \pi_i$ equals 1, -1 or 0, respectively, one sees that members of $\mathcal{I}_n$ for which the minimum oper is achieved are in one-to-one correspondence with the set of Motzkin paths of length $n-1$, which establishes the first statement. Further, this bijection shows that the minimal members of $\mathcal{I}_{n,j}$ are synonymous with first quadrant lattice paths containing u, d and h steps and ending at the point $(n-1, j-1)$, and hence are enumerated by the array $A064189[n-1, j-1]$ from [19]. □

Remark: The subset of $\mathcal{I}_n$ whose members satisfy $|\pi_{i+1} - \pi_i| \leq 1$ for $1 \leq i \leq n-1$ are studied in greater detail in [17], where they are referred to as *smooth* inversion sequences.

2.3 Average outer perimeter

In this subsection, we find an explicit formula for the average outer perimeter of a member of $\mathcal{I}_n$. To do so, let $AE_y(x,v) = \frac{\partial}{\partial y} AE(x,v) \mid_{q=1}$, $AN_y(x,v,u) = \frac{\partial}{\partial y} AN(x,v,u) \mid_{q=1}$ and $AP_y(x,v,u) = \frac{\partial}{\partial y} AP(x,v,u) \mid_{q=1}$, where y denotes either the q, x or v variable. By differentiating both sides of (4) with respect to q, and setting $q=1$, we obtain

$$\begin{aligned}
AE_q(x,v) = {} & 2(3-10v+11v^2-4v^3)x^2 + ((v-1)^3+1-(1-v)^2x)AE_q(xv,1) \\
& + x(1-v)^2 AE_q(x,1) + 2x(1-v)^2 AE_v(xv,1) + v^2x^2(1-v)AE_v(x,v) \\
& + (2v(1-v)^2 + (v^2x - v^3 + v^2 - v + 1)x)AE(x,v) - x(1-v)^2 AE(xv,1) \\
& + x((1-3v)(1-v) - v^2x)AE(x,1) - 2vx^2(1-v)^2 AE_x(xv,1) \\
& + \frac{1}{2} vx^2(1-v)^2 AE_{vv}(vx,1),
\end{aligned} \tag{5}$$

where $AE_{vv}(x,1) = \frac{\partial^2}{\partial v^2} AE(x,v) \mid_{q=v=1}$ and $AE(x,v)$ is used here to denote $AE(x,v) \mid_{q=1}$. To solve the functional equation (5) for $AE_q(x,v)$, we use the fact

$$AE(x,v) \mid_{q=1} = \sum_{n\geq 2} (n-1)! \frac{1-v^{n-1}}{1-v} x^n,$$

which follows from the definitions, and further, make a guess that

$$AE_q(x,1) = 6x^2 + \sum_{n\geq 3} (n-2)! \left(\frac{n^3+12n^2-9n+4}{8} + (n-1)H_{n-1} \right) x^n,$$

where $H_n = \sum_{k=1}^n \frac{1}{k}$ is the n-th harmonic number.

Then, by (5), we obtain the following general formula for $AE_q(x,v)$:

$$\begin{aligned} AE_q(x,v) = 6x^2 - \frac{(11v^2+v-8)x^3}{1-v} &+ \sum_{n\geq 3}(n-1)!H_n x^{n+1}\frac{1-v^n}{1-v} \\ &+ \sum_{n\geq 3}(n-2)!x^n\left(\frac{x(n^3+4n^2+23n-20)}{8(v-1)} - \frac{vx^2(n-1)(n-2)(n-3)}{6(v-1)}\right. \\ &+ \left.\frac{a_n(x,v)v^{n-1}+3b_n(x,v)}{24(1-v)^4}\right), \end{aligned}$$

where

$$\begin{aligned} a_n(x,v) &= 24v^5x^2 - 72v^4x^2 - 60v^4x + 72v^3x^2 + 204v^3x - 96v^2x^2 - 48v^3 - 204v^2x + 24vx^2 \\ &\quad + 96v^2 + 84vx - 48v - 24x \\ &\quad - vx(v-1)(44v^3x - 88v^2x - 21v^2 + 44vx + 42v - 24x - 21)n \\ &\quad + 12vx(v-1)^3(2vx+5)n^2 - vx(v-1)^3(4vx-3)n^3, \\ b_n(x,v) &= -8v^3x^2 + 12v^3x + 24v^2x^2 + 16v^3 - 36v^2x - 32v^2 + 20vx + 16v + 4x \\ &\quad + x(v-1)(8v^2x - 15v^2 + 14v + 1)n - 12x(v-1)^3n^2 - x(v-1)^3n^3. \end{aligned}$$

This formula for $AE_q(x,v)$ can be shown to yield the expression that we guessed above for $AE_q(x,1)$, as required, upon taking the limit as v approaches 1. By Theorem 7, one can find explicit formulas for $AN_q(x,v,u)$, $AP_q(x,v,v)$ and $AP_q(x,v,u)$ (as the expressions are rather lengthy, we omit them), which implies the following result.

Theorem 3 *We have*

$$AE_q(x,1) + AN_q(x,1,1) + AP_q(x,1,1) = \sum_{n\geq 2} n!\left(\frac{n^2+15n+2}{8} + H_n + \frac{1}{n}\right)x^n.$$

That is, the average outer perimeter of members of $\mathcal{I}_n$ *for* $n \geq 2$ *is given by*

$$\frac{n^2+15n+2}{8} + H_n + \frac{1}{n}.$$

It is also possible to explain the foregoing result directly without recourse to recurrences or generating functions.

Combinatorial proof of Theorem 3: We may assume $n \geq 4$, as the formula is apparent for $n = 2, 3$. Throughout, let $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_n$. In analogy to a right lateral square defined in the proof of Lemma 1 above, a *left lateral* square will refer to the one bordering the left side of the first column of π. Note first that there are $n!$ left lateral, $(n-1)!(1+2\cdots+n) = \frac{(n+1)!}{2}$ right lateral and $(2n)n!$ boundary squares (high and low) altogether within the members of $\mathcal{I}_n$. We now count two additional classes of squares that are defined as follows. By a *right* (*left*) *border* outer perimeter square, we mean one that is neither a right (left) lateral nor high boundary square and that borders the right (left) side of some column. Note that it is possible for a square in column i of π for some $2 \leq i \leq n-1$ to be both a right and left border square, provided $\pi_i + 1 < m = \min\{\pi_{i-1}, \pi_{i+1}\}$, in which case there are $m - \pi_i - 1$ such squares in column i. Our strategy will be to find separate expressions for the totals of the right and left border squares taken over all the members of $\mathcal{I}_n$ and then subtract from the sum of these expressions the number of squares that are both right and left border to correct for double counting.

