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Controlling a demographic wave in defined contribution pension systems

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Abstract

A common problem in systems and control theory is to provide an approximation to non-linear systems. We provide a novel approach as a general solution to this problem originally conceived by Gamkrelidze. We consider and solve a general approximation problem which provides the fundamentals for various switching-type systems thus encompassing a wide range of systems theory problems.

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1. Introduction

Vertical integration is one of the key elements of modern industrial production [1,2]. We witness the widespread penetration of industrial sensors and monitoring devices embedded in the production chain resulting in a vast amount of digital data on plant level activity. This, together with the control and system theory provides a unique chance to implement a decentralised automated production system resulting in improved efficiency and higher output as production optimisation requires optimal systems. In the following we provide an important element in optimal system control applicable to a wide range of state-of-the-art systems.

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Optimising linear control systems shows similarities with linear programming as it was noticed by Pontryagin and others. Starting with the qualitative analysis of optimal control, Gamkrelidze [3] considered the

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = \xi,$$

linear, time varying (LTV) system for controls over the $u(t) \in \overset{k}{\underset{1}{\times}}[0,1]$ cube. He proved that if $u : [0,T] \to \overset{k}{\underset{1}{\times}}[0,1] = U$ is piecewise continuous, then for every $\varepsilon > 0$ there exists a piecewise continuous control $v : [0,T] \to U$, which takes its values from the vertices of the cube U and the solution

$$\dot{y}(t) = A(t)y(t) + B(t)v(t), \quad y(0) = \xi$$

satisfies the condition

$$\| x(t) - y(t) \| < \varepsilon.$$

Thus, even though the $u(t)$ and $v(t)$ controls might show significant pointwise differences, the respective trajectories will still remain uniformly close.

We will use a simple application in the following sections to demonstrate the novel approach presented in the approximation theorems. This application is a simple Buck-Boost converter and is widely applied [4,5].

2. Applications

We will demonstrate the proposed approximation problem through a well-known engineering application. Let's consider the so called Buck-Boost converter circuit:

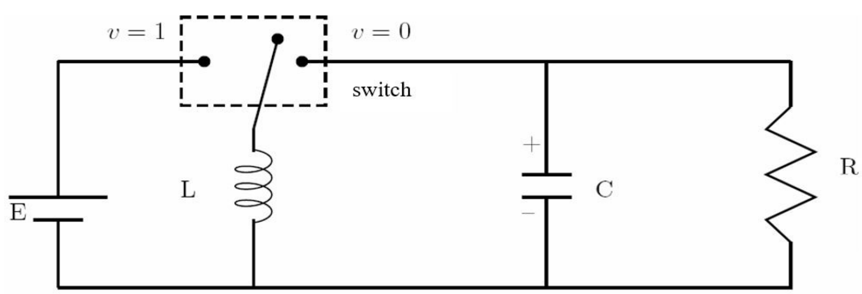

Figure 1. A Buck-Boost converter circuit

The function $v(t)$ describes the state of the switch and can take discrete values $\{0, 1\}$, as it is visible on Figure 1. The ideal behaviour can be characterised by a piecewise continuous function $u : [0,T] \to [0,1]$ which is described by the following system of differential equations

$$\begin{aligned} L\dot{x}_1 &= (1-u)x_2 + uE \\ C\dot{x}_2 &= -(1-u)x_1 - \frac{x_2}{R} \end{aligned} \tag{3}$$

where x_1 denotes current flowing through the coil and x_2 the voltage drop on the capacitor.

Function u which describes the ideal behaviour cannot be generated by turning the switch on and off, only a good approximation can be achieved. For example, for a given precision $\varepsilon > 0$ a function $v(t)$ can be assigned describing a set of switches,

$$\begin{aligned} L\dot{y}_1 &= (1-v)y_2 + vE \\ C\dot{y}_2 &= -(1-v)y_1 - \frac{y_2}{R} \end{aligned} \tag{4}$$

where the resulting y_1 and y_2 fulfil the conditions

$$\left|x_1(t) - y_1(t)\right| < \varepsilon, \left|x_2(t) - y_2(t)\right| < \varepsilon.$$

It is visible that our simple example is nonlinear, does not have any connection with the optimal control, and is a typical problem in systems theory. Mathematically, however it is of a very similar nature as the approximation problem considered by Gamkrelidze.

