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Comparison of disposal strategies in linear reverse logistics models

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Abstract

In this paper we investigate disposal activities in a reverse logistics models with linear costs structure. The two models have different disposal strategy. The first model analyzes a continuous disposal strategy, i.e. disposal can happen in every time in the planning horizon. This type of models was investigated in earlier works. The second possible disposal strategy assumes that disposal is only at the end of the planning horizon. After characterization of the optimal strategies of the two models we compare the costs and the trajectories of the optimal paths. The result of the paper is that the relevant costs are higher in case of disposal at the end of the planning horizon. It is cost efficient to dispose off the not necessary products inside of the planning horizon.

Mathematics Subject Classifications (2020). 90, 93

Keywords. Remanufacturing, reverse logistics, optimal policies, optimal control, disposal, inventory

1 Introduction

Reverse logistics models with optimal control were investigated extensively in the literature. Some of the models analyze situations with disposal activity [1], [2], [5], but there are models without it [4]. If we assume that there are a possibility to choose disposal options, then there is a question which of them to choose. The used products can be disposed off continuously or only at the end in the planning horizon. In this paper we investigate these two methods of waste disposal in linear models.

Solution of reverse logistics models with continuous waste disposal is known [1], [5], and the optimal trajectories can be calculated in a simple way. This solution will be compared to a new model with disposal at the end of the planning horizon. This comparison makes it possible to decide which control mechanism to select.

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The paper will organize as follows. First we present the known models with continuous disposal and we offer a simple algorithm to construct the optimal trajectory. Then we analyze the model with disposal at the end of the planning horizon, and we compare the optimal solutions. It will be shown that the continuous disposal is more effective than the other. In the last section we summarize our investigations.

2 The models

We will investigate a two-store reverse logistics model with continuous disposal. The model can be represented as an optimal control problem with two state variables (inventory status in the first and second store) and with three control variables (rates of manufacturing, remanufacturing and disposal). The objective is to minimize the sum of the holding costs in the stores and costs of the manufacturing, remanufacturing and disposal. The following parameters are in our model:

- T: length of the planning horizon,
- $S(t)$: the rate of demand, continuous differentiable,
- r: the return rate in the second store ($0 \leq r \leq 1$),
- τ: the delay of the return, non-negative,
- $R(t)$: the rate of return, continuous differentiable, $R(t) = rS(t-\tau)$
- h_1: the inventory holding costs in the first store,
- h_2: the inventory holding costs in the second store,
- p_m: production cost for the manufacturing,
- p_r: production cost for the remanufacturing,
- p_d: cost for the disposal.

Decision variables:

- $I_1(t)$: inventory level in the first store, non-negative,
- $I_2(t)$: inventory level in the second store, non-negative,
- $P_r(t)$: rate of the remanufacturing, non-negative,
- $P_m(t)$: rate of the manufacturing (production), non-negative,
- $P_d(t)$: rate of the disposal in case of continuous disposal, non-negative.

We will make two assumptions about cost parameters of the holding and production, remanufacturing and disposal costs.

Assumption 1. The unit holding cost for the newly manufactured and remanufactured products are higher than that of returned and used items: $h_1 > h_2$.

Assumption 2. The sum of unit production and unit disposal costs are higher than that of unit remanufacturing cost: $p_m + p_d > p_r$.

The first assumption underlines that the holding costs for new and remanufactured products are higher than that of used items. If the costs of inventory holding are defined as a percent of the sales/purchasing price, then a new product has a higher holding cost. The second assumption secures that the

