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Original Paper

APPROXIMATE SOLUTION OF OPTIMAL PULSE CONTROL PROBLEM ASSOCIATED WITH THE HEAT CONDUCTION PROCESS

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We consider the approximate solution of the control problem with minimum energy for an object described by the heat equation, with the process described by the linear equation of parabolic type and the system controlled by impulsive external influences. Our optimal control problem deals with finding a control parameter belonging to the class of admissible controls that provides the desired temperature distribution in a finite time with minimal energy consumption (energy consumption is described by the quadratic functional). Previous works dedicated to optimal impulse control problems have mostly used the Pontryagin's maximum principle. However, from a practical point of view, this approach does not lead to satisfactory results. This is due to the fact that the corresponding boundary value problems in this case have no solution in a traditional class of absolutely continuous trajectories. In this work, we propose a method based on the moment relations. We seek for the approximate solution of the corresponding boundary value problem in the form of finite Fourier sum and state our optimal control problem in a finite-dimensional phase space. As a result, we obtain an optimal impulse control problem in a finite-dimensional function space. Taking into account the given condition for a finite time, we reduce the obtained problem to the L-problem of moments. Thus, the problem of finding a control parameter is reduced to the solution of the system of Fredholm integral equations of the first kind, with the norm of the sought solution not exceeding a given number. By Levi's theorem, every element of Hilbert space can be represented by the sum of the elements of two orthogonal subspaces. This assertion makes it possible to find control parameters in analytical form. We also establish the convergence of the chosen approximation.

Keywords: *optimal impulse control of parabolic system, heat transfer process, Fourier method, moment problem.*

INTRODUCTION

In practice, the control parameters act at given moments of time, or in isolated spatial points (Dykhta *et al.,* 2003; Aliyev *et al.,* 2010). Such problems are called optimal impulse control problems. In previous works (Bressan *et al.*, 1991; 1994; Dykhta, 1995; Yegorov, 2004), the optimal impulse control problems for lumped parameter systems have been considered. These problems have been considered for distributed parameter systems, when the impulse effect occurs only in one point (Yegorov, 1978; Mamedov, 2006).

In the case of lumped systems, the method of L-problem of moments has been used to solve the considered problems (Mamedov *et al*., 2017). In the simplest cases, the author succeeded to determine the structure of the set of optimal solutions. Optimal control problems with nonlocal boundary conditions have also been considered (Abdullayev, 2017; Aida-zade *et al.,* 2017; Abdullayev, 2018; Abdullayev *et al,* 2018; Aida-zade *et al.,* 2020). Similar problems have been considered for systems with focused parameters (Dykhta *et al.,* 2014; Sharifov, 2014; Samsonyuk, 2015).

Most of the previous works have considered the optimal impulse control problems with lumped parameters, using mainly the maximum principle method. This creates some problems with the construction of conjugate problem and control parameter. In this work, we treat the optimal impulse control problem with distributed parameters. Namely, we consider the process of heat transfer in a rod. Impulse effects occur at the given moments of time and in isolated spatial points. The quality criterion is a quadratic functional.

We propose a different approach to solving our problem. In other words, we introduce a concept of a generalised solution of the corresponding mixed problem and we construct the finite-dimensional approximation of this solution in the form of truncated Fourier series. Thus, we obtain the integral representation for the coefficients of this series. This provides us with a system of integral equations with respect to control parameters. Thus, the problem is reduced to finding a vector of minimum length from these moment relations. Applying the theorem on orthogonal expansion of normed space, we define every approximation of the control parameter and the corresponding value of the functional in analytical form. We also prove the convergence of chosen approximation. Finally, we give an example to illustrate our method.

Note that the optimal impulse control problems are one of the most important parts of optimisation theory. These problems deal with dynamical processes with discontinuous trajectories and generalised impulse-type controls. The main motivation for considering these problems is modelling of processes controlled during time intervals so short that they can be idealised into instantaneous time intervals. In practice, such problems occur in laser technology, robotics, rocket dynamics, quantum electronics, ecology, economics, etc. In spite of the practical importance of impulse control problems, they have been scarcely studied for distributed systems.