3. Problem formulation

Herewith we consider and solve a general approximation problem which provides the fundamentals for further switching-type systems encompassing a wide range of systems theory problems. The Buck-Boost converter is considered as the paradigm of switching systems.

Introducing LPV systems from a Buck-Boost converter.
The system is non-linear, thus it cannot be fitted in the framework of the Gamkrelidze approximation theorem. Instead of linearising we introduce a parameter to replace the $\binom{x_1}{x_2}$ state making the system formally linear. After dividing by the physical constants L and C, (3) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= \frac{1}{L}x_2 + \left(\frac{E}{L} - p_2\frac{1}{L}\right)u, \\ \dot{x}_2 &= -\frac{1}{C}x_1 - \frac{1}{RC}x_2 + \frac{1}{C}p_1 u, \end{aligned} \tag{5}$$

or in matrix form,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & \frac{1}{RC} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L}p_2 \\ \frac{1}{C}p_1 \end{pmatrix} u. \tag{6}$$

Considering the $\binom{p_1}{p_2}$ parameter vector as the $\binom{x_1}{x_2}$ state variable, we arrive to the non-linear system in (4). We can immediately raise the problem of adopting a Gamkrelidze-type approximation theorem to LPV or LPTV systems, while in the same time giving a generalisation.

It is well visible that the outlined problem is purely of system theory and not of optimisation theory. We will call the approximating systems which will be piecewise continuous constant parametric linear systems, similarly to the Buck-Boost switches switching systems.

Let $U \subset R^{k_1}$ be a convex polyhedron and $P \subset R^{k_1}$ be a compact set. We assume that

$$A : P \times [0, T] \to R^{n \times n}$$

$$B : P \times [0, T] \to R^{n \times k_1}$$

are satisfying the uniform Lipschitz-condition in t with L_1 and L_2 parameters, respectively.

We substitute the p parameter of the

$$\dot{x} = A(p, t)x + B(p, t)u \tag{7}$$

LPTV system with a state-time-dependent variable. For this consider an open set $D \subset R^n \times [0, T]$, and the $p : D \to P$ parameter-function. We assume that function p is uniformly Lipschitz-continuous with a Lipschitz-constant L_3. To have an easy picture of the structure of such a set D, consider the following figure:

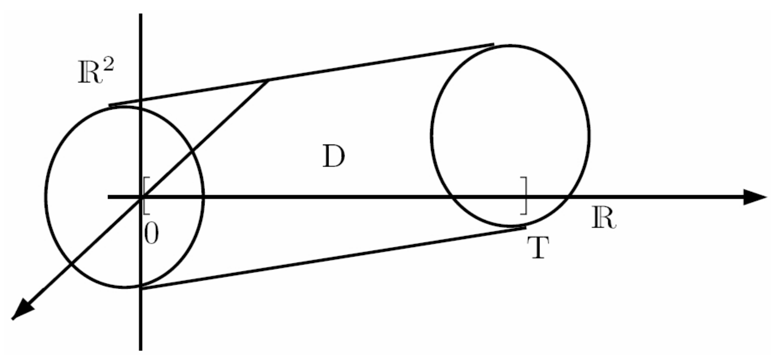

Figure 2.

The basis and top shape of the cylinder-like object belongs to the D-domain, but the constituents of the cylinder do not, due to the openness in $R^2 \times [0, T]$.

4. Approximation Theorem

In the followings we outline an approximation theorem. In the theorem the approximation of control u can contain large discrete-point errors, since the approximating v control considers the vertices of the U convex polyhedron, but the respective trajectories are uniformly close.