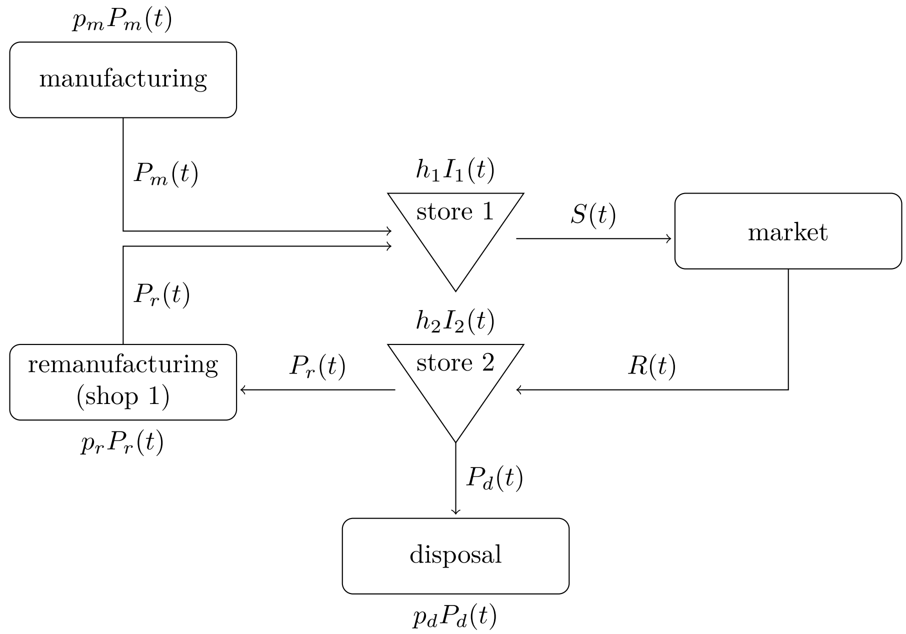

Figure 1: The material and cost flow of the continuous disposal model

remanufacturing is more effective than the manufacturing of a new product and disposal of a used and returned item.

The functioning and the cost flow of the system is presented in Figure 1. The customers demand (outflow) is satisfied from the store 1. The manufactured and remanufactured items are transported to store 1 (inflow). The returned items are collected (inflow) in store 2, in order to either remanufacture or dispose of (outflow). The subsystem store 2 can be called as reverse production-inventory system. It is assumed that not all items return, and the return process occurs with a fix delay.

The continuous disposal model can be written in the following form:

$$\int_0^T \left[h_1 I_1(t) + h_2 I_2(t) + p_m P_m(t) + p_r P_r(t) + pdP_d(t)\right] \mathrm{d}t \to \min \tag{1}$$

such that

$$\begin{bmatrix} \dot{I}_1(t) \\ \dot{I}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix} + \begin{bmatrix} -S(t) \\ R(t) \end{bmatrix}, \qquad \begin{bmatrix} I_1(0) \\ I_2(0) \end{bmatrix} = \begin{bmatrix} I_{10} \\ 0 \end{bmatrix}, \tag{2}$$

$$\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

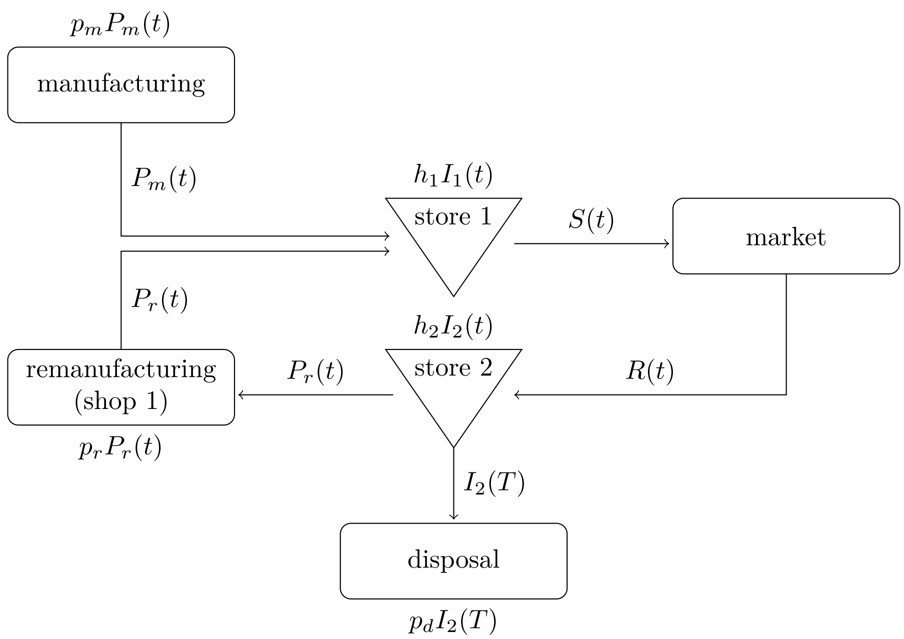

Figure 2: The material and cost flow of the model with disposal at the end of the planning horizon