PROBLEM STATEMENT

Let a controlled process be described by the function *u (t, x)*, which satisfies the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(t,x) + F(t,x,p) \qquad (1)$$

inside the domain $\overline{Q} = [0 \le x \le 1, 0 \le t \le T]$ and the initial and boundary conditions

$$u(0,x) = u^0(x), \qquad (2)$$

$$\frac{\partial u(t,0)}{\partial x} = 0, \quad \frac{\partial u(t,1)}{\partial x} + \alpha u(t,1) = 0, \qquad (3)$$

on the boundary $\overline{Q}$, where $f(t, x)$, $F(t, x, p)$ and $u^0(x)$ are the given functions, and p is a control parameter.

We will consider the optimal control problems for the following different special cases:

a) $F = \sum_{i=1}^{m} \delta(t - t_i) p_i(x)$, where $0 < t_1 < t_2 < ... < t_m < T$ are the fixed time points, $\delta(t)$ is the Dirac function, and $p_i(x)$, $i = \overline{1, m}$ are the control functions.

The class of admissible controls is defined as the set of all vector functions $\overline{p}(x) = (p_1(x), p_2(x). ..., p_m(x))$ belonging to the space $L_2^m(0,1)$.

b) $F = \sum_{j=1}^{l} \delta(x - x_j) p_j(t)$, where $0 < x_1 < x_2 < ... < x_l < l$ are the given observation points, and $p_j(t)$, $j = \overline{1, l}$, are the control functions.

The class of admissible controls is defined as a set of all vector functions $\overline{p}(t) = (p_1(t), p_2(x). ..., p_l(t))$ belonging to the space $L_2^m(0,T)$.

c) $F = \sum_{i=1}^{m} \sum_{j=1}^{l} \delta(t - t_i) - \delta(x - x_j) p_{ij}(t)$, where $0 < t_1 < t_2 < ... < t < T$ are the given time points, $0 < x_1 < x_2 < ... < x_m < 1$ are the observation points, and p_{ij} 's are the numerical parameters. Admissible control parameters are arbitrary real numbers.

Thus, in every one of the above cases it can be assumed that the admissible controls belong to the class of quadratically summable functions. Therefore, if $f(t,x) \in L_2(Q)$, $u^0(x) \in L_2(0,1)$, then every admissible control defines a unique generalised solution of the boundary value problem (1)-(3), which possesses the following properties (Mamedov, 2006):

1) $u(t,x) \in L_2(Q)$

2) formally calculated derivative $\frac{\partial u(t,x)}{\partial x}$ belongs to $L_2(0,1)$ for almost every $t \in [0,T]$.

3) for every function $\Phi(t,x) \in W_2^1(Q)$, the function $u(t,x)$ satisfies the integral identity

$$\int_0^1 u(t,x)\Phi(t,x)\Big|_{t=t_1}^{t=t_2} dx - \int_{t_1}^{t_2}\int_0^1 \{u(t,x)\frac{\partial \Phi(t,x)}{\partial t} - \frac{\partial u}{\partial x}\cdot\frac{\partial \Phi}{\partial x} +$$
$$+[f(t,x) + F(t,x.p)]\Phi\} dxdt + \alpha\int_{t_1}^{t_2} u(t,1)\Phi(t,1)dt = 0, \qquad (4)$$

where t_1 and t_2 are arbitrary time moments with $0 \le t_1 \le t_2 \le T$.

4) for every function $g(x) \in L_2(0,1)$, the equality

$$\lim_{t \to +0} \int_0^1 [u(t,x) - u^0(x)] g(x) dx = 0 \qquad (5)$$

holds.

Let $\varphi(x)$ be a given function from $L_2(0,1)$. The considered optimal control problem consists in finding the control function such that the corresponding solution $u(t,x)$ of the problem (1)–(3) satisfies the condition

$$u(T,x) = \varphi(x) \qquad (6)$$

and provides the minimum of the functional

$$I[p]=\|p\|^2. \tag{7}$$

In particular, in case a) the functional (7) becomes

$$I[\bar{p}]=\|\bar{p}\|^2_{L_2^m(0,1)}=\int_0^1\sum_{i=1}^m p_i^2(x)dx, \tag{8}$$

in case b)

$$I[\bar{p}]=\|\bar{p}\|^2_{L_2^l(0,T)}=\int_0^T\sum_{j=1}^l p_j^2(t)dt, \tag{9}$$

and in case c)

$$I[p]=\sum_{i=1}^m\sum_{j=1}^l p_{ij}^2. \tag{10}$$

Thus, we see that in every case the quality criteria are the quadratic functionals. As the set of values is bounded from below, we can assert the existence of minimal values of these functionals.

