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Results on E-quasiconvex Functions for a linear map E

Ayache Benhadid *
Department of Mathematics, Faculty of Mathematics and Computer Science
Batna 2 University, Batna 05078, Algeria
and
Mahmoud Brahimi†
Department of Mathematics, Faculty of Mathematics and Computer Science
Batna 2 University, Batna 05078, Algeria

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Abstract

It is well known that convexity theory and its generalization play an important role and have many applications in mathematical programming. In the previous article [2], the authors studied E-convex and semi-E-convex functions. In the present article, we deal with a class of E-quasiconvex functions and give a weak condition for a lower semi-continuous function on $\mathbb{R}^n$ to be an E-quasiconvex function.

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1 Introduction

Recently, generalizations on convexity have attracted the attention of many researchers. E-convexity is one of them, mainly due to Youness [7], Following this Yu-Ru Syau and E. Stanley Lee [8] introduced the concept of E-quasiconvex functions and strictly E-quasiconvex functions. This generalizations is important for quantitative and qualitative studies in non linear programming problems or applied mathematics see [2, 3, 5, 6, 8]. In the present paper, we deal with the lower semi-continuous function on $\mathbb{R}^n$ under a weak condition to be an E-quasiconvex function.

*e-mail: a.benhadid@univ-batna2.dz
†e-mail: m.brahimi@univ-batna2.dz

2 Preliminaries

Let M be a nonempty subset of $\mathbb{R}^n$ and let E be a mapping from : $\mathbb{R}^n$ to $\mathbb{R}^n$. We recall:

DEFINITION 1 *[7] A set $M \subseteq \mathbb{R}^n$ is said to be E-convex in $\mathbb{R}^n$ if*

$$\lambda E(a) + (1-\lambda)E(b) \in M,$$

for all $a, b \in M$ and $\lambda \in [0,1]$.

DEFINITION 2 *[7] Let $M \subseteq \mathbb{R}^n$ be a nonempty E-convex set, a real-valued function $f : M \to \mathbb{R}$ is said to be E-convex on M if*

$$f\left(\lambda E(a) + (1-\lambda)E(b)\right) \leq \lambda f\left(E\left(a\right)\right) + (1-\lambda) f\left(E\left(b\right)\right),$$

for all $a, b \in M$ and $\lambda \in [0,1]$.

DEFINITION 3 *[8] Let $M \subseteq \mathbb{R}^n$ be a nonempty E-convex set, a real-valued function $f : M \to \mathbb{R}$ is said to be quasi-E-convex on M if*

$$f\left(\lambda E(a) + (1-\lambda)E(b)\right) \leq \max\left\{f\left(E(a)\right), f\left(E(b)\right)\right\},$$

for all $a, b \in M$ and $\lambda \in [0,1]$.

The main tool in this search is the following Proposition concerning the lower semi-continuous function property which shall be used in the sequel.

PROPOSITION 1 *[4] A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous if and only if $\forall \lambda \in \mathbb{R}$, the set $\{z \in \mathbb{R}^n : f(z) \leq \lambda\}$ is closed.*

DEFINITION 4 *Let $a, b \in \mathbb{R}^n$. The line segment $[a, b]$ (with endpoints a and b is the segment*

$$\{\lambda a + (1-\lambda)b : 0 \leq \lambda \leq 1\}.$$

If $a \neq b$, the interior $]a, b[$ of $[a, b]$ is the segment

$$\{\lambda a + (1-\lambda)b : 0 < \lambda < 1\}.$$

In the same way, we can define $[a, b)$ and $(a, b]$.

3 Main results

PROPOSITION 2 *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map. The function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-E-convex on $\mathbb{R}^n$ if and only if the level set K_α is E-convex on $\mathbb{R}^n$ for each $\alpha \in \mathbb{R}$. Where $K_\alpha = \{x \in \mathbb{R}^n : f(E(x)) \leq \alpha\}$.*

Proof. Suppose that f is quasi-E- convex, let $x_1, x_2 \in K_\alpha$, $\lambda \in [0,1]$. We have

$$\begin{aligned} f\left(E\left(\lambda E\left(x_1\right)+(1-\lambda) E\left(x_2\right)\right)\right) &= f\left(\lambda E\left(x_1\right)+(1-\lambda) E\left(x_2\right)\right), \\ &\leq \max\left\{f\left(E(x_1)\right), f\left(E(x_2)\right)\right\}, \\ &\leq \alpha. \end{aligned}$$