$$\begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

It is assumed that the initial inventory level in store 2 is equal to zero. If not, then there exists no solution.

The model with disposal at the end of the planning horizon can be written in the following form:

$$\int_0^T \left[h_1 I_1(t) + h_2 I_2(t) + p_m(t) P_m(t) + p_r(t) P_r(t)\right] \mathrm{d}t + p_d I_2(T) \rightarrow \min \tag{5}$$

such that

$$\begin{bmatrix} \dot{I}_1(t) \\ \dot{I}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} P_m(t) \\ P_r(t) \end{bmatrix} + \begin{bmatrix} -S(t) \\ R(t) \end{bmatrix}, \qquad \begin{bmatrix} I_1(0) \\ I_2(0) \end{bmatrix} = \begin{bmatrix} I_{10} \\ 0 \end{bmatrix}, \tag{6}$$

$$\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} P_m(t) \\ P_r(t) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

The aim of the paper is to compare the optimal solutions of the posed models.

3 The model with continuous disposal

3.1 Solution of model (1)–(4)

To solve the problem, we apply the Pontryagin's maximum principle. In order to optimize the goal functional, we must introduce two auxiliary function: Hamilton function and Lagrange function. The Lagrange function is an extension of the Hamilton function with the inventory constraints.

Hamilton function

$$H\left(I_1(t), I_2(t), P_m(t), P_r(t), P_d(t), \psi_1(t), \psi_2(t), t\right) = \left[\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} + \begin{bmatrix} p_m & p_r & p_d \end{bmatrix} \begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix}\right] + \begin{bmatrix} \psi_1(t) & \psi_2(t) \end{bmatrix} \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix} + \begin{bmatrix} -S(t) \\ R(t) \end{bmatrix} \right\}$$

The next following Lagrange function is a useful tool for construction of adjoint variables.

Lagrange function

$$L\left(I_1(t), I_2(t), P_m(t), P_r(t), P_d(t), \psi_1(t), \psi_2(t), t\right) = -\left[\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} + \begin{bmatrix} p_m & p_r & p_d \end{bmatrix} + \begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix}\right] + \begin{bmatrix} \psi_1(t) & \psi_2(t) \end{bmatrix} \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix} \begin{bmatrix} -S(t) \\ R(t) \end{bmatrix} \right\} + \begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix}$$

Our problem is a convex (linear) optimal control problem. The advantage of a convex problem that necessary conditions are sufficient, as well. So it is true [3], [6]:

THEOREM 1 *In order for* $\left\{I_1^0(t), I_2^0(t), P_m^0(t), P_r^0(t), P_d^0(t)\right\}$ *to be optimal the problem (1)–(4) it is necessary and sufficient that there exist functions* $\psi_1(t)$ *and* $\psi_2(t)$*, where for all* $0 \leq t \leq T$ *we have* $\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(a) *stationarity for the inventory levels:*

$$\begin{aligned}\begin{bmatrix}\dot{\psi}_1(t)\\ \dot{\psi}_2(t)\end{bmatrix} &= -\frac{\partial L\left(I_1^0(t), I_2^0(t), P_m^0(t), P_r^0(t), P_d^0(t), \psi_1(t), \psi_2(t), \lambda_1(t), \lambda_2(t), t\right)}{\partial \begin{bmatrix}I_1(t)\\ I_2(t)\end{bmatrix}},\\ \begin{bmatrix}\dot{\psi}_1(t)\\ \dot{\psi}_2(t)\end{bmatrix} &= \begin{bmatrix}h_1\\ h_2\end{bmatrix} - \begin{bmatrix}\lambda_1(t)\\ \lambda_2(t)\end{bmatrix},\end{aligned}$$