To find the average number of right border squares, suppose $\pi = \pi_1 \cdots \pi_n$ has a right border square in column i occurring at height ℓ. Then it is necessary and sufficient that $4 \le i \le n$, $3 \le \ell \le i-1$, $1 \le \pi_i \le \ell - 2$ and $\ell \le \pi_{i-1} \le i-1$. It is seen that the probability that there is a right border square in column i at height ℓ within a randomly chosen member of $\mathcal{I}_n$ is given by $\frac{\ell-2}{i}\left(1-\frac{\ell-1}{i-1}\right)$. Summing over $3 \le \ell \le i$ gives $\sum_{\ell=3}^{i} \frac{\ell-2}{i}\left(1-\frac{\ell-1}{i-1}\right)$ right border squares on average lying in column i within the members of $\mathcal{I}_n$. Note that

$$\sum_{\ell=3}^{i} \frac{\ell-2}{i}\left(1-\frac{\ell-1}{i-1}\right) = \frac{1}{i}\sum_{\ell=1}^{i-2}\ell - \frac{1}{i(i-1)}\sum_{\ell=1}^{i-2}\ell(\ell+1) = \left(\frac{1}{i}-\frac{1}{i(i-1)}\right)\sum_{\ell=1}^{i-2}\ell - \frac{1}{i(i-1)}\sum_{\ell=1}^{i-2}\ell^2$$
$$= \frac{i-2}{i(i-1)}\cdot\frac{(i-1)(i-2)}{2} - \frac{1}{i(i-1)}\cdot\frac{(i-1)(i-2)(2i-3)}{6} = \frac{(i-2)(i-3)}{6i}.$$

Summing over $4 \le i \le n$ then yields

$$\sum_{i=4}^{n} \frac{(i-2)(i-3)}{6i} = \frac{1}{6}\sum_{i=2}^{n}\left(i-5+\frac{6}{i}\right) = \frac{n^2-9n-4}{12} + H_n$$

right border squares on average in members of $\mathcal{I}_n$ for all $n \ge 4$.

We now count left border squares in $\mathcal{I}_n$. Note that a left border square occurs in column i of π at height ℓ if and only if $2 \le i \le n-1$, $3 \le \ell \le i+1$, $1 \le \pi_i \le \ell-2$ and $\ell \le \pi_{i+1} \le i+1$. Thus, the probability that there is a left border square in column i at height ℓ is given by $\frac{\ell-2}{i}\left(1-\frac{\ell-1}{i+1}\right)$. Summing over $3 \le \ell \le i+1$ gives

$$\sum_{\ell=3}^{i+1} \frac{\ell-2}{i}\left(1-\frac{\ell-1}{i+1}\right) = \frac{1}{i}\sum_{\ell=1}^{i-1}\ell - \frac{1}{i(i+1)}\sum_{\ell=1}^{i-1}\ell(\ell+1) = \frac{1}{i+1}\left(\frac{i(i-1)}{2} - \frac{(i-1)(2i-1)}{6}\right)$$
$$= \frac{i-1}{6},$$

and hence there are an average of $\sum_{i=2}^{n-1}\frac{i-1}{6} = \frac{(n-1)(n-2)}{12}$ left border squares in members of $\mathcal{I}_n$.

Finally, observe that a square in column i of π at height ℓ is both a right and left border square if and only if $4 \le i \le n-1$, $3 \le \ell \le i-1$, $1 \le \pi_i \le \ell-2$, $\ell \le \pi_{i-1} \le i-1$ and $\ell \le \pi_{i+1} \le i+1$. Considering all possible ℓ then gives on average $\sum_{\ell=3}^{i-1}\frac{\ell-2}{i}\left(1-\frac{\ell-1}{i-1}\right)\left(1-\frac{\ell-1}{i+1}\right)$ squares in column i in members of $\mathcal{I}_n$ that are both right and left border. Note that

$$\sum_{\ell=3}^{i-1}\frac{\ell-2}{i}\left(1-\frac{\ell-1}{i-1}\right)\left(1-\frac{\ell-1}{i+1}\right)$$
$$= \frac{1}{i}\sum_{\ell=3}^{i}(\ell-2) - \frac{2}{i^2-1}\sum_{\ell=3}^{i}(\ell-1)(\ell-2) + \frac{1}{i(i^2-1)}\sum_{\ell=3}^{i}(\ell-1)^2(\ell-2)$$
$$= \frac{1}{i}\sum_{\ell=1}^{i-2}\ell - \frac{2}{i^2-1}\sum_{\ell=1}^{i-2}\ell(\ell+1) + \frac{1}{i(i^2-1)}\sum_{\ell=1}^{i-2}\ell(\ell+1)^2$$
$$= \frac{1}{i(i+1)} + \frac{1}{i(i^2-1)}\sum_{\ell=1}^{i-1}\left((i^2-2i)\ell - 2(i-1)\ell^2 + \ell^3\right) = \frac{1}{i(i+1)} + \frac{i-4}{12},$$

and summing over $4 \le i \le n-1$ gives on average

$$\sum_{i=4}^{n-1} \frac{1}{i(i+1)} + \sum_{i=4}^{n-1} \frac{i-4}{12} = \frac{1}{4} - \frac{1}{n} + \frac{(n-4)(n-5)}{24} = \frac{n^2-9n+26}{24} - \frac{1}{n}$$

squares in members of $\mathcal{I}_n$ for $n \ge 2$ that are both right and left border. Combining the formulas found above, we have that the average outer perimeter on $\mathcal{I}_n$ is given by

$$\begin{aligned}
&1 + \frac{n+1}{2} + 2n + \frac{n^2-9n-4}{12} + H_n + \frac{(n-1)(n-2)}{12} - \left(\frac{n^2-9n+26}{24} - \frac{1}{n}\right) \\
&= \frac{n^2+15n+2}{8} + H_n + \frac{1}{n},
\end{aligned}$$

as desired. □

3 Distribution of inner perimeter

Let $b(n,i,j)$ denote the distribution of the inner perimeter statistic on $\mathcal{I}_{n,i,j}$ and let $b(n,j) = \sum_{i=1}^{n-1} b(n,i,j)$ for $n \ge 2$ be the corresponding distribution on $\mathcal{I}_{n,j}$, with $b(1,1) = q$. Let $b(n,i,j) = b(n,j) = 0$ in all cases where the set over which the distribution is taken is empty.

Then the $b(n,i,j)$ satisfy the following recurrence relations.

Lemma 3 *If $n \ge 2$, then*

$$b(n,i,j) = \sum_{k=1}^{j-1} q^{j-k+1} b(n-1,k,i) + q \sum_{k=j}^{n-2} b(n-1,k,i), \quad 1 \le j < i \le n-1, \tag{6}$$

$$b(n,1,j) = q^j b(n-1,1), \quad 1 \le j \le n, \tag{7}$$

$$b(n,i,j) = \sum_{k=1}^{i-1} q^{j-k+1} b(n-1,k,i) + \sum_{k=i}^{n-2} q^{j-i+2} b(n-1,k,i), \quad 2 \le i \le j \le n, \tag{8}$$

with initial value $b(1,1) = q$.