Approximation Theorem. Assume that for a piecewise continuous $u : [0, T] \to U$ control, and for a uniformly global Lipschitz-continuous state-time-dependent parameter function $p : D \to \mathcal{P}$, t in t, and for a $\xi \in R^n$ initial condition there is a solution for the

$$\begin{aligned} \dot{x}(t) &= A\left(p(x(t),t),t\right)x(t) + B\left(p(x(t),t),t\right)u(t), \\ x(0) &= \xi \end{aligned} \tag{8}$$

initial value problem. Then, there is a $\varepsilon_0 > 0$ for which for every $0 < \varepsilon < \varepsilon_0$ the following exist:

1) $\delta > 0$
2) a piecewise constant $v : [0,T] \to \mathcal{U}$ which takes the values of the vertices of the U convex polyhedron,
3) $q : D \to \mathcal{P}$ piecewise constant state-time-dependent parameter function, that for all $\eta \in R^n$ initial conditions satisfying $\| \xi - \eta \| < \delta$ the solution of the initial value problem of
$$\begin{aligned} \dot{y}(t) &= A\left(q(y(t),t),t\right)y(t) + B\left(q(y(t),t),t\right)v(t), \\ y(0) &= \eta \end{aligned} \tag{9}$$

has the whole $[0,T]$ interval as a domain, and there $\|x(t) - y(t)\| < \varepsilon$ holds.

Before providing the proof, we prove an approximation lemma which will replace matrix-norm estimations during the proof of the theorem.

Approximation lemma. Let $U \subset R^{k_1}$ be a convex finite polyhedron $U : [0,T] \to \mathcal{U}$, a piecewise continuous function, and $\mathbf{B} : [0,T] \to R^{n\times k_1}$ a piecewise continuous matrix function. Then for any $\bar{\varepsilon} > 0$ there exists a $v : [0,T] \to \mathcal{U}$ piecewise continuous approximation function, which takes values in the vertices of $\mathcal{U}$, and

$$\left\| \int_0^t B(\tau)\left(u(\tau) - v(\tau)\right)d\tau \right\| < \bar{\varepsilon} \tag{10}$$

Proof. Let $d_{\mathcal{U}}$ be the diameter of the $\mathcal{U}$ polyhedron, that is $d = \max \| u_{l_1} - u_{l_2} \|$, where u_{l_1}, u_{l_2} run along the vertices of the $\mathcal{U}$ polyhedron, furthermore

$$|B| = \sup_{t\in[0,T]} \| B(t) \| \,.$$

Since u and B are piecewise continuous, thus for all $\bar{\varepsilon} > 0$ there exist such $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = T$ points in $[0,T]$ that on the $[t_{i-1}, t_i) \subset [0,T]$ intervals both u and B are continuous, and

$$\begin{aligned} &\| u(t) - u(t_{i-1}) \| < \frac{\bar{\varepsilon}}{3T|B|}, \quad \text{if} \quad t \in [t_{i-1}, t_i), \\ &\| B(t) - B(t_{i-1}) \| < \frac{\bar{\varepsilon}}{3Td_{\mathcal{U}}}, \quad \text{if} \quad t \in [t_{i-1}, t_i), \\ &(t_i - t_{i-1})|B|d_{\mathcal{U}} < \frac{\bar{\varepsilon}}{3T}. \end{aligned}$$

Denote the vertices of the $\mathcal{U}$ polyhedron with $u_1, u_2, \ldots, u_L$. Since $\mathcal{U}$ is convex, therefore there exists at least one convex combination $u_1, u_2, \ldots, u_L$ of the vertices for which

$$u(t_{i-1}) = \sum_{l=1}^{L} \lambda_l^i u_l \qquad \left(\sum_{l=1}^{L} \lambda_l^i = 1,\ \lambda_l^i \geq 0 \right).$$

Let‘s partition the $[t_{i-1}, t_i)$ interval in $\lambda_1^i, \lambda_2^i \ldots \lambda_L^i$ proportions to sub-intervals, allowing for 0 length: $t_{i1} = t_{i0} \leq t_{i1} \leq \ldots t_{iL} = t$ where

$$t_{il} = t_{i-1} + (t_i - t_{i-1}) \left(\sum_{l_1=1}^{l} \lambda_{l_1}^i \right).$$

Let’s define the approximating function $v : [0, T] \to \mathcal{U}$ on the sub-interval $t \in [t_{il-1}, t_{il}) \subset [t_{i-1}, t_i)$ with the value $v(t) = u_l$ from the vertex. Let $t \in [0, T)$ be arbitrary, then $t \in [t_{i-1}, t_i)$ for an i^{th} sub-interval.