(b) *maximum principle for the manufacturing, remanufacturing and disposal rates:*

$$\max_{\begin{bmatrix}0\\0\\0\end{bmatrix} \le \begin{bmatrix}P_m(t)\\ P_r(t)\\ P_d(t)\end{bmatrix}} \left\{H\left(I_1^0(t), I_2^0(t), P_m(t), P_r(t), P_d(t)\psi_1(t), \psi_2(t), t\right)\right\} =$$

$$H\left(I_1^0(t), I_2^0(t), P_m^0(t), P_r^0(t), P_d^0(t), \psi_1(t), \psi_2(t), t\right)$$

$$\max_{0\le P_m(t)} \left\{\left(\psi_1(t) - p_m\right) P_m(t)\right\} = \left(\psi_1(t) - p_m\right) P_m^0(t) = 0$$

$$\max_{0\le P_r(t)} \left\{\left(\psi_1(t) - \psi_2(t) - p_r\right) P_r(t)\right\} = \left(\psi_1(t) - \psi_2(t) - p_r\right) P_r^0(t) = 0$$

$$\max_{0\le P_d(t)} \left\{-\left(\psi_2(t) + p_d\right) P_d(t)\right\} = -\left(\psi_2(t) + p_d\right) P_d^0(t) = 0$$

(c) *discontinuity of the adjoint variables in some points:*

$$\begin{bmatrix}\psi_1\left(t_j^-\right)\\ \psi_2\left(t_j^-\right)\end{bmatrix} \ge \begin{bmatrix}\psi_1\left(t_j^+\right)\\ \psi_2\left(t_j^+\right)\end{bmatrix}, \quad t_j \in (0,T), \quad j \in J, \quad \begin{bmatrix}I_1^0\left(t_j\right)\\ I_2^0\left(t_j\right)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

(d) *constraint qualification for the inventory levels:*

$$\begin{bmatrix}\lambda_1(t) & \lambda_2(t)\end{bmatrix}\begin{bmatrix}I_1^0(t)\\ I_2^0(t)\end{bmatrix} = 0, \quad \begin{bmatrix}\lambda_1(t) & \lambda_2(t)\end{bmatrix} \ge \begin{bmatrix}0 & 0\end{bmatrix},$$

(e) *terminal conditions:*

$$\begin{bmatrix}\psi_1(T) & \psi_2(T)\end{bmatrix}\begin{bmatrix}I_1^0(T)\\ I_2^0(T)\end{bmatrix} = 0, \quad \begin{bmatrix}\psi_1(T) & \psi_2(T)\end{bmatrix} \ge \begin{bmatrix}0 & 0\end{bmatrix}.$$

Condition (a) is the moving rule of the adjoint variables. Conditions (d) are constraint qualification for the inventory status, and conditions (e) are the terminal conditions for the state variables. The adjoint variables can be interpreted, as the shadow prices in stores 1 and 2. From conditions (b) the optimal manufacturing and remanufacturing rates can be written in simpler way:

$$P_m^0(t) = \begin{cases}0 & \psi_1(t) \le p_m\\ \text{undefined} & \psi_1(t) = p_m\end{cases},$$

$$P_r^0(t) = \begin{cases} 0 & \psi_1(t) - \psi_2(t) \le p_r \\ \text{undefined} & \psi_1(t) - \psi_2(t) = p_r \end{cases},$$

and

$$P_d^0(t) = \begin{cases} 0 & \psi_2(t) \ge -p_d \\ \text{undefined} & \psi_2(t) = -p_d \end{cases}.$$

Now we will construct the optimal trajectory.

3.2 A simple forward algorithm to construct the optimal trajectory

In an earlier work [1] we have proven some properties of the optimal solution. These results are summarized before constructing the optimal path. The optimal trajectory has the following properties.

LEMMA 1 *The optimal controls satisfy* $P_m^0(t)P_d^0(t) = 0$ *for every* $t \in [0, T]$.