APPLYING THE L-PROBLEM OF MOMENTS

Consider in $L_2(0,1)$ the orthonormal system of functions $X_n(x)=\frac{\cos\lambda_n x}{\sqrt{\omega_n}}$, $n=1,2,\ldots$, where λ_n 's are the eigenvalues of the boundary value problem

$$X''(x)+\lambda^2X(x)=0,\quad 0<x<1,$$
$$X'(0)=0,\quad X'(1)+aX(1)=0, \tag{11}$$

and the positive roots of the equation $\lambda tg\lambda=\alpha$, while $\omega_n=\frac{\alpha+\alpha^2+\lambda_n^2}{2(\alpha^2+\lambda_n^2)}$ is a normalising factor. Taking this into account, we will seek the approximate solution of the problem (1)–(3) in the form of finite Fourier series

$$u^N(t,x)=\sum_{n=1}^N u_n^N(t)X_n(x),\ u_n^N(t)=\int_0^1 u^N(t,x)X_n(x)dx. \tag{12}$$

Now let us consider each case separately.

a) Let the function F in the equation (1) be defined as in a). Let us multiply both sides of the equation (1) by $X_n(x)$ and integrate over x from 0 to 1. Considering the boundary conditions (3), (11) and the form of the function F, we see that the coefficients $u_n^N(t)$ satisfy the following system of ordinary differential equations:

$$\frac{du_n^N(t)}{dt}=-a^2\lambda_n^2u_n^N(t)+\sum_{i=1}^m\delta(t-t_i)p_{in}+f_n(t),$$
$$n=1,2,\ldots,N. \tag{13}$$

As the function (12) must satisfy the condition (2), we have

$$u_n^N(0)=u_n^0, \tag{14}$$

where $f_n(t)$, p_{in} and u_n^0 are the Fourier coefficients of the functions $f(t, x)$, $P_i(x)$ and $u_n^0(x)$, respectively. Then the functional (8) becomes

$$I[\bar{p}]=\sum_{n=1}^\infty I_n[\bar{p}],\ \ I_n[\bar{p}]=\sum_{i=1}^m p_{in}^2. \tag{15}$$

As the Fourier coefficients $u_n^N(t)$ are defined from the system (13) independently of each other, it suffices to consider the minimisation problem for the functional $I_n[\bar{p}]$.

Thus, the problem is reduced to determining the numerical parameters $\bar{p}=(p_{1n},p_{2n},\ldots,p_{mn})\in E^m$ such that the corresponding solutions of the problem (13)–(14) satisfy the conditions

$$u_n^N(T)=\varphi_n \tag{16}$$

and the functional $I_n[\bar{p}]$ takes the smallest possible value for every fixed n, where φ_n's are the Fourier coefficients of the function $\varphi(x)$.

By Cauchy formula, the solution of the equation (13) with the initial condition (14) can be represented as follows:

$$u_n^N(t)=u_n^0e^{-a^2\lambda_n^2t}+\int_0^t\left[f_n(\tau)+\sum_{i=1}^m\delta(\tau-t_i)p_{in}\right]e^{-a^2\lambda_n^2(t-\tau)}d\tau.$$

Then the condition (16) can be rewritten as follows:

$$u_n^0e^{-a^2\lambda_n^2T}+\int_o^T\left[f_n(t)+\sum_{i=1}^m\delta(t-t_i)p_{in}\right]e^{-a^2\lambda_n^2(T-t)}dt=\varphi_n.$$

Hence, we obtain the following system of equations:

$$\sum_{i=1}^m p_{in}e^{-a^2\lambda_n^2(T-t_i)}=\psi_n,\quad n=1,2,\ldots,N, \tag{17}$$

where

$$\psi_n\varphi_n-u_n^0e^{-a^2\lambda_n^2T}-\int_0^T f_n(t)e^{-a^2\lambda_n^2(T-t)}dt. \tag{18}$$

Thus, every admissible control, which provides satisfaction of condition (16), certainly satisfies the moment relations (17).