Proof. In each case, we append a column of size j to $\rho \in \mathcal{I}_{n-1,k,i}$ to obtain $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_{n,i,j}$. First suppose $n \ge 3$ and $2 \le i \le j \le n$, with $i < n$. If $1 \le k \le i-1$, then in going from ρ to π, one introduces j inner perimeter squares (i.e., those in the appended column), while at the same time, taking away from the inner perimeter the $k-1$ squares in the final column of ρ at heights $2, 3, \ldots, k$. Thus, we get a contribution of $q^{j-k+1} b(n-1,k,i)$ for such π and summing over $k \in [i-1]$ accounts for the first summation on the right side of (8). On the other hand, if $i \le k \le n-2$, then appending a column of size j to ρ always results in their being a net increase of $j-(i-2)$ in the iper value, as the middle $i-2$ squares in the final column of ρ cease to be part of the inner perimeter. Considering all $i \le k \le n-2$ yields the second sum on the right side and completes the proof of (8). On the other hand, if $i = 1$, then no inner perimeter squares of ρ are lost when j is appended, regardless of the value of k, which implies (7). Now assume $1 \le j < i \le n-1$. If $1 \le k \le j-1$, then reasoning as in the first part of the proof above of (8) yields the first sum on the right side of (6). On the other hand, if $j \le k \le n-2$, then the squares at heights $2, 3, \ldots, j$ in the final column of ρ are lost from the inner perimeter when j is appended, resulting in their being an increase of 1 in the iper value. This accounts for the second part of formula (6) and completes the proof. □

Analogous to before, we define the following generating functions

$$BN(x,v,u,q)=\sum_{n\geq 3}\sum_{j=1}^{n-2}\sum_{i=j+1}^{n-1} b(n,i,j)x^n v^{i-1}u^{i-j-1}$$

and

$$BP(x,v,u,q)=\sum_{n\geq 2}\sum_{j=1}^{n}\sum_{i=1}^{j} b(n,i,j)x^n v^{i-1}u^{j-i}.$$

Let $b(n)=\sum_{i=1}^{n-1}\sum_{j=1}^{n} b(n,i,j)=\sum_{j=1}^{n} b(n,j)$ for $n\geq 2$, with $b(1)=q$. Then $b(n)$ gives the distribution on all of $\mathcal{I}_n$ for the inner perimeter statistic. Note that the generating function of $b(n)$ for $n\geq 1$ is given by $qx+BN(x,1,1,q)+BP(x,1,1,q)$.

By rewriting (6)–(8) in terms of generating functions (we omit the details), we obtain the following system of functional equations.

THEOREM 4 *We have*

$$\begin{aligned}
BN(x,v,u,q) &= \frac{qx}{u-q}\left(\frac{q}{u}BP(x,v,uv,q)-BP(x,v,qv,q)\right)\\
&+\frac{qx}{1-u}\left(\frac{1}{u}BP(x,v,vu,q)-BP(x,vu,vu,q)+\frac{1}{vu^2}BN(x,v,\frac{1}{vu},q)\right.\\
&\left.-\frac{1}{vu}BN(x,vu,\frac{1}{vu},q)\right)\\
&+\frac{qx}{vu^2(1-u)}\left(u^2BN(x,v,\frac{1}{v},q)-BN(x,v,\frac{1}{vu},q)\right),\\
BP(x,v,u,q) &= q^2(qu+1)x^2+\frac{qx}{1-qu}\left(BP(x,v,qv,q)-quBP(qux,\frac{v}{qu},\frac{v}{u},q)\right)\\
&-\frac{q(1-q)x}{1-qu}\left(BP(x,v,0,q)-quBP(qux,\frac{v}{qu},0,q)\right)\\
&+\frac{q^2x}{v(1-qu)}\left(BN(x,v,\frac{1}{v},q)-q^2u^2BN(qux,\frac{v}{qu},\frac{qu}{v},q)\right)\\
&+\frac{q(1-q)x}{1-qu}(C(x,q)-quC(qux,q))\\
&+\frac{q(1-q)x}{1-qu}(BP(x,0,0,q)-quBP(qux,0,0,q)),
\end{aligned}$$

where $C(x,q)$ denotes the free coefficient of u in the generating function $uBN(x,\frac{1}{u},u,q)$.

We are unable to solve the preceding system explicitly for $BN(x,v,u,q)$ and $BP(x,v,u,q)$. Further, the inner perimeter statistic, unlike the outer, is not sign-balanced on the set $\mathcal{I}_n$ for $n\geq 2$, since taking $u=v=1$ and $q=-1$ gives the series expansion

$$\sum_{n\geq 2} b(n)\mid_{q=-1} x^n=-2x^4+6x^5-16x^6+40x^7-76x^8+148x^9-1232x^{11}+8776x^{12}+\cdots.$$

We remark that we did not find this sequence (or its absolute value) in the OEIS [19] nor have we found an explicit expression for it. In the next two subsections, we consider the degree and minimum q-exponent of the polynomial $b(n)$, the first and last coefficients of $b(n)$ and the value of the derivative of $b(n)$ evaluated at $q=1$.

3.1 Maximum and minimum inner perimeter

By induction on n and Lemma 3, we have the following formula for $d(b(n,i,j))$.

THEOREM 5 *For all* $m \geq 1$,

$$d(b(3m,i,j)) = \begin{cases} 3m^2 - m + j, & i = 1 \leq j \leq 3m, \\ 3m^2 - m + 1, & 2 \leq i \leq m,\ 1 \leq j \leq i-1, \\ 3m^2 - m + 2 + j - i, & 2 \leq i \leq m,\ i \leq j \leq 3m, \\ 3m^2 - 2m + 1 + i, & m+1 \leq i \leq 3m-1,\ 1 \leq j \leq m, \\ 3m^2 - 3m + 1 + j + i, & m+1 \leq i \leq 3m-1,\ m+1 \leq j \leq 3m, \end{cases}$$

$$d(b(3m+1,i,j)) = \begin{cases} 3m^2 + m + j, & i = 1 \leq j \leq 3m+1, \\ 3m^2 + m + 1, & 2 \leq i \leq m,\ 1 \leq j \leq i-1, \\ 3m^2 + m + 2 + j - i, & 2 \leq i \leq m,\ i \leq j \leq 3m+1, \\ 3m^2 + 1 + i, & m+1 \leq i \leq 3m,\ 1 \leq j \leq m, \\ 3m^2 - m + 1 + j + i, & i = m+1 \leq j \leq 3m+1, \\ 3m^2 - m + j + i, & m+2 \leq i \leq 3m,\ m+1 \leq j \leq 3m+1, \end{cases}$$

and

$$d(b(3m+2,i,j)) = \begin{cases} 3m^2 + 3m + 1 + j, & i = 1 \leq j \leq 3m+2, \\ 3m^2 + 3m + 2, & 2 \leq i \leq m+1,\ 1 \leq j \leq i-1, \\ 3m^2 + 3m + 3 + j - i, & 2 \leq i \leq m+1,\ i \leq j \leq 3m+2, \\ 3m^2 + 2m + 1 + i, & m+2 \leq i \leq 3m+1,\ 1 \leq j \leq m+1, \\ 3m^2 + m + j + i, & m+2 \leq i \leq 3m+1,\ m+2 \leq j \leq 3m+2. \end{cases}$$

Concerning the degree of the polynomial $b(n)$, which corresponds to the maximum iper value of a member of $\mathcal{I}_n$, we have the following result which is a consequence of Theorem 5.