The meaning of the lemma is that manufacturing and disposal do not take place in a point of time. This property makes easier to construct the optimal path. The next lemma characterizes the intervals where only remanufacturing takes place.

LEMMA 2 *The maximal length of time* τ *for only remanufacturing is* $\tau' = \frac{p_m + p_d - p_r}{h_2}$.

The optimal trajectory has a cyclical character. A cycle $[t_1, t_4]$ consists of three parts with the control variables:

1. $P_m^0(t) = S(t) - R(t) \ge 0$, $P_r^0(t) = R(t)$, $P_d^0(t) = 0$, $t \in [t_1, t_2]$,
2. $P_m^0(t) = 0$, $P_r^0(t) = S(t)$, $P_d^0(t) = R(t) - S(t) \ge 0$, $t \in [t_2, t_3]$,
3. $P_m^0(t) = 0$, $P_r^0(t) = S(t)$, $P_d^0(t) = 0$, $t \in [t_3, t_4]$,

In this case the length of interval $[t_3, t_4]$ is not greater than τ'. It is possible that there is no disposal. The optimal inventory level is zero in store 1 except at the beginning of the planning horizon. Now we are able to build the optimal trajectory. We will use the property of the optimal trajectory that the sum of the inventory levels in both stores must be minimal. Let us investigate the model in the following form with the known manufacturing, remanufacturing and disposal rates:

$$I_1(t) + I_2(t) = I_{10} + \int_0^t [P_m(t) - P_d(t)]\,\mathrm{d}t - \int_0^t [S(t) - R(t)]\,\mathrm{d}t \ge 0,$$

$$I_1(t) = I_{10} + \int_0^t [P_m(t) + P_r(t)]\,\mathrm{d}t - \int_0^t S(t)\,\mathrm{d}t.$$

The forward algorithm consists now of two steps. In the first step we calculate the optimal rate of manufacturing, and in the second step the optimal disposal rate. It is easy to construct the optimal production rate. The production rate is equal to $S(t) - R(t)$ if it is positive. (See figure 3.) We must calculate the optimal manufacturing rate for each cycle. Then we calculate the optimal disposal rate in each cycle. We must push the line right until τ' is not achieved.

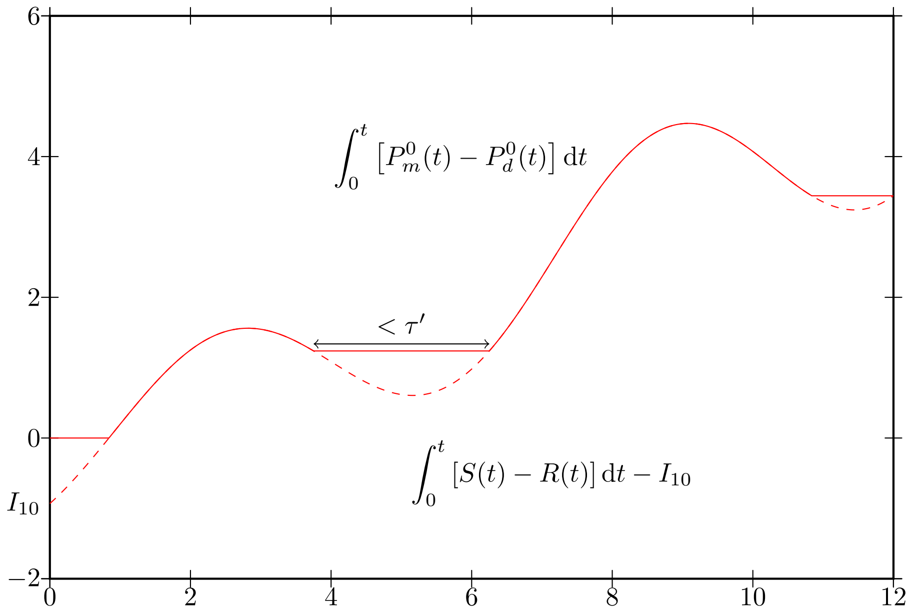

Figure 3: The construction of the optimal production and disposal rate

To illustrate the algorithm we use the next data: $T = 12$ month, $S(t) = (1+0.05t)\cdot(1+sin(t))$, $r = 0.7$, $\tau = 1$, $R(t) = rS(t-\tau)$, $h_1 = 4$, $h_2 = 2$, $p_m = 8$, $p_r = 6$, $p_d = 9$. For this case $\tau' = 2.5$ month, i.e. only remanufacturing can appear 2.5 months long. The optimal production, disposal and remanufacturing rates are