Then, for $t < T$,

$$\left\| \int_0^t B(\tau)(u(\tau) - v(\tau)) d\tau \right\| \leq \left\| \int_{t_{i-1}}^t B(\tau) \left(u(\tau) - v(\tau) \right) d\tau \right\| + \\ + \sum_{j=1}^{i-1} \left\| \int_{t_{j-1}}^{t_j} B(\tau) \left(u(\tau) - v(\tau) \right) d\tau \right\|.$$

We estimate seperately the first element and the sum of the right side:

$$\left\| \int_0^t B(\tau) \left(u(\tau) - (\tau) \right) d\tau \right\| \leq \int_{t_{i-1}}^t \parallel B(\tau) \parallel \cdot \parallel u(\tau) - v(\tau) \parallel d\tau \leq \\ \leq (t - t_{i-1}) |B| d_U \leq (t_i - t_{i-1}) |B| d_U < \frac{\bar{\varepsilon}}{3}.$$

Simple calculations yield the estimation

$$\left\| \int_0^t B(\tau) \left(u(\tau) - v(\tau) \right) d\tau \right\| \leq \bar{\varepsilon}$$

proving our approximation lemma. □

Approximation theorem. Proof.
Assume that

$$\xi \in R^n, \quad \text{and} \quad \parallel \xi - \zeta \parallel < \frac{\varepsilon}{6} \exp(-LT),$$

where the L constant can be defined using

$$|A| = \sup \parallel A(p,t) \parallel, |B| = \sup \parallel B(p,t) \parallel, |u| = \sup \parallel u(t) \parallel,\ |x| = \sup \parallel x(t) \parallel$$

as follows:

$$L = L_1 L_3 |x| + L_2 L_3 |u| \cdot |A|,$$

where $L = L_1 L_3$ are the Lipschitz-constans introduced to the A,B,P as global functions of t, respectively.

According to our conditions the initial value problem has a solution on the complete $[0, T]$. Since $D \subset R^n \times [0, T]$ is an open set, therefore there exists a $\varepsilon_0 > 0$, for which the closed ε_0 radius neighbourhood of $x : [0, T] \to R^n$ is a subset of D. Then an $0 < \varepsilon$ can be selected arbitrarily under the condition $\varepsilon < \varepsilon_0$. Let v be an arbitrary control for now, on the following problem:

$$\begin{aligned} &\dot{z}(t) = A(p(z(t), t), t)z(t) + B(p(z(t), t), t)v(t), \\ &z(0) = \zeta \end{aligned} \tag{11}$$

We can compare the z solution of this problem on the semi-open domain $[0, T_0)$ of the interval $[0, T]$ with the solution of $x : [0, T] \to R^n$. For this reason we integrate equations (8) and (11) and estimate the deviations of the solutions,

$$\begin{aligned} &\| \, x(t) - z(t) \, \| \le \| \, \xi - \zeta \, \| + \\ &+ \left\| \int_0^t \Big(A(p(x(\tau), \tau), \tau) x(\tau) - A(p(z(\tau)(\tau), \tau) z(\tau) \Big) d\tau \right\| + \\ &+ \left\| \int_0^t \Big(B(p(x(\tau), \tau), \tau) u(\tau) - B(p(z(\tau)(\tau), \tau) v(\tau) \Big) d\tau \right\| \le \\ &\le + \| \, \xi - \zeta \, \| \left\| \int_0^t \Big(A(p(x(\tau), \tau), \tau) x(\tau) - A(p(z(\tau)(\tau), \tau) x(\tau) \Big) d\tau \right\| + \\ &+ \left\| \int_0^t \big(A(p(z(\tau), \tau), \tau \big) \, (x(\tau) - z(\tau)) \, d\tau \right\| + \\ &+ \left\| \int_0^t \Big(B(p(x(\tau), \tau), \tau) u(\tau) - B(p(z(\tau)(\tau), \tau) u(\tau) \Big) d\tau \right\| + \\ &+ \left\| \int_0^t \Big(B(p(z(\tau), \tau) \, (u(\tau) - v(\tau)) \, d\tau \right\| . \end{aligned}$$

Let us estimate each of the last four elements separately.