Denote

$$\bar{p}_n=(p_{1n},p_{2n},\ldots,p_{mn}),\ \bar{e}_n=(e^{-a^2\lambda_n^2(T-t)},\ldots,e^{-a^2\lambda_n^2(T-t_m)}).$$

Then the moment relations (17) can be rewritten as follows:

$$(\bar{p}_n,\bar{e}_n)=\psi_n. \tag{19}$$

Consequently, we are required to find the vector $\bar{p}_n$ of minimal length, which satisfies the condition (19).

Denote by H the subspace of the space E^n, consisting of elements of the form

$$\bar{q}_n=\alpha_n\bar{e}_n, \tag{20}$$

where α_n is an arbitrary real coefficient, and the vector $\bar{e}_n$ is defined by (19). Then, by Levi's theorem, every element $\bar{p}_n \in E^m$ can be uniquely represented in the following form (Yegorov, 2004):

$$\bar{p}_n = \bar{q}_n + \bar{g}_n; \quad \bar{q}_n \in H, \quad \bar{g}_n \perp H,$$

and

$$\|\bar{p}_n\| = \|\bar{q}_n\| + \|\bar{g}_n\|.$$

Hence it follows that

$$(\bar{p}_n, \bar{e}_n) = (q_n, \bar{e}_n),$$

where the component $\bar{q}_n$ of the element $\bar{p}_n$ does not affect whether this element satisfies the condition (19). Also, from (20) it follows that the component g_n does not affect the norm of the element $\bar{p}_n$.

Thus, if the considered optimal control problem has a solution, then it belongs to H, i.e. it can be represented in the form

$$\bar{p}_n = \alpha_n \bar{e}_n \text{ or } p_{in} = \alpha_n e^{-a^2\lambda_n^2(T-t_i)}, \quad i = \overline{1,m} \tag{21}$$

To determine the unknown coefficient in the obtained relation, let's substitute it in (19).

$$(\alpha_n \bar{e}_n, \bar{e}_n) = \psi_n \quad \text{or} \quad \alpha_n (\bar{e}_n, \bar{e}_n) = \psi_n.$$

Hence, we obtain

$$\alpha_n = \frac{\psi_n}{(\bar{e}_n, \bar{e}_n)} = \frac{\psi_n}{\sum_{k=1}^{m} e^{-2a^2\lambda_n^2(T-t_k)}}.$$

and

$$p_{in} = \frac{\psi_n e^{-a^2\lambda_n^2(T-t_i)}}{\sum_{k=1}^{m} e^{-2a^2\lambda_n^2(T-t_k)}}, \quad i = \overline{1,m}; \quad n = \overline{1,N}. \tag{22}$$

b) In this case, the function F in the equation (1) is defined as in b). Then, similar to the case a), to define the coefficients $u_n^N(t)$ we obtain the following system of ordinary differential equations:

$$\frac{du_n^N(t)}{dt} = -a^2\lambda_n^2 u_n^N(t) + \sum_{i=1}^{l} X_n(x_i) p_i(t) + f_n(t), \quad n = 1,2,\ldots,N, \tag{23}$$

with the initial condition (14).

Thus, the corresponding optimal control problem is reduced to determining the vector function

$$\bar{p}(t) = (p_1(t), p_2(t), \ldots, p_l(t)) \in L_2^l(0,T)$$

such that the corresponding solution of the problem (13)–(14) satisfies the conditions (14) and the functional (9) takes the smallest possible value.