$$P_m^0(t) = \begin{cases} 0 & t < 0.8403 \\ S(t) - R(t) & 0.8403 \leq t < 2.8176 \\ 0 & 2.8176 \leq t < 6.2514 \\ S(t) - R(t) & 6.2514 \leq t < 9.0842 \\ 0 & 9.0842 \leq t \leq 12 \end{cases},$$

$$P_d^0(t) = \begin{cases} 0 & t < 2.8176 \\ R(t) - S(t) & 2.8176 \le t < 3.7514 \\ 0 & 3.7514 \le t < 9.0842 \\ R(t) - S(t) & 9.0842 \le t < 10.8364 \\ 0 & 10.8364 \le t \le 12 \end{cases},$$

and

$$P_r^0(t) = \begin{cases} R(t) & t < 2.8176 \\ S(t) & 2.8176 \le t < 6.2514 \\ R(t) & 6.2514 \le t < 9.0842 \\ S(t) & 9.0842 \le t \le 12 \end{cases}.$$

The optimal inventory levels can be calculated with the control variables. The minimal costs are 110.622.

4 The model with disposal at the end of the planning horizon

4.1 Solution of model 5–8

To solve the problem, we apply the Pontryagin's maximum principle. In order to optimize the goal functional, we must introduce two auxiliary function in this case, too: Hamilton function and Lagrange function. The Lagrange function is an extension of the Hamilton function with the inventory constraints.

Hamilton function

$$\begin{aligned} &H\left(I_1(t), I_2(t), P_m(t), P_r(t), \psi_1(t), \psi_2(t), t\right) = \\ &\qquad - \left[\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} + p_m \quad p_r \begin{bmatrix} P_m(t) \\ P_r(t) \end{bmatrix} \right] + \\ &\qquad\qquad \begin{bmatrix} \psi_1(t) & \psi_2(t) \end{bmatrix} \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{matrix} P_m(t) \\ P_r(t) \end{matrix} + \begin{bmatrix} -S(t) \\ R(t) \end{bmatrix} \right\} \end{aligned}$$

The next following Lagrange function is a useful tool for construction of adjoint variables.

Lagrange function

$$L\left(I_1(t), I_2(t), P_m(t), P_r(t), P_d(t), \psi_1(t), \psi_2(t), t\right) =$$
$$-\left[\begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} + \begin{bmatrix} p_m & p_r \end{bmatrix}\begin{bmatrix} P_m(t) \\ P_r(t) \end{bmatrix}\right] +$$
$$\begin{bmatrix} \psi_1(t) & \psi_2(t) \end{bmatrix}\left\{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} P_m(t) \\ P_r(t) \end{bmatrix} + \begin{bmatrix} -S(t) \\ R(t) \end{bmatrix}\right\} +$$
$$\begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix}\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix}$$

Our problem is a convex (linear) optimal control problem. The advantage of a convex problem that necessary conditions are sufficient, as well. So it is true [3], [6]:

THEOREM 2 *In order for* $\left\{I_1^0(t), I_2^0(t), P_m^0(t), P_r^0(t)\right\}$ *to be optimal the problem (5)–(8) it is necessary and sufficient that there exist functions* $\psi_1(t)$ *and* $\psi_2(t)$*, where for all* $0 \le t \le T$ *we have* $\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(a) stationarity for the inventory levels:

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = -\frac{\partial L\left(I_1^0(t), I_2^0(t), P_m^0(t), P_r^0(t), \psi_1(t), \psi_2(t), \lambda_1(t), \lambda_2(t), t\right)}{\partial \begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix}},$$