$$\left\|\int_0^t \Big(A(p(x(\tau),\tau),\tau) - A(p(z(\tau)(\tau),\tau)\Big)x(\tau)d\tau\right\| \le$$
$$\le \int_0^t \left\|A\left(p(x(\tau),\tau),t\right) - A\left(p(z(\tau),z),\tau\right)\right\| \cdot \parallel x(\tau) \parallel d\tau \le$$
$$\le L_1 L_3 |x| \int_0^t \parallel x(\tau) - z(\tau) \parallel d\tau.$$
$$\left\|\int_0^t A(p(z(\tau),\tau),\tau)\left(x(\tau) - z(\tau)\right)d\tau\right\| \le |A| \int_0^t \left\|x(\tau) - z(\tau)\right\| d\tau$$
$$\left\|\int_0^t \big(B(p(x(\tau),\tau),\tau) - B\left(p(z(\tau),\tau),\tau\right)\big)u(\tau)d\tau\right\| \le$$
$$\le \int_0^t \left\|B\left(p(x(\tau),\tau),\tau\right) - B\left(p(z(\tau),\tau),\tau\right)\right\| \cdot \left\|u(\tau)\right\| d\tau \le$$
$$\le L_2 L_3 |u| \int_0^t \left\|x(\tau) - z(\tau)\right\| d\tau.$$

The last element can not be easily estimated to justify our approximation theorem. Therefore, we introduce the following notation:

$$L = L_1 L_3 |x| + L_2 L_3 |u| + |A|,$$
$$\psi(t) = \left\|\int_0^t B\left(p(z(\tau),\tau)\right)\left(u(\tau) - v(\tau)\right)d\tau\right\|,$$
$$\varphi(t) = \parallel x(t) - z(t) \parallel .$$

Then, we get

$$\varphi(t) \le \varphi(0) + L\int_0^t \varphi(\tau)d\tau + \psi(t) \tag{12}$$

We substitute (12) for :

$$\varphi(t) \le \varphi(0) + L\int_0^t \left(\varphi(0) + L\int_0^t \varphi(\tau_2)d\tau_2 + \psi(\tau_1)\right)d\tau_1 + \psi(t).$$

We repeat this with the resulting inequality, to get that

$$\begin{aligned}&\| x(t)-z(t) \| \leq \exp Lt \| \xi-\zeta \| + \\ &+\left\| \int_0^t B\left(p(z(t),\tau),\tau\right)\left(u(\tau)-v(\tau)\right)d\tau \right\| + \\ &+L\int_0^t \exp L(t-\tau_1)\left\| \int_0^{\tau_1} B\left(p(z(\tau_2)),\tau_2\right)\left(u(\tau_2)-v(\tau_2)\right)d\tau_2 \right\| d\tau_1.\end{aligned} \tag{13}$$

Assuming that

$$\| \xi-\zeta \| < \frac{\delta}{2} < \frac{\varepsilon}{6}\exp(-LT), \quad \text{azaz} \quad \bar{\delta} < \frac{\varepsilon}{3}\exp(-LT)$$

$$\psi(t) = \left\| \int_0^t B\left(p(z(\tau),\tau),\tau\right)\left(u(\tau)-v(\tau)\right)d\tau \right\| < \frac{\varepsilon}{6},$$

and, for having the third element to be smaller than $\frac{\varepsilon}{6}$, it is necessary, that

$$\psi(t) = \left\| \int_0^t B\left(p(z(\tau),\tau)\tau\right)\left(u(\tau)-v(\tau)\right)d\tau \right\| \leq \frac{\varepsilon}{6(\exp LT-1)}$$

also holds. Applying our approximation lemma using the function

$$B(t) = B\left(p(z(t),t),t\right)$$

there exists a control $v:[0,T]\to\mathcal{U}$ which can have values only from the vertices of the $\mathcal{U}$ convex polyhedron, is piecewise constant and $\psi(t) < \frac{\varepsilon}{6}$.

Thus

$$\psi(t) < \min\left\{\frac{\varepsilon}{6(\exp LT-1)},\ \frac{\varepsilon}{6}\right\},$$

from which follows, that

$$\| x(t)-z(t) \| < \frac{\varepsilon}{2}.$$

From this follows from the bound-to-bound continuity of the solutions that for the piecewise constant control $v:[0,T]\to\mathcal{U}$ taking its range in the vertices of the $\mathcal{U}$ convex polyhedron, the respective $z:[0,T]\to R^n$ has the complete $[0,T]$ as its domain. Therefore we need to examine if such a $v:[0,T]-\mathcal{U}$ exists, as stated in the approximation theorem, for which