By Cauchy formula, the solution of the system (23) with the initial condition (14) can be represented as follows:

$$u_n^N(t) = u_n^0 e^{-a^2\lambda_n^2 t} + \int_0^t \left[f_n(\tau) + \sum_{i=1}^{l} X_n(x_i) p_i(\tau) \right] e^{-a^2\lambda_n^2(t-\tau)} d\tau.$$

Then from (16) it follows that

$$u_n^0 e^{-a^2\lambda_n^2 t} + \int_0^T \left[f_n(\tau) + \sum_{i=1}^{l} X_n(x_i) p_i(t) \right] e^{-a^2\lambda_n^2(t-t)} dt$$

or

$$\int_0^T \sum_{i=1}^{l} p_i(t) X_n(x_i) e^{-a^2\lambda_n^2(t-t)} dt = \psi_n, \quad n = 1,2,\ldots,N. \tag{24}$$

where ψ_n is defined by (18).

Consequently, every admissible control that provides satisfaction of the condition (16) must satisfy the moment relation (24).

Denote

$$\bar{p}(t) = (p_1(t), p_2(t), \ldots, p_l(t)),$$
$$\bar{e}_n(t) = (X_n(x_1) e^{-a^2\lambda_n^2(T-t)}, \ldots, X_n(x_1) e^{-a^2\lambda_n^2(T-t)}).$$

Then, using scalar product in $L_2^l(0,T)$, the relation (24) can be rewritten as follows:

$$(\bar{p}(t), \bar{e}_n(t)) = \psi_n, \quad n = 1,2,\ldots,N, \tag{25}$$

Thus, we are required to find the vector function $\bar{p}(t)$ with minimal norm satisfying the condition (25).

Denote by H the subspace of the space $L_2^l(0,T)$ consisting of elements of the form

$$\bar{q}^N(t) = \sum_{k=1}^{N} \alpha_k \bar{e}_k(t), \tag{26}$$

where α_n are the unknown coefficients, and the vectors $\bar{e}_n(t)$ are defined by (25). Then, again by Levi's theorem, every element $p(t) \in L_2^l(0,T)$ can be uniquely represented in the form

$$\bar{p}(t) = \bar{q}^N(t) + \bar{g}^N(t), \quad \bar{q}^N(t) \in H, \quad g^N(t) \perp H,$$

and

$$\|\bar{p}(t)\| = \|\bar{q}^N(t)\| + \|\bar{g}^N(t)\|.$$

Hence it follows that

$$(\bar{p}(t), \bar{e}_n(t)) = (\bar{q}^N(t), \bar{e}_n(t)).$$

So, if the considered optimal control problem has a solution, then it belongs to H, i.e. it has the form (26):

$$\bar{p}(t) = \sum_{k=1}^{N} \alpha_k \bar{e}_k(t) \quad \text{or} \quad p_i(t) = \sum_{k=1}^{N} \alpha_k X_k(x_i) e^{-a^2\lambda_k^2(T-t)}, \quad i = \overline{1,l} \tag{27}$$

To determine the unknown coefficients α_k, let's substitute the value (26) in (25). Then we obtain the following system of algebraic equations

$$\sum_{k=1}^{N}(e_n(t), e_k(t))\alpha_k = \psi_n,$$

or

$$\sum_{k=1}^{N} M_{nk}\alpha_k = \psi_n, \quad n = 1,2,\ldots,N, \tag{28}$$

where

$$M_{nk} = (e_n(t), e_k(t)) = \int_0^T \sum_{i=1}^{l} X_n(x_i)X_k(x_i)e^{-a^2(\lambda_n^2+\lambda_k^2)(T-t)}dt$$

The matrix $M = \{M_{nk}\}_{n=\overline{1,N}}^{k=\overline{1,N}}$ is obviously symmetric, and its determinant is the Gram determinant of the vectors $e_1(t)$, $e_2(t),\ldots, e_N(t)$. As these vectors are linearly independent, the determinant of the matrix M is different from zero, and the system (28) has a unique solution.

Thus, in this case the optimal control is defined in the form (27), where the coefficients are the solution of the system (28).

c) In this case, the function F in the equation (1) is defined as in c). Then, similar to previous cases, to define the coefficients $u_n^N(t)$ we obtain the following system of ordinary differential equations:

$$\frac{du_n^N(t)}{dt} = -a^2\lambda_n^2 u_n(t) + \sum_{i=1}^{m}\sum_{j=1}^{l}\delta(t-t_i)X(x_j)p_{ij} + f_n(t), \quad n = 1,2,\ldots,N, \tag{29}$$

with the initial condition (14).