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} - \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix},$$

(b) maximum principle for the manufacturing, remanufacturing and disposal rates:

$$\max_{\begin{bmatrix} 0 \\ 0 \end{bmatrix} \le \begin{bmatrix} P_m(t) \\ P_r(t) \end{bmatrix}} \left\{H\left(I_1^0(t), I_2^0(t), P_m(t), P_r(t), \psi_1(t), \psi_2(t), t\right)\right\} =$$
$$H\left(I_1^0(t), I_2^0(t), P_m^0(t), P_r^0(t), \psi_1(t), \psi_2(t), t\right)$$

$$\max_{0 \le P_m(t)} \left\{\left(\psi_1(t) - p_m\right) P_m(t)\right\} = \left(\psi_1(t) - p_m\right) P_m^0(t) = 0$$

$$\max_{0 \le P_r(t)} \left\{\left(\psi_1(t) - \psi_2(t) - p_r\right) P_r(t)\right\} = \left(\psi_1(t) - \psi_2(t) - p_r\right) P_r^0(t) = 0$$

(c) discontinuity of the adjoint variables in some points:

$$\begin{bmatrix} \psi_1\left(t_j^-\right) \\ \psi_2\left(t_j^-\right) \end{bmatrix} \ge \begin{bmatrix} \psi_1\left(t_j^+\right) \\ \psi_2\left(t_j^+\right) \end{bmatrix}, \quad t_j \in (0, T), \quad j \in J, \quad \begin{bmatrix} I_1^0\left(t_j\right) \\ I_2^0\left(t_j\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

(d) constraint qualification for the inventory levels:

$$\begin{bmatrix}\lambda_1(t) & \lambda_2(t)\end{bmatrix}\begin{bmatrix}I_1^0(t)\\ I_2^0(t)\end{bmatrix} = 0, \quad \begin{bmatrix}\lambda_1(t) & \lambda_2(t)\end{bmatrix} \geq \begin{bmatrix}0 & 0\end{bmatrix},$$

(e) terminal conditions:

$$\psi_1(T)I_1^0(T) = 0, \quad [\psi_2(t) - p_d]\, I_2^0(t) = 0, \quad \begin{bmatrix}\psi_1(T) & \psi_2(T)\end{bmatrix} \geq \begin{bmatrix}0 & 0\end{bmatrix}.$$

Condition (a) is the moving rule of the adjoint variables. Conditions (d) are constraint qualification for the inventory status, and conditions (e) are the terminal conditions for the state variables. The adjoint variables can be interpreted, as the shadow prices in stores 1 and 2. From conditions (b) the optimal manufacturing and remanufacturing rates can be written in simpler way:

$$P_m^0(t) = \begin{cases} 0 & \psi_1(t) \leq p_m \\ \text{undefined} & \psi_1(t) = p_m \end{cases},$$

$$P_r^0(t) = \begin{cases} 0 & \psi_1(t) - \psi_2(t) \leq p_r \\ \text{undefined} & \psi_1(t) - \psi_2(t) = p_r \end{cases}.$$

4.2 A forward algorithm for the optimal trajectory

It is easy to see that the optimal production and remanufacturing strategy is the following except beginning of the planning horizon:

1. $P_m^0(t) = S(t) - R(t) \geq 0$, $P_r^0(t) = R(t)$, $t \in [t_1, t_2]$,
2. $P_m^0(t) = 0$, $P_r^0(t) = S(t)$, $t \in [t_2, t_3]$.

