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THE SPATIAL VARIATION OF SEISMIC GROUND MOTIONS UNDER THE MAGNIFYING GLASS OF THE COHERENCY FUNCTION REGARDING THE SMART-1 ARRAY LOCATED IN LOTUNG, TAIWAN

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Abstract: The presented topic contains the calculation procedures and the signals processing concepts used in the studies conducted in the presented report. The spatial variation of seismic ground motions is a phenomenon which regards the modification in terms of the frequency content and phase offset. This phenomenon assumes that the waves pass through soil strata that modify the arrival time of the waves. This idea is called wave passage effect. The incoherence of the waves is caused by the fact that when the waves pass through the soil, they encounter objects which determine them to reflect or refract from the main direction of propagation.

Keywords: dense array, power spectral density, cross spectral density, autocovariance function, cross covariance function

1. Introduction

In earthquake engineering is used the concept of seismic recording (accelerograms), which represents the variation of acceleration of the ground in free field, caused by earthquakes. Seismic sources all over the world are studied using accelerometers placed where there exists seismic activity.

Researches all around the world have thought of the following phenomenon that may arise from the study of accelerograms: The accelerograms from earthquakes are enough to describe the spatial and temporal variation of seismic motions?[1],[2],[3]

The answer to this question was negative because researchers have concluded that it is necessary for the earthquake engineering community to have dense arrays for recording seismic events during earthquakes, in order to study the phenomenon related to spatial variation of seismic ground motions.

The phenomenon of spatial variation of seismic ground motions affects a portfolio of structures such as mat foundations that have large in plane dimensions. These mat foundations can be found in nuclear power plants where there are certain equipment standing on them. Another category of structures which are sensitive to this problem are dams, bridges which extend over large distances, buried pipelines, tunnels.

In 1978, at a workshop in Hawaii, researchers gathered there took the decision of installing dense arrays of sensors in certain highly seismic active regions over the globe in order to understand this phenomenon [1],[2],[3]

The first dense array of sensors was built in LOTUNG TAIWAN, the array is called SMART-1. The bidimensional array has a number of 37 stations spread along three circles counting each 12 stations and a central station. The central station is called C00, and the inner ring has stations denoted by I01 to I12, the middle ring has stations denoted by M01 to M12, and the outer ring has stations denoted by O01 to O12.[8]

The distances from the center to the three circles are 200m,1000m and 2000m, and all accelerometers are tri-axial (N-S, E-Vertical). The Smart-1 Array is functioning since 1980 and has recorded a large number of earthquakes through time, over 20 events with moment magnitude of 4 to 7.8 and epicentral distances between 5km to 80 km. [8]

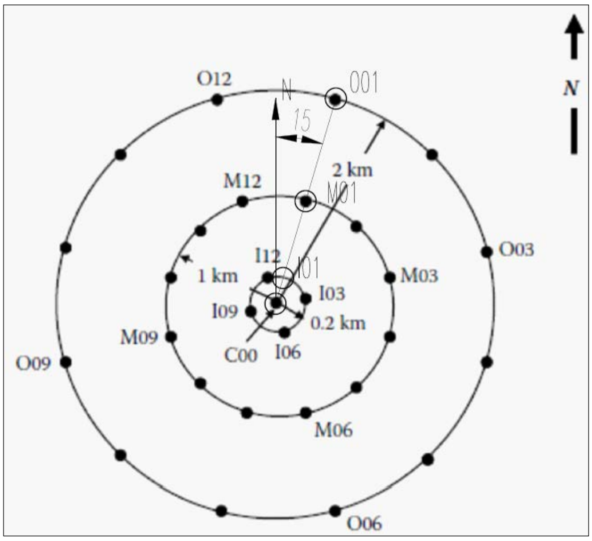

Fig. 1- The spreading of sensors from the SMART-1. The center station, The Inner, Middle, and Outer ring.

The aim of this report is to present the calculation regarding the coherency function for the event 40 from 20.05.1986, more specifically the stations C00, I01, M01, O01 associated with the N-S direction of ground movement.