$$\begin{aligned}\psi(t) = \left\| \int_0^t B\left(p(z(\tau),\tau),\tau\right)\left(u(\tau)-v(\tau)\right)d\tau \right\| \leq \\ \leq \min\left\{\frac{\varepsilon}{6(\exp LT-1)},\ \frac{\varepsilon}{6}\right\}\end{aligned}$$

holds. For this, apply the approximation lemma on the function

$$t \mapsto B\left(p(z(t),t),t\right) = B(t)$$

and on the number

$$\bar{\varepsilon} = \min\left\{\frac{\varepsilon}{6(\exp LT - 1)},\ \frac{\varepsilon}{6}\right\}$$

which asserts that with the constructed v control the solutions x and z of the initial value problems (8) and (11) satisfy $\| x(t) - z(t) \| < \frac{\varepsilon}{2}$, thus z also has its domain on the whole $[0,T]$.

Let us compare now the solutions y and z of initial value problems (9) and (11). We would like to choose the piecewise constant parameter-function $q : D \to \mathcal{P}$ in equation (9) that $\| y(t) - z(t) \| < \frac{\varepsilon}{2}$ would hold assuming $\| \eta - \xi \| < \bar{\bar{\delta}}$ for some $\bar{\bar{\delta}} > 0$ which would be smaller than $\bar{\delta}$. Then, $\delta > 0$ in the first approximation theorem could be chosen as $\delta = \min\left\{\bar{\delta}, \bar{\bar{\delta}}\right\}$. Stating the estimation the usual way and assuming that $\| \zeta - \eta \| < \bar{\bar{\delta}}$, and $p \parallel (x,t) - q(x,t) \| < \bar{\varepsilon}, \forall (x,t) \in D$ we find that for the function $\phi(t) = \| z(t) - y(t) \|$ and the constants

$$L = L_1 L_3 \left(|x| + \frac{\varepsilon}{2}\right) + L_2 L_3 |v| + |A|,$$
$$M = \left(L_1(|x| + \frac{\varepsilon}{2}) + L_2|v|\right) T$$

the inequality

$$\varphi(t) \leq \varphi(0) + L \int_0^t \varphi(\tau) d\tau + M$$

holds. The iterative process applied previously multiple times yields

$$\varphi(t) \leq (\phi(0) + M\xi) \exp(-LT).$$

This is almost the original form of the Gronwall-Bellman-lemma. If we want this deviation to be smaller than $\frac{\varepsilon}{2}$, then we have to choose the $\bar{\bar{\delta}} < \frac{\varepsilon}{4} \exp(-LT)$, and

$$\bar{\varepsilon} < \frac{\varepsilon}{4M} \exp(-LT)$$

constraints. This, together with the inequality

$$\| x(t) - y(t) \| \leq \| x(t) - z(t) \| + \| z(t) - y(t) \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

proves our aproximation theorem. ∎

5. Conclusions

Let us now notice that although we required that the A and B matrix functions to be linearly dependent from the parameter, we can, for all $(x,t) \to A(x(t),t), (x,t) \to B(x(t),t)$, satisfying the smoothness condition transform the

$$\dot{x} = A(x,t)x + B(x,t)n$$

system to an LPV system. For this consider the following

$$A(x,t) = (a_{ij}(x,t))_{i,j=1}^{n}\,,\ B(x,t) = (b_{ij}(x,t))_{i=1,j=1}^{n,k_1}$$

matrices. Replace $a_ij(x,t)$ with the p_{ij} parameters, and $b_{ij}(x,t)$ with the q_{ij} parameters. Thus, we defined the $A(p)$ and $B(q)$ parameter-variable matrices, which are obviously linear in p,q parameters, which is nothing else but the linearity of matrix addition. With this we get the

$$\dot{x}A(p)x + B(q)u$$

LPV system. Using the substitutions $p_{ij} = a_{ij}(x,t)$ and $q_{ij} = b_{ij}(x,t)$ trivially the original system is recovered. Obviously, not too much is gained with this LTV-ification, under such general circumstances the system will not show any interesting qualities.

Our example with the converter well illustrates the range of applicability of our approximation theorem specific to real life applications [7],[8].

In systems theory analysis this theorem is quite promising, e.g. in stability, controllability and observability [6],[9].

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