Thus $\lambda E\left(x_1\right)+(1-\lambda) E\left(x_2\right) \in K_\alpha$.
For the converse part, let $\alpha = \max\left\{f\left(E(x_1)\right), f\left(E(x_2)\right)\right\}$. Then $x_1, x_2 \in K_\alpha$, let $\lambda \in [0,1]$. Using the fact that K_α is E-convex, we get

$$f\left(\lambda E(x_1)+(1-\lambda)E(x_2)\right) \leq \alpha.$$

This completes the proof. □

Lemma 1 *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map.*
Consider $\overline{x} \in [(E\left(x\right), (E\left(y\right)]$. Then

$$E\left(\overline{x}\right) = \overline{x}.$$

Proof. Let $\overline{x} \in [(E\left(x\right), (E\left(y\right)]$, then there exist $\alpha \in [0,1]$, such that $\overline{x} = \alpha E\left(x\right) + (1-\alpha)E\left(y\right)$. Using the fact that E is linear and idempotent map, we have

$$\begin{aligned} E\left(\overline{x}\right) &= E(\alpha(E\left(x\right)+(1-\alpha)(E\left(y\right)) \\ &= \alpha E\left(x\right)+(1-\alpha)E\left(y\right) \\ &= \overline{x}. \end{aligned}$$

This completes the proof. □

Theorem 1 *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map, and $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi continuous. Then f is quasi-E-convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in]0,1[$ (α depends on x, y) such that*

$$f\left(\alpha E(x)+(1-\alpha)E(y)\right) \leq \max\left\{f\left(E(x)\right), f\left(E(y)\right)\right\}.$$

Proof. By Proposition (2), it is sufficient to show that

$$K_\lambda = \left\{x \in \mathbb{R}^n : f\left(E(x)\right) \leq \lambda\right\}$$

is E-convex on $\mathbb{R}^n$ for each $\lambda \in \mathbb{R}$. By contradiction, suppose that there exists a real number λ_0 such that the set $\left\{x \in \mathbb{R}^n : f\left(E(x)\right) \leq \lambda_0\right\}$ is not E-convex. Thus, there exist $\alpha_0 \in]0,1[$ and $x, y \in K_{\lambda_0}$, such that $\alpha_0 E\left(x\right)+(1-\alpha_0)E\left(y\right) \notin K_{\lambda_0}$.
Let $x_0 = \alpha_0 E\left(x\right) + (1-\alpha_0)E\left(y\right)$, then $x_0 \notin K_{\lambda_0}$. Using the fact that E is an idempotent map, we see that $E\left(x\right), E\left(y\right) \in K_{\lambda_0}$. Let

$$A = K_{\lambda_0} \cap [(E\left(x\right), x_0] \text{ and } B = K_{\lambda_0} \cap [(E\left(y\right), x_0]$$

Since f is a lower semi-continuous function, by Proposition (1), K_{λ_0} is a closed subset of $\mathbb{R}^n$. Consequently, A and B are bounded and closed subsets of $\mathbb{R}^n$. Also we have $x_0 \notin A$, and $x_0 \notin B$. Thus, there exist $Z_A \in A$ and $Z_B \in B$ such that,

$$\min_{Z \in A} \|Z - x_0\| = \|Z_A - x_0\|$$

and

$$\min_{Z \in B} \|Z - x_0\| = \|Z_B - x_0\|.$$

Hence, we have

$$]Z_A, Z_B[\cap K_{\lambda_0} = \emptyset \tag{1}$$

On the other hand, by the hypothesis of the theorem, for $Z_A, Z_B \in \mathbb{R}^n$ there exists an $\alpha \in]0,1[$ such that,

$$f(\alpha E(Z_A) + (1-\alpha)E(Z_B)) \leq \max\{f(E(Z_A)), f(E(Z_B))\}. \tag{2}$$

Since $Z_A \in K_{\lambda_0}$, $Z_B \in K_{\lambda_0}$. We have,

$$f(E(Z_A)) \leq \lambda_0 \text{ and } f(E(Z_B)) \leq \lambda_0. \tag{3}$$

Combining (2) and (3), we obtain,

$$f(\alpha E(Z_A) + (1-\alpha)E(Z_B)) \leq \lambda_0 \tag{4}$$

By Lemma (1) we have $E(Z_A) = Z_A$ and $E(Z_B) = Z_B$. Using (4) we get

$$f(\alpha Z_A + (1-\alpha)Z_B) \leq \lambda_0.$$

So,

$$\alpha Z_A + (1-\alpha)Z_B \in K_{\lambda_0},$$

which contradicts (1). Thus, we conclude that $K_\lambda = \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ is E-convex on $\mathbb{R}^n$ for each $\lambda \in \mathbb{R}$. ☐

Corollary 1 *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map, and $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi continuous . Then f is quasi-E-convex if and only if there exists an $\alpha \in]0,1[$ such that for all $x, y \in \mathbb{R}^n$,*

$$f(\alpha E(x) + (1-\alpha)E(y)) \leq \max\{f(E(x)), f(E(y))\}.$$

If we take $E = Id_{\mathbb{R}^n}$,, then we find the known results about quasiconvex functions.

Theorem 2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi continuous. Then f is quasiconvex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in]0,1[$ (α depends on x, y) such that*

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}.$$

Theorem 3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi continuous. Then f is quasiconvex if and only if there exists an $\alpha \in]0,1[$ such that for all $x, y \in \mathbb{R}^n$*

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}.$$

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