Then, the optimal control problem is reduced to determining the numbers $p_{i,j}$, $i = 1,2,...m$; $j = 1,2,...l$ such that the corresponding solution of the equation (29) with the initial condition (14) satisfies the conditions (16) and the functional (10) takes the smallest possible value.

By Cauchy formula, the solution of the equation (29) with the initial condition (14) has the following form:

$$u_n^N(t) = u_n^0 e^{-a^2\lambda_k^2 t} + \int_0^t \left(\sum_{i=1}^{m}\sum_{j=1}^{l}\delta(\tau-t_i)X_n(x_j)p_{ij} + f_n(\tau)\right)e^{-a^2\lambda_k^2(t-\tau)}d\tau, \tag{30}$$

Then

$$u_n^N(t) = u_n^0 e^{-a^2\lambda_k^2 T} + \sum_{i=1}^{m}\sum_{j=1}^{l}X_n(x_j)p_{ij}e^{-a^2\lambda_k^2(T-t_i)} + \int_0^T f_n(t)e^{-a^2\lambda_n^2(T-t)}dt.$$

From the condition (16) it follows that the control parameters satisfy the following moment relations:

$$\sum_{i=1}^{m}\sum_{j=1}^{l}X_n(x_j)p_{ij}e^{-a^2\lambda_k^2(T-t_i)} = \psi_n, \tag{31}$$

where ψ_n's are defined by (18). To rewrite the equality (31) in the form of scalar product, let's introduce the following vectors:

$$\overline{p}_n = (p_{11},\ldots,p_{1l},p_{21},\ldots,p_{2l},\ldots,p_{m1},\ldots,p_{ml}),$$

$$v_n = (v_{11}^n,\ldots,v_{1l}^n,v_{21}^n,\ldots,v_{2l}^n,\ldots,v_{m1}^n,\ldots,v_{ml}^n),$$

where $v_{ij}^n = X_n(x_j)e^{-a^2\lambda_n^2(T-t_i)}$.

Then the moment relation (31) can be rewritten as follows:

$$(\overline{p}_n, \overline{v}_n) = \psi_n. \tag{32}$$

Consequently, we are required to find the vector of minimal length, which satisfies the condition (32). Similar to the previous cases, we seek the solution of the obtained problem in the form

$$\overline{p}_n = \alpha_n \overline{v}_n,$$

where α_n are the unknown coefficients, and to find them we substitute this value in the equality (32). Then we obtain

$$(\alpha_n\overline{v}_n, \overline{v}_n) = \psi_n \text{ or } \alpha_n(\overline{v}_n, \overline{v}_n) = \psi_n$$

Hence

$$\alpha_n = \frac{\psi_n}{(\overline{v}_n, \overline{v}_n)} = \frac{\psi_n}{\sum_{i=1}^{m}\sum_{j=1}^{l}X_n^2(x_j)e^{-2a^2\lambda_n^2(T-t_i)}}$$

and the control parameters are

$$p_{ij} = \alpha_n X_n(x_j)e^{-a^2\lambda_n^2(T-t_i)}$$

Note that the convergence of the chosen approximation follows from the convergence of the Fourier series.

CONVERGENCE OF THE CHOSEN APPROXIMATION

In cases a) and b), every component of the approximation of optimal control coincides with the exact solution. Therefore, it suffices to consider the case b). We can write

$$\left\|p_i(t) - p_i^N(t)\right\|_{L_2(0.T)} = \int_0^T \sum_{k=N+1}^{\infty}\alpha_k X_k(x_i)e^{-a^2\lambda_k^2(T-t)}dt \le$$

$$\le \int_0^T \sum_{k=N+1}^{\infty}\alpha_k^2 X_k^2(x_i)\cdot\sum_{k=N+1}^{\infty}e^{-2a^2\lambda_k^2(T-t)}dt =$$

$$= \sum_{k=N+1}^{\infty}\alpha_k^2 X_k^2(x_i)\cdot\sum_{k=N+1}^{\infty}\frac{1}{2a^2\lambda_k^2}(1-e^{-2a^2\lambda_k^2}) <$$

$$< \sum_{k=N+1}^{\infty}\alpha_k^2 X_k^2(x_i)\cdot\sum_{k=N+1}^{\infty}\frac{1}{2a^2\lambda_k^2}.$$