COROLLARY 4 *Let* $u_m^{(k)}$ *denote the maximum inner perimeter of a member of* $\mathcal{I}_{3m+k}$. *Then we have* $u_m^{(k)} = 3m^2 + (2k+3)m + \binom{k+1}{2}$ *for* $k = 0, 1, 2$ *and all* $m \geq 1$.

Combinatorial proof of Corollary 4: Let $v_m^{(k)}$ denote the number of members of $\mathcal{I}_{3m+k}$ such that $\operatorname{oper}(\pi) = u_m^{(k)}$. We will establish the formula for $u_m^{(k)}$ in Corollary 4 and at the same time show further that

$$v_m^{(0)} = 1, v_m^{(1)} = 3 \text{ and } v_m^{(2)} = 2, \quad m \geq 1.$$

By a *maximal* member of $\mathcal{I}_n$, we mean here one for which the greatest value of the inner perimeter on $\mathcal{I}_n$ is achieved. Let $\mathcal{I}_n^*$ denote the set of maximal members of $\mathcal{I}_n$, with $v_n = |\mathcal{I}_n^*|$. Then for $3 \leq n \leq 5$, we have $v_3 = 1$, $v_4 = 3$ and $v_5 = 2$, with the corresponding sets $\mathcal{I}_n^*$ given by $\{123\}$, $\{1134, 1224, 1234\}$ and $\{12145, 12245\}$. Note that members of $\mathcal{I}_n^*$ must end in n, as the sequence of iper values of $\pi^{(n,\ell)}$ for $1 \leq \ell \leq n$ is seen to be weakly increasing, ending in an ascent. Consider forming $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_n$ from $\pi' = \pi_1 \cdots \pi_{n-3} \in \mathcal{I}_{n-3}$ where $n \geq 6$ by appending $\pi_{n-2}\pi_{n-1}\pi_n$. Then we have $d := \operatorname{iper}(\pi) - \operatorname{iper}(\pi') \leq 2n$ for all $\pi \in \mathcal{I}_n$. To prove this, we may assume $\pi_n = n$ and consider two general cases as follows. First suppose that at most two out of three squares in columns $n-2$, $n-1$ and n at height ℓ for each $\ell \in [2, n-2]$ belong to the inner perimeter of π. Further, the three squares at height 1 in these columns are always part of the inner perimeter, as are the top two squares in column n since $\pi_n = n$ and the top square in column $n-1$ if $\pi_{n-1} = n-1$. Then the maximum possible value of d is given by $2(n-3) + 6 = 2n$ in this case. Note that such a difference is indeed achieved when $\pi_{n-2} = 1$, $\pi_{n-1} = n-1$ and $\pi_n = n$. Now suppose for some $\ell \in [2, n-2]$ that the three squares in columns $n-2$, $n-1$ and n at height ℓ all belong to the inner perimeter of π. Then it is seen that there is only one such ℓ and it must correspond to $\min\{\pi_{n-1}, \pi_{n-2}\}$ when it exists. Further, the existence of such an ℓ implies $\pi_{n-1} < n-1$. But then the top square in column $n-1$ fails to belong to the inner perimeter of π and $d \leq 2n$ once again.

We now claim that if $\pi_{n-1} < n-1$ within $\pi \in \mathcal{I}_n$ where $n \geq 5$, then π cannot be maximal. To prove this, we may assume $\pi_n = n$ and $\pi' \in \mathcal{I}^*_{n-3}$, for if either fails to hold, then π cannot be maximal, as either condition failing to hold would imply that $\text{iper}(\pi)$ is strictly less than the iper value of the member of $\mathcal{I}_n$ obtained by appending $1(n-1)n$ to a member of $\mathcal{I}^*_{n-3}$. If $\min\{\pi_{n-1}, \pi_{n-2}\} = 1$, then no such $\ell \in [2, n-2]$ as described above exists. Then $\pi_{n-1} < n-1$ implies $d < 2n$ and thus π cannot be maximal. So assume $\pi_{n-1}, \pi_{n-2} \geq 2$. If $\pi_{n-2} = 2$ and $\pi_{n-1} = n-2$, then $n \geq 5$ and $\pi_n = n$ implies the square at height two in column $n-1$ is not counted in $\text{iper}(\pi)$. Then no height ℓ as described above would exist in this case and once again $d < 2n$. If $\pi_{n-2} = 2$ and $\pi_{n-1} < n-2$, then such a height ℓ may exist in this case, but this is offset by the fact that there would be at least one height (namely, $n-2$) where only one of the final three squares at that height is counted in $\text{iper}(\pi)$, and hence $d < 2n$ in this case also. On the other hand, if $\pi_{n-2} \geq 3$, then $\pi_{n-1} \geq 2$ and π' ending in $n-3$ (being maximal) would imply that the square at height two in column $n-2$ is not part of the inner perimeter of π. Considering separately when $\pi_{n-1} = 2$ or $\pi_{n-1} > 2$ leads again to the conclusion that $d < 2n$ in either case, which completes the proof of the claim.

Then each $\pi \in \mathcal{I}^*_n$ for $n \geq 6$ is obtained by appending $a(n-1)n$ to $\pi' \in \mathcal{I}^*_{n-3}$ for some $a \geq 1$. If $a > 1$, then the square at height two in the final column of π' would cease to be part of the inner perimeter after $a(n-1)n$ is appended, as $n \geq 6$ implies the penultimate letter of π' is at least two. This would cause for π not to be maximal, so we must have $a = 1$. Thus, all members of $\mathcal{I}^*_n$ for $n \geq 6$ arise by appending $1(n-1)n$ to members of $\mathcal{I}^*_{n-3}$. This implies $v_n = v_{n-3}$ for $n \geq 6$, which yields the second statement. Further, we have that members of $\mathcal{I}^*_n$ for $n \geq 6$ are of the form $\pi = \rho 1(\ell-1)\ell \cdots 1(n-1)n$, where $\ell \in \{6, 7, 8\}$ with $n \equiv \ell \pmod 3$ and $\rho \in \mathcal{I}^*_3$, $\mathcal{I}^*_4$ or $\mathcal{I}^*_5$, whichever is appropriate. Computing the inner perimeter of the maximal π in each case of n mod 3 yields the formula given for $u^{(k)}_n$ and completes the proof. □

By Theorem 5, we obtain the following formula for $d(b(n, j))$.