2. Signal Processing concepts regarding the Coherency Function of seismic ground motions

The whole calculations regarding the coherency function are based in a wide sense upon two assumptions:

The Stationarity Assumption

This assumption considers that the statistical indicators of the stochastic process are independent of absolute time, they being only a function of the shift time" τ".

In a strict sense stationarity needs that all cumulative distribution functions to be equal regardless the time shift" τ" (positive or negative time shift) [7]

$$F_{X_1 X_2 \ldots X_n,}(x_1, x_2 \ldots x_n; t_1, t_2 \ldots t_n) = F_{X_1 X_2 \ldots X_n,}(x_1, x_2 \ldots x_n; t_1 + \tau, t_2 + \tau \ldots t_n + \tau) \quad (1)$$

The Ergodicity Assumption

This assumption considers that the individual statistics of the parts of a stochastic process are the same as the ensemble statistics. Obviously, the information contained in the existing data, the duration of the process is reduced, there exists bias but the process could be considered ergodic.

An ergodic and stationary process has parts of the process representative for the ensemble of the process.[8]

2.1. Stochastic representations of seismic accelerograms

In order to obtain stochastic characteristics of the ground motions for each station, the accelerograms $a(\vec{r_j}, t)$ are assumed to be time variant. Hence that each value of the stochastic process at a certain time moment is a random variable. [8]

$\vec{r_j} = \left(x_j, y_j, z_j\right)^T, j = \overline{1, N}$ represents the position of the station.

The stochastic process is defined by it mean and autocorrelation function.

$$m_a(\vec{r_j}, t) = E\{a(\vec{r_j}, t)\} \quad (2)$$

$$R_{aa}(\vec{r_j};t_1,t_2) = E[\{a(\vec{r_j},t_1) - m_a(\vec{r_j},t_1)\}\{a(\vec{r_j},t_2) - m_a(\vec{r_j},t_2)\}$$
$$= E\{a(\vec{r_j},t_1)a(\vec{r_j},t_2)\} - m_a(\vec{r_j},t_1)m_a(\vec{r_j},t_2) \quad (3)$$

The ergodicity assumption is very important due to the fact that a certain part of the recording of the ground motion is sufficient for the whole ensemble of data.

The mathematical expression of the autocovariance function is

$$\hat{R}_{jj}(\tau) = \begin{cases} \frac{1}{T}\int_0^{T-|\tau|} a_j(t)\, a_j(t+\tau)dt, |\tau| \leq T \\ 0, |\tau| > T \end{cases} \quad (4)$$

One can observe that this mathematical function is an even function. [6]

$$\hat{R}_{jj}(\tau) = \hat{R}_{jj}(-\tau) \quad (5)$$

Another important assumption is that of Gaussian processes.

In a wide sense the stationarity assumption applies only to the mean and to the autocorrelation function of the stochastic process. If the gaussian assumption is also made then the process is completely defined by its first and second moments.[8]

Hence the mean of a stochastic process is constant. The case of accelerograms which are processed for studies it is necessary that they have the mean equal to zero, if not they must be demeaned.

The stationarity assumption has a particular feature: Since there is not any dependence of the absolute time, but only of the time shift, this means that the time history does not have any start or finish and hence its statistical properties remain the same for infinite time.

2.1.1. Power spectral density

The power spectral density and the autocovariance function are a Weiner-Khinchin transformation. [7]

$$\widehat{R_{JJ}}(\tau) = \int_{-\infty}^{+\infty} \widehat{S_{JJ}}(\omega)e^{i\omega t}d\omega \Leftrightarrow \widehat{S_{JJ}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{R_{JJ}}(\tau)e^{-i\omega t}dt \quad (6)$$

Alternatively, the power spectral density can be evaluated in the frequency domain as follows:

Let there be

$$A_j(\omega) = \Lambda_j(\omega)\exp\,[i\phi_j(\omega)] \quad (7)$$

The above formula represents the Fourier Transform of the accelerogram $a_j(t)$.