The right-hand side of the obtained inequality converges to zero as $N \to \infty$, because each factor is a remainder term of a convergent series, i.e.

$$\lim_{N\to\infty}\left\|p_i(t) - p_i^N(t)\right\|_{L_2(0,T)} = 0.$$

Example. To illustrate the efficiency of the suggested algorithm, let's consider the following model problem of optimal control:

$$u_t = u_{xx} + \sum_{i=1}^{4} \delta(t-t_i)p_i(x) + f(t,x); 0 < t < 1, 0 < x < 1; \quad (33)$$

$$u(0,x) = u^0(x) \quad (34)$$

$$u_x(t,0) = 0, \quad u_x(t,1) - 1.5u(t,1) = 0, \quad (35)$$

where

$$t_i = 0{,}2i; \; f(t,x) = (2\cos x + \beta x^2 - 2\beta)e^t - (\beta x^2 + 2\beta)e^{-t},$$

$$u^0(x) = 2\cos x = \beta x^2, \; \beta = 2\sin 1 + 3\cos 1.$$

The optimal control problem consists in finding the admissible control $\bar{p}(x) = (p_1(x), p_2(x), p_3(x), p_4(x))$ such that the corresponding solution *u(t, x)* of the problem (33)–(35) satisfies the condition

$$u(1,x) = \varphi(x), \quad (36)$$

where $\varphi(x) = (\cos x + \beta x^2)ch1$ and the functional

$$I[p] = \int_0^1 \sum_{i=1}^{4} p_i^2(x)dx \quad (37)$$

takes the smallest possible value.

The analytic solution of the considered problem has the following form:

$$u(t,x) = (\cos x + \beta x^2)cht + \sum_{i=1}^{4} \delta(t-t_i)x^3; \; p_i(x) = 3x; \; i = \overline{1,4}.$$

The following numerical values have been obtained (Table 1, Fig. 1), where $u^{(n)}(t, x)$ is a corresponding n-th iteration of the approximate solution of the boundary value problem (33) – (35), control parameter and minimising functional, while $u^*(t, x)$, $p^*(x)$ and $I^*[p]$ are the exact values of these quantities. Tables 1–3 show the results of numerical calculations.

It is easily seen from Figure 1 that the permissible error becomes significantly less after the third iteration.

DISCUSSION OF RESULTS

The obtained results show that for some classes of optimal pulse control problems with distributed parameters the method of moments is more efficient. Unlike the maximum principle method, our method allows constructing an efficient algorithm for iterations of control parameter.

The results of numerical experiment show the constructivism of this method and provide a possibility to obtain fairly high accuracy results.

CONCLUSIONS

We studied the approximate solution of the minimum energy control problem for a parabolic system. The system is controlled by the impulsive external influences. In different