Corollary 5 *Let $n \geq 1$ and $1 \leq j \leq n$. Then we have*

$$d(b(n,j)) = \begin{cases} 3m^2 + \max\{m, j\}, & \text{if } n = 3m; \\ 3m^2 + 2m + \max\{m+1, j\}, & \text{if } n = 3m+1; \\ 3m^2 + 4m + 1 + \max\{m+1, j\}, & \text{if } n = 3m+2. \end{cases}$$

Combinatorial proof of Corollary 5: We may assume $n \geq 5$, as the formula may be verified directly for $1 \leq n \leq 4$. Given $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_n$, let $\pi^{(i,\ell)}$ be as in the proof of Lemma 2 above and $\widetilde{\Pi}^{(i)} = (\text{iper}(\pi^{(i,\ell)})^i_{\ell=1}$ denote the corresponding sequence of iper values, starting with $\ell = 1$. Note that $\widetilde{\Pi}^{(i)}$ where $3 \leq i \leq n$ starts with an ascent, i.e., $\text{iper}(\pi^{(i,1)}) < \text{iper}(\pi^{(i,2)})$, if and only if $\min\{\pi_{i-1}, \pi_{i-2}\} = 1$ or 2 and $\min\{\pi_{i+1}, \pi_{i+2}\} = 1$ or 2, where we take $\pi_{n+1} = \pi_{n+2} = 1$ when $i = n$ or $n-1$ and a minimum of 2 can occur in either case only when $\pi_{i-1} = 2$ or $\pi_{i+1} = 2$. Further, $\widetilde{\Pi}^{(i)}$ has an ascent at index $\ell \geq 2$, i.e., $\text{iper}(\pi^{(i,\ell)}) < \text{iper}(\pi^{(i,\ell+1)})$ for some $\ell \in [2, i-1]$, if and only if $\ell > \min\{\pi_{i-1}, \pi_{i+1}\}$ and either (I) $\ell \geq M-1$, where $M = \max\{\pi_{i-1}, \pi_{i+1}\}$, or (II) $\ell \leq M-2$ and increasing the i-th entry of π from ℓ to $\ell+1$ does not eliminate from the inner perimeter a square at height $\ell+1$ in either column $i-1$ or $i+1$. From the preceding characterizations for when an ascent occurs, it is seen that $\widetilde{\Pi}^{(i)}$ cannot contain a descent anywhere to the right of its first ascent (if it exists). Moreover, if an ascent in $\widetilde{\Pi}^{(i)}$ occurs at index ℓ for some $\ell \geq 2$, then such an ascent must be followed by another ascent. Thus, the maximum value of $\widetilde{\Pi}^{(i)}$ is always achieved at $\ell = 1$ or $\ell = i$ (possibly both).

Let $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_{n,j}$, where $n \geq 5$ and $1 \leq j \leq n$. By the preceding, in determining the maximum inner perimeter achieved by a member of $\mathcal{I}_{n,j}$, we may restrict attention to those π such that $\pi_i \in \{1, i\}$ for $1 \leq i \leq n-1$. Note that if π is of the form $\pi = \pi'1(n-2)(n-1)j$, then the sequence $\widetilde{\Pi}^{(n-3)}$ is (weakly) decreasing, as increasing the $(n-3)$-rd entry of π from ℓ to $\ell+1$ would eliminate from the inner perimeter the square at height $\ell+1$ in column $n-2$. Thus, we may assume $\pi_{n-3} = 1$ if $\pi_{n-2} = n-2$ and $\pi_{n-1} = n-1$ within $\pi \in \mathcal{I}_{n,j}$ when maximizing the inner perimeter of π. Therefore, in order to ascertain $d(b(n,j))$, one need only consider $\pi \in \mathcal{I}_{n,j}$ having one of the following three forms: (i) $\pi = \alpha 1 j$, (ii) $\pi = \beta 1(n-1)j$ or (iii) $\pi = \gamma 1(n-2)(n-1)j$. Clearly, we may take α, β and γ to be maximal in order to maximize $\text{iper}(\pi)$ in each case.

Let $u_n = d(b(n))$ denote the largest inner perimeter of a member of $\mathcal{I}_n$. Clearly, if $j = 1$ or $j = n$, then we have $d(b(n,1)) = u_{n-1} + 1$ and $d(b(n,n)) = u_n$ in these cases, so we may assume $2 \leq j \leq n-1$. Then the maximum value of $\text{iper}(\pi)$ for π of the forms (i), (ii) and (iii) above are given by $u_{n-2} + j + 1$, $u_{n-3} + n + j$ and $u_{n-4} + 2n - 1 + \delta_{j,n-1}$, respectively. We must then determine the largest of these three quantities. To do so, we consider cases on n mod 3 and recall the formula for u_n from Corollary 4. If $n = 3m$ for some $m \geq 2$, then the three quantities work out respectively to $u_{3(m-1)+1} + j + 1 = 3m^2 - m + j$, $u_{3(m-1)} + 3m + j = 3m^2 + j$ and $u_{3(m-2)+2} + 6m - 1 + \delta_{j,3m-1} = 3m^2 + m + \delta_{j,3m-1}$. If $2 \leq j \leq m$, then the largest of the three quantities is given by $3m^2 + m$, whereas if $m < j \leq 3m-1$, then it is $3m^2 + j$. Thus, in general, we have $d(b(3m,j)) = 3m^2 + \max\{m, j\}$ for $2 \leq j \leq 3m-1$, with this formula seen to hold also for $j = 1$ and $j = 3m$. This establishes the first formula stated above for $d(b(n,j))$. Similar arguments can be given in the $n = 3m+1$ and $n = 3m+2$ cases. $\square$

Remark: One can extend the combinatorial proof of Corollary 5 to realize the formulas for $d(b(n,i,j))$ in Theorem 5, though a more intricate analysis is needed.

The smallest q-exponent on the other hand of a term appearing in the polynomial $b(n)$, which corresponds to the minimum iper value of a member of $\mathcal{I}_n$, is clearly equal to n and is achieved only by the inversion sequence $11\cdots1$. Moreover, the lowest exponent of a term appearing in $b(n,i,j)$ and $b(n,j)$ is given by $n+i+j-2$ and $n+j-1$, respectively. Note that $\pi \in \mathcal{I}_{n,i,j}$ for which the minimum iper value is achieved can be decomposed as $\pi = 1^a \alpha i j$ if $2 \leq i \leq j$, where $a \geq 1$ and α is a (strictly) decreasing possibly empty sequence in $[2, i-1]$, or as $\pi = 1^a \beta i j$ if $1 \leq j < i$, where $a \geq 1$ and β is decreasing in $[2, j]$. Thus, there are 2^{i-2} members $\pi \in \mathcal{I}_{n,i,j}$ for which $\text{iper}(\pi) = n+i+j-2$ if $2 \leq i \leq j$ and 2^{j-1} such members of $\mathcal{I}_{n,i,j}$ if $1 \leq j < i$, with only a single $\pi \in \mathcal{I}_{n,1,j}$ for which $\text{iper}(\pi) = n+j-1$. Summing over i shows that the coefficient of the lowest degree term in $b(n,j) = \sum_{i=1}^{n-1} b(n,i,j)$ always equals one, which is easily realized directly.