The inventory levels can be calculated by the help of the optimal production and remanufacturing rates. The optimal trajectory is determined, as above, with the sum of the inventory levels. In this case we must choose the production rate which is minimal. (See figure 4.) We apply the data of the previous section. The optimal production and remanufacturing rates are the following:

$$P_m^0(t) = \begin{cases} 0 & t < 0.8403 \\ S(t) - R(t) & 0.8403 \leq t < 2.8176 \\ 0 & 2.8176 \leq t < 6.5088 \\ S(t) - R(t) & 6.5088 \leq t < 9.0842 \\ 0 & 9.0842 \leq t \leq 12 \end{cases},$$

and

$$P_r^0(t) = \begin{cases} R(t) & t < 2.8176 \\ S(t) & 2.8176 \leq t < 6.5088 \\ R(t) & 6.5088 \leq t < 9.0842 \\ S(t) & 9.0842 \leq t \leq 12 \end{cases}.$$

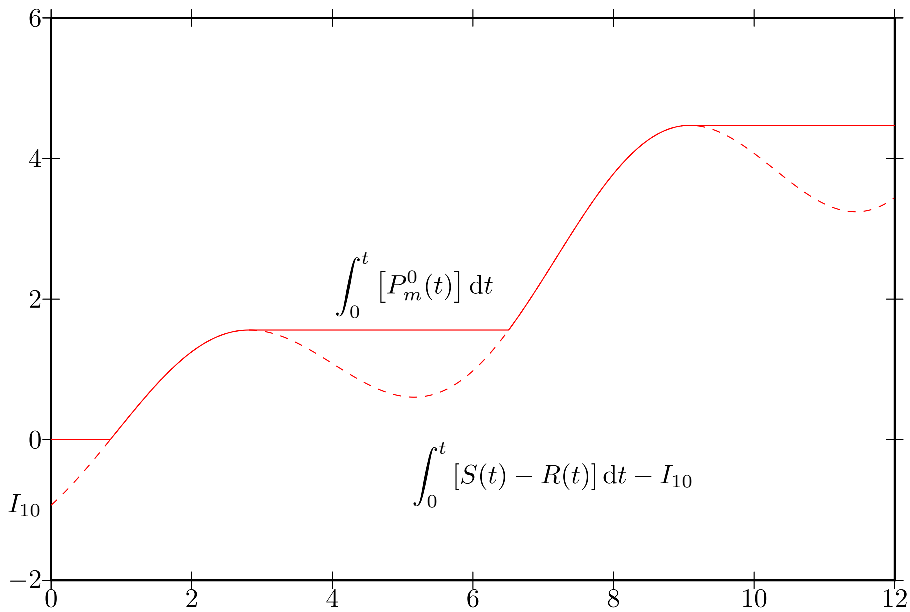

Figure 4: The construction of the optimal production rate in the second model

The minimal costs are 112.769. These costs are higher than that of continuous disposal. We look for the case of this difference. In the next section we prove that the continuous disposal results always a lower cost.

5 Comparison of the optimal solutions of the models

The numerical analysis has shown that the continuous disposal has a lower cost than that of disposal at the end of the planning horizon. It is not an accident, and we will prove it.

THEOREM 3 *The continuous disposal gives a not greater cost than the model with disposal at the end of the planning horizon.*

Proof. Let us assume that optimal solution of model with continuous disposal is $(P_m^c(t), P_r^c(t), P_d^c(t), I_1^c(t), I_2^c(t))$ and for the other $(P_m^e(t), P_r^e(t), I_1^e(t), I_2^e(t))$. We know that the optimal solution of the second model is possible solution in the first model, but the optimal solution in the first model is not a possible solution in the second model. It means that the set of the possible solutions is greater in the first model than in the second. So

$$\int_0^T \left[h_1 I_1^c(t) + h_2 I_2^c(t) + p_m P_m^c(t) + p_r P_r^c(t) + p_d P_d^c(t)\right] \mathrm{d}t \leq$$
$$\int_0^T \left[h_1 I_1^e(t) + h_2 I_2^e(t) + p_m P_m^e(t) + p_r P_r^e(t) + p_d \cdot 0\right] \mathrm{d}t + p_d \cdot I_2^e(T),$$

and this fact proves the theorem. The strict equality is true, if in the continuous case the optimal disposal rate is zero in the planning horizon. □

6 Conclusion

In this paper we have compared two linear reverse logistics model with different disposal option. An algorithm was constructed to determine the optimal trajectories in both models. We have shown that the continuous disposal along the planning horizon results a better solution.

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