$$I^{(1)}[p] = 20.164, \; I^{(2)}[p] = 18.785, \; I^{(3)}[p] = 15.543,$$

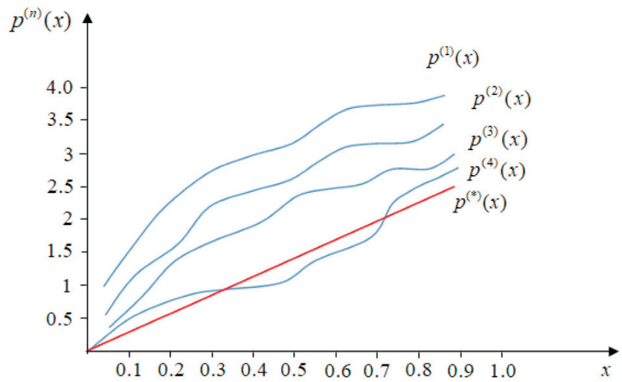

Fig. 1. Graph of iterations for control parameter

Table 1. First iteration results

	$u^{(1)}(t,x)$								
x_i / t_j	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	5.108	5.311	5.650	6.138	6.787	7.610	8.621	9.831	11.255
0.2	5.156	5.347	5.652	6.070	6.604	7.254	8.020	8.903	9.906
0.3	5.229	5.437	5.786	6.288	6.955	7.801	8.838	10.079	11.539
0.4	5.240	5.319	5.518	5.837	6.278	6.840	7.524	8.332	9.265
0.5	5.433	5.653	6.023	6.554	7.260	8.154	9.248	10.557	12.092
0.6	5.566	5.780	6.127	6.605	7.217	7.963	8.844	9.862	11.018
0.7	5.726	5.965	6.367	6.945	7.713	8.682	9.867	11.280	12.936
0.8	5.907	6.143	6.529	7.064	7.749	8.586	9.576	10.719	12.019
0.9	6.117	6.382	6.831	7.476	8.330	9.406	10.718	12.280	14.105
	$p_1^{(1)}(x) = p_2^{(1)}(x) = p_3^{(1)}(x)$								
	1.701	2.345	2.761	2.897	3.179	3.620	3.751	3.792	4.103

Table 2. Second iteration results

$t_j \backslash x_i$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	$u^{(2)}(t,x)$								
0.1	4.108	4.311	4.650	5.138	5.787	6.610	7.621	8.831	10.255
0.2	4.156	4.347	4.652	5.070	5.604	6.254	7.020	7.903	8.906
0.3	4.229	4.437	4.786	5.288	5.955	6.801	7.838	9.079	10.539
0.4	4.319	4.518	4.837	5.278	5.840	6.524	7.332	8.265	9.323
0.5	4.433	4.653	5.023	5.554	6.260	7.154	8.248	9.557	11.092
0.6	4.566	4.780	5.127	5.605	6.217	6.963	7.844	8.862	10.018
0.7	4.726	4.965	5.367	5.945	6.713	7.682	8.867	10.280	11.936
0.8	4.907	5.143	5.529	6.064	6.749	7.586	8.576	9.719	11.019
0.9	5.117	5.382	5.831	6.476	7.330	8.406	9.718	11.280	13.105
	$p_1^{(2)}(x)=p_2^{(2)}(x)=p_3^{(2)}(x)$								
	1.052	1.421	1.988	2.317	2.616	3.251	3.274	3.510	3.682

Table 3. Third iteration results

$t_j \backslash x_i$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	$u^{(3)}(t,x)$								
0.1	3.108	3.311	3.650	4.138	4.787	5.610	6.621	7.831	9.255
0.2	3.156	3.347	3.652	4.070	4.604	5.254	6.020	6.903	7.906
0.3	3.229	3.437	3.786	4.288	4.955	5.801	6.838	8.079	9.539
0.4	3.319	3.518	3.837	4.278	4.840	5.524	6.332	7.265	8.323
0.5	3.433	3.653	4.023	4.554	5.260	6.154	7.248	8.557	10.092
0.6	3.566	3.780	4.127	4.605	5.217	5.963	6.844	7.862	9.018
0.7	3.726	3.965	4.367	4.945	5.713	6.682	7.867	9.280	10.936
0.8	3.907	4.143	4.529	5.064	5.749	6.586	7.576	8.719	10.019
0.9	4.117	4.382	4.831	5.476	6.330	7.406	8.718	10.280	12.105
	$p_1^{(3)}(x)=p_2^{(3)}(x)=p_3^{(3)}(x)$								
	0.751	1.207	1.496	1.756	2.098	2.278	2.261	2.971	3.105

cases, using the L-problem of moments, we succeeded to find optimal control in analytical form.

To illustrate the convergence of the chosen approximation, we gave an example and conducted numerical calculations.

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APTUVENS OPTIMĀLAS IMPULSA KONTROLES PROBLĒMAS RISINĀJUMS, KAS SAISTĪTS AR SILTUMA VADĪTSPĒJU

Darbā tiek pētīta minimālās enerģijas kontroles aptuvenā risinājuma problēma objektam, kas aprakstīts ar siltuma vienādojumu. Speciāli tiek analizēts gadījums, kad process ir atkarīgs no t.s. impulsīvām ekstremalitātēm. Autori atrod analītisko vienādojumu dažām īpašām kontroles parametru vērtībām un pierāda to konverģenci izvēlētajām aproksimācijām.