3.2 Average inner perimeter

THEOREM 6 *The average inner perimeter of members of $\mathcal{I}_n$ for $n \geq 2$ is given by*

$$\frac{3n^2 + 41n - 28}{24} - \frac{H_n}{2} + \frac{1}{n}.$$

Proof. Let $t(n,i,j) = \frac{d}{dq} b(n,i,j) \mid_{q=1}$ and note $b(n,i,j) \mid_{q=1} = (n-2)!$. Define $t(n,j) = \frac{d}{dq} b(n,j) \mid_{q=1} = \sum_{i=1}^{n-1} t(n,i,j)$. Differentiating both sides of (6)–(8) with respect to q, and letting $q=1$, gives the following formulas for $t(n,i,j)$ where $n \geq 3$:

$$t(n,i,i) = t(n-1,i) + \left(\binom{i+1}{2} + 2n - 3 - 2i\right)(n-3)! - \delta_{i,1}(n-2)!, \quad 1 \leq i \leq n-1, \tag{9}$$

$$t(n,i,j) = t(n-1,i) + \left(\binom{j+1}{2} + (j-i+2)(n-1-(j+i+1)/2)\right)(n-3)!, \quad 2 \leq i < j \leq n, \tag{10}$$

$$t(n,1,j) = t(n-1,1) + j(n-2)!, \quad 2 \leq j \leq n, \tag{11}$$

$$t(n,i,j) = t(n-1,i) + \left(\binom{j+1}{2} + n - 2 - j\right)(n-3)!, \quad 1 \leq j < i \leq n-1. \tag{12}$$

Summing (10) and (12) over i for a fixed $j \in [2, n-1]$ yields respectively

$$\begin{aligned}
&\sum_{i=2}^{j-1} t(n,i,j) \\
&= \sum_{i=2}^{j-1} t(n-1,i) + \left((n-3+j)\binom{j+1}{2} - \frac{(j+1)(j^2-4)}{2} + \binom{j}{3} - 3n + 3\right)(n-3)!, \\
&\sum_{i=j+1}^{n-1} t(n,i,j) = \sum_{i=j+1}^{n-1} t(n-1,i) + (n-1-j)\left(\binom{j+1}{2} + n - 2 - j\right)(n-3)!.
\end{aligned}$$

To the sum of the last two equations, we add (9) with i replaced by j and (11), which yields

$$t(n,j) = t(n-1) + \left((n-2)^2 + j^2(n-2) - \frac{j(j-1)(2j-1)}{6} + \delta_{j,n}\right)(n-3)!, \quad 1 \leq j \leq n, \tag{13}$$

after several algebraic steps, where $t(n) = \frac{d}{dq} b(n) \mid_{q=1} = \sum_{j=1}^{n} t(n,j)$. Note that the $j=1$ and $j=n$ cases of (13) require a separate argument. Summing (13) over $1 \leq j \leq n$, we obtain

$$\begin{aligned}
t(n) - nt(n-1) &= \left(n(n-2)^2 + (n-2)\sum_{j=1}^{n} j^2 - \sum_{j=1}^{n}\binom{j}{3} - \sum_{j=1}^{n}\frac{j(j^2-1)}{6} + 1\right)(n-3)! \\
&= \left(n(n-2)^2 + \frac{n(n+1)(n-2)(2n+1)}{6} - \binom{n+1}{4} - \frac{n^2(n+1)^2}{24}\right. \\
&\quad \left. + \frac{n(n+1)}{12} + 1\right)(n-3)! \\
&= \frac{3n^3 + 16n^2 - 25n - 6}{12}(n-2)!.
\end{aligned}$$

An induction on $n \geq 2$ using the last equality now gives

$$t(n) = \left(\frac{3n^2 + 41n - 28}{24} - \frac{H_n}{2} + \frac{1}{n}\right) n!,$$

as desired. □

It is also possible to provide a combinatorial explanation of the prior result, which like the proof of Theorem 3 above, features a double counting argument.

Combinatorial proof of Theorem 6: We may assume $n \geq 4$, as the formula is apparent for $n = 2, 3$. First note that there are $n \cdot n!$ inner perimeter squares occurring at height 1 within all the members of $\mathcal{I}_n$ and also the same number occurring at the top of a column within a bargraph. From this, we must subtract the number of squares that correspond to a column of height one within an inversion sequence, of which it is seen that there are $H_n \cdot n!$. Thus, there are $(2n - H_n)n!$ inner perimeter squares altogether that occur at the top or bottom of a column. To count the remaining inner perimeter squares in $\mathcal{I}_n$, we divide them into two general classes as follows. By a *right* (*left*) *border* inner perimeter square, we mean one that does not occur at the top or the bottom of a column of a bargraph and whose right (left) side borders the boundary of the bargraph.

Let $\pi = \pi_1 \cdots \pi_n \in \mathcal{I}_n$. To count the right border squares in $\mathcal{I}_n$, first note that π has a right border square at height ℓ in column i where $i < n$ if and only if $3 \leq i \leq n-1$, $2 \leq \ell < \pi_i \leq i$ and $1 \leq \pi_{i+1} \leq \ell - 1$. Further, π has a right border square at height ℓ in column n if and only if $3 \leq \ell + 1 \leq \pi_n \leq n$. Considering all possible i and ℓ, this yields on average

$$\sum_{i=3}^{n-1}\sum_{\ell=2}^{i-1}\frac{\ell-1}{i+1}\left(1-\frac{\ell}{i}\right)+\sum_{\ell=2}^{n-1}\left(1-\frac{\ell}{n}\right)=\sum_{i=3}^{n-1}\left(\frac{i-1}{i(i+1)}\sum_{\ell=1}^{i-2}\ell-\frac{1}{i(i+1)}\sum_{\ell=1}^{i-2}\ell^2\right)+\frac{1}{n}\sum_{\ell=1}^{n-2}\ell$$
$$=\sum_{i=3}^{n-1}\frac{(i-1)(i-2)}{6(i+1)}+\frac{(n-1)(n-2)}{2n}=\frac{n^2-9n-4}{12}+H_n+\frac{(n-1)(n-2)}{2n}$$
$$=\frac{n^2-3n-22}{12}+H_n+\frac{1}{n}$$

right border squares in members of $\mathcal{I}_n$. Note that π has a left border square at height ℓ in column i where $i < n$ if and only if $3 \leq i \leq n-1$, $2 \leq \ell < \pi_i \leq i$ and $1 \leq \pi_{i-1} \leq \ell - 1$. This yields

$$\sum_{i=3}^{n-1}\sum_{\ell=2}^{i-1}\frac{\ell-1}{i-1}\left(1-\frac{\ell}{i}\right)=\sum_{i=3}^{n-1}\frac{i-2}{6}=\frac{(n-2)(n-3)}{12}$$

left border squares on average within the members of $\mathcal{I}_n$, excluding those occurring in the final column.

From the total of the previous two cases, we must subtract the number of squares that are counted twice, i.e., those whose right and left sides both touch the boundary of a bargraph and do not occur in the last column or at the top or bottom of a column. Note that π has an inner perimeter square at height ℓ in column i where $i < n$ that is both a right and left border square if and only if the two sets of necessary conditions specified above for i and ℓ are simultaneously satisfied. Considering all possible i and ℓ gives on average $\sum_{i=3}^{n-1}\sum_{\ell=2}^{i-1}\frac{(\ell-1)^2}{i^2-1}\left(1-\frac{\ell}{i}\right)$ such squares in members of $\mathcal{I}_n$. Note that

$$\sum_{\ell=2}^{i-1}\frac{(\ell-1)^2}{i^2-1}\left(1-\frac{\ell}{i}\right)=\frac{1}{i^2-1}\sum_{\ell=1}^{i-2}\ell^2-\frac{1}{i(i^2-1)}\sum_{\ell=1}^{i-2}\ell^2(\ell+1)=\frac{1}{i(i+1)}\sum_{\ell=1}^{i-1}\ell^2-\frac{1}{i(i^2-1)}\sum_{\ell=1}^{i-1}\ell^3$$
$$=\frac{(i-1)(2i-1)}{6(i+1)}-\frac{i(i-1)}{4(i+1)}=\frac{(i-1)(i-2)}{12(i+1)},$$

and hence

$$\sum_{i=3}^{n-1}\sum_{\ell=2}^{i-1}\frac{(\ell-1)^2}{i^2-1}\left(1-\frac{\ell}{i}\right)=\sum_{i=3}^{n-1}\frac{(i-1)(i-2)}{12(i+1)}=\frac{n^2-9n-4}{24}+\frac{H_n}{2}.$$

Combining the previous cases, and subtracting the number of twice counted squares, implies that the average iper value on $\mathcal{I}_n$ is given by

$$\begin{aligned}
&2n - H_n + \left(\frac{n^2-3n-22}{12} + H_n + \frac{1}{n}\right) + \frac{(n-2)(n-3)}{12} - \left(\frac{n^2-9n-4}{24} + \frac{H_n}{2}\right)\\
&= \frac{3n^2+41n-28}{24} - \frac{H_n}{2} + \frac{1}{n},
\end{aligned}$$

as desired. □

4 Appendix

We derive in this section the functional equation for $AE(x,v)$ given in Theorem 1.

First note that by (1)–(3), we have

$$\begin{aligned}
AE(x,v) &= q^2\sum_{n\geq2}\sum_{j=1}^{n-1} a(n-1,j)x^n v^{j-1} = q^6x^2 + q^2x\sum_{n\geq2}\sum_{j=1}^{n}\sum_{i=1}^{n-1} a(n,i,j)x^n v^{j-1}\\
&= q^6x^2 + q^2xAE(x,v) + q^2xAP(x,v,v) + \frac{q^2x}{v}AN(x,v,\frac{1}{v}),
\end{aligned} \tag{14}$$

$$\begin{aligned}
AN(x,v,u) &= q\sum_{n\geq3}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=1}^{n-2} a(n-1,k,j)x^n v^{j-1}u^{j-i-1}\\
&= qx\sum_{n\geq3}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} a(n,j,j)x^n v^{j-1}u^{j-i-1}\\
&\quad + qx\sum_{n\geq2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{j-1} a(n,k,j)x^n v^{j-1}u^{j-i-1}\\
&\quad + qx\sum_{n\geq4}\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{k=j+1}^{n-1} a(n,k,j)x^n v^{j-1}u^{j-i-1}\\
&= qx\sum_{n\geq3}\sum_{j=2}^{n-1}\sum_{i=1}^{j-1} a(n,j,j)x^n v^{j-1}u^{j-i-1} + qx\sum_{n\geq2}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\sum_{k=1}^{j-1} a(n,k,j)x^n v^{j-1}u^{j-i-1}\\
&\quad + qx\sum_{n\geq4}\sum_{j=2}^{n-2}\sum_{i=1}^{j-1}\sum_{k=j+1}^{n-1} a(n,k,j)x^n v^{j-1}u^{j-i-1}\\
&= qx\sum_{n\geq2}\sum_{j=1}^{n-1} a(n,j,j)x^n v^{j-1}\frac{1-u^{j-1}}{1-u} + qx\sum_{n\geq2}\sum_{j=2}^{n}\sum_{k=1}^{j-1} a(n,k,j)x^n v^{j-1}\frac{1-u^{j-1}}{1-u}\\
&\quad + qx\sum_{n\geq3}\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1} a(n,k,j)x^n v^{j-1}\frac{1-u^{j-1}}{1-u}\\
&= \frac{qx}{1-u}\left(AE(x,v) - AE(x,uv) + AP(x,v,v) - AP(x,uv,uv)\right)\\
&\quad + \frac{qx}{1-u}\left(\frac{1}{v}AN(x,v,\frac{1}{v}) - \frac{1}{uv}AN(x,uv,\frac{1}{uv})\right)
\end{aligned} \tag{15}$$

and

$$
\begin{aligned}
AP(x,v,u) &= q^7ux^2 + \sum_{n\geq 3}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\sum_{k=1}^{i} q^{2j-2i+1}a(n-1,k,i)x^nv^{i-1}u^{j-i}\\
&\quad + \sum_{n\geq 3}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\sum_{k=i+1}^{j-1} q^{2j-k-i+2}a(n-1,k,i)x^nv^{i-1}u^{j-i}\\
&\quad + \sum_{n\geq 4}\sum_{j=2}^{n-2}\sum_{i=1}^{j-1}\sum_{k=j}^{n-2} q^{j-i+2}a(n-1,k,i)x^nv^{i-1}u^{j-i}\\
&= q^7ux^2 + qx\sum_{n\geq 2}\sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{j=i+1}^{n+1} q^{2j-2i}a(n,k,i)x^nv^{i-1}u^{j-i}\\
&\quad + q^2x\sum_{n\geq 3}\sum_{i=1}^{n-2}\sum_{k=i+1}^{n-1}\sum_{j=k+1}^{n+1} q^{2j-k-i}a(n,k,i)x^nv^{i-1}u^{j-i}\\
&\quad + q^2x\sum_{n\geq 3}\sum_{i=1}^{n-2}\sum_{k=i+1}^{n-1}\sum_{j=i+1}^{k} q^{j-i}a(n,k,i)x^nv^{i-1}u^{j-i}\\
&= q^7ux^2 + \frac{q^3ux}{1-q^2u}\sum_{n\geq 2}\sum_{i=1}^{n}\sum_{k=1}^{i} a(n,k,i)x^nv^{i-1}(1-q^{2n+2-2i}u^{n+1-i})\\
&\quad + \frac{q^4ux}{1-q^2u}\sum_{n\geq 3}\sum_{i=1}^{n-2}\sum_{k=i+1}^{n-1} a(n,k,i)x^nv^{i-1}(q^{k-i}u^{k-i}-q^{2n+2-k-i}u^{n+1-i})\\
&\quad + \frac{q^3ux}{1-qu}\sum_{n\geq 3}\sum_{i=1}^{n-2}\sum_{k=i+1}^{n-1} a(n,k,i)x^nv^{i-1}(1-q^{k-i}u^{k-i})\\
&= q^7ux^2 + \frac{q^3ux}{1-q^2u}\Bigg(AE(x,v) - AE(q^2ux,\frac{v}{q^2u}) + AP(x,v,v)\\
&\quad - AP(q^2ux,\frac{v}{q^2u},\frac{v}{q^2u})\Bigg) + \frac{q^5u^2x}{v(1-q^2u)}\left(AN(x,v,\frac{qu}{v}) - AN(q^2ux,\frac{v}{q^2u},\frac{qu}{v})\right)\\
&\quad + \frac{q^3ux}{v(1-qu)}\left(AN(x,v,\frac{1}{v}) - quAN(x,v,\frac{qu}{v})\right). \qquad (16)
\end{aligned}
$$

It is then possible to express the generating functions $AN(x,v,u)$ and $AP(x,v,u)$ in terms of $AE(x,v)$.

Theorem 7 *We have*

$$AN(x,v,u) = \frac{1}{q(1-u)}(AE(x,v) - AE(x,uv)), \qquad (17)$$

$$AP(x,v,v) = -q^4x + \frac{1-q^2x}{q^2x}AE(x,v) + \frac{AE(x,v)-AE(x,1)}{q(1-v)} \qquad (18)$$

and

$$\begin{aligned}AP(x,v,u) &= \frac{(q^2(q-1)ux+(qu-v)(v-1))qu}{(v-qu)(1-q^2u)(1-v)}AE(x,v)+\frac{q^3(1-q)u^2x}{(1-qu)(1-q^2u)(1-v)}AE(x,1)\\ &-\frac{q^4u^2x}{(1-q^2u)(v-q^2u)}AE(q^2ux,1)\\ &+\frac{q^6(q-1)u^3x-(v-q^2u)(v-qu)}{q(1-q^2u)(v-q^2u)(v-qu)}AE(q^2ux,\frac{v}{q^2u})\\ &+\frac{q^4u^2x}{(1-q^2u)(v-qu)}AE(q^2ux,\frac{1}{q})+\frac{q^3(1-q)u^2x}{(1-q^2u)(v-qu)(1-qu)}AE(x,qu). \end{aligned} \tag{19}$$

Moreover,

$$AE(x,v)=\frac{q^6x^2+q^2xAP(x,v,v)+\frac{q^2x}{v}AN(x,v,\frac{1}{v})}{1-q^2x}. \tag{20}$$

Proof. By (14), we have (20), which leads to (17), by (15). Hence, by (17) and (20), we obtain (18). Substituting (18) into (16), and making use of (17), then yields (19). □

Using Theorem 7, one can now obtain the functional equation formula given in Theorem 1 after several algebraic steps.

References

[1] M. Archibald, A. Blecher and A. Knopfmacher, Parameters in inversion sequences, *Math. Slovaca* 73(3) (2023), 551–564.

[2] A. Blecher, C. Brennan and A. Knopfmacher, The inner site-perimeter of compositions, *Quaest. Math.* 43(1) (2020), 55–66.

[3] A. Blecher, C. Brennan, A. Knopfmacher and T. Mansour, The site-perimeter of words, *Trans. Combin.* 6(2) (2017), 37–48.

[4] A. Blecher, A. Knopfmacher and T. Mansour, Alphabetic points and records in inversion sequences, *Discrete Math. Lett.* 10 (2022), 85–90.

[5] M. Bousquet-Mélou and A. Rechnitzer, The site-perimeter of bargraphs, *Adv. in Appl. Math.* 31 (2003), 86–112.

[6] W. Cao, E. Y. Jin and Z. Lin, Enumeration of inversion sequences avoiding triples of relations, *Discrete Appl. Math.* 260 (2019), 86–97.

[7] S. Corteel, M. A. Martinez, C. D. Savage and M. Weselcouch, Patterns in inversion sequences I, *Discrete Math. Theor. Comput. Sci.* 18(2) (2016), Art. #2.

[8] D. Kim and Z. Lin, Refined restricted inversion sequences, *Sém. Lothar. Combin.* 78B (2017), Art. #52.

[9] Z. Lin, Restricted inversion sequences and enhanced 3-noncrossing partitions, *European J. Combin.* 70 (2018), 202–211.

[10] Z. Lin and and S. H. F. Yan, Vincular patterns in inversion sequences, *Appl. Math. Comput.* 364 (2020), Art. 124672.

[11] T. Mansour, Border and tangent cells in bargraphs, *Discrete Math. Lett.* 1 (2019), 26–29.

[12] T. Mansour, The perimeter and the site-perimeter of set partitions, *Electron. J. Combin.* 26 (2019), Art. P2.30.

[13] T. Mansour and A. Sh. Shabani, Enumerations on bargraphs, *Discrete Math. Lett.* 2 (2019), 65–94.

[14] T. Mansour and M. Shattuck, Pattern avoidance in inversion sequences, *Pure Math. Appl.* 25 (2015), 157–176.

[15] T. Mansour and M. Shattuck, Statistics on bargraphs of inversion sequences of permutations, *Discrete Math. Lett.* 4 (2020), 37–44.

[16] T. Mansour and M. Shattuck, Counting inversion sequences by parity successions and runs, *Discrete Appl. Math.* 330 (2023), 24–39.

[17] T. Mansour and M. Shattuck, Enumeration of smooth inversion sequences and proof of a recent related conjecture, *J. Difference Equ. Appl.* 29(3) (2023), 270–296.

[18] M. Martinez and C. Savage, Patterns in inversion sequences II: inversion sequences avoiding triples of relations, *J. Integer Seq.* 21 (2018), Art. 18.2.2.

[19] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, 2020. Available at https://oeis.org.

[20] R. P. Stanley, *Enumerative Combinatorics, Volume I*, Cambridge University Press, Cambridge, UK (1997).

[21] C. Yan and Z. Lin, Inversion sequences avoiding pairs of patterns, *Discrete Math. Theor. Comput. Sci.* 22(1) (2020), Art. 6539.

[22] S. H. F. Yan, Bijections for inversion sequences, ascent sequences and 3-nonnesting set partitions, *Appl. Math. Comput.* 325 (2018), 24–30.