The power spectral density can be expressed as:

$$\widehat{S_{JJ}}(\omega) = \frac{2\pi}{T}A_j(\omega)^* A_j(\omega) \quad (8)$$

The marked term represents the complex conjugate of the Fourier transform of the accelerogram $a_j(t)$.

This approach offers information about the physical interpretation of the power spectral density function which is the amplitude of the Fourier transform of the accelerogram scaled with the term $\frac{2\pi}{T}$. The term T represents the duration of the accelerogram considered into analysis.

$\widehat{S_{JJ}}(\omega)$is called periodogram.

The periodogram is a real, asymptotic indicator of the power spectral density function. The variance of the periodogram can be smoothed in time or frequency domain.

2.1.2. Estimation of the power spectral density through smoothing of the periodogram.

The smoothed power spectral density can be obtained through applying a window function to the autocorrelation function as follows:

$$\overline{S_{JJ}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} w(\tau)\widehat{R_{JJ}}(\tau)e^{-i\omega t}dt \tag{9}$$

$$\overline{S_{JJ}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{R_{JJ}}(\tau)e^{-i\omega t}dt \tag{10}$$

$\overline{R_{JJ}}(\tau) = w(\tau)\widehat{R_{JJ}}(\tau)$ represents the smoothed autocorrelation function and $w(\tau)$ is the smoothing window with the following properties:

$$w(0) = 1 \tag{11}$$

$$w(\tau) = w(-\tau) \tag{12}$$

$$w(\tau) = 0, |\tau| \geq L\Delta t, L\Delta t < T \tag{13}$$

Equivalent the smoothed power spectral density can be evaluated directly in the frequency domain through the following convolution expression:

$$\overline{S_{JJ}}(\omega) = \int_{-\infty}^{+\infty} W(u)\widehat{S_{JJ}}(\omega - u)du \tag{14}$$

$W(u)$ represents the Fourier Transform of the smoothing window $w(\tau)$.

$$w(\tau) = \int_{-\infty}^{+\infty} W(\omega)e^{i\omega\tau}d\tau \tag{15}$$

$$W(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} w(\tau)e^{-i\omega\tau}d\tau \tag{16}$$

From the above expressions we can conclude the following properties of the smoothing window:

$$\int_{-\infty}^{+\infty} W(\omega)d\omega = w(0) = 1 \tag{17}$$

$$W(\omega) = W(-\omega) \tag{18}$$

For discrete frequencies the equation has the following form:

$$\overline{S_{JJ}}^{M}(\omega_n) = \sum_{m=-M}^{+M} W(m\Delta\omega)\widehat{S_{JJ}}(m\Delta\omega + \omega_n) \tag{19}$$

$$\overline{S_{JJ}}^{M}(\omega_n) = \frac{2\pi}{T}\sum_{m=-M}^{+M} W(m\Delta\omega){\Lambda_j}^2(m\Delta\omega + \omega_n) \tag{20}$$

$$\Delta\omega = \frac{2\pi}{T}\ represents\ the\ discretisation\ interval\ of\ the\ frequency$$

$$\omega_n = n\Delta\omega\ represents\ the\ discrete\ frequency$$

$$W(m\Delta\omega)\ is\ the\ spectral\ window.$$

2M+1 is the number of points of the spectral window.

We must observe the fact that various smoothing windows conduct to the same results as long as the frequency bandwidth is the same. [8]

Through the smoothing operation the variance of the estimator is reduced at a fraction of the periodogram's depending on the bandwidth of the smoothing window defined as follows:[6]

$$\text{Bandwidth} = B = \left[\int_{-\infty}^{+\infty} w(\tau)^2 d\tau\right]^{-1} = \left[2\pi\int_{-\infty}^{+\infty} W(\omega)^2 d\omega\right]^{-1} \tag{21}$$

The following equation states that the product between the bandwidth and the variance are constant. This means that the larger the bandwidth the smaller the variance and vice-versa.

2.2. Bivariate stochastic processes

So far, we have analyzed time series which have been represented by processes which were independently analyzed. In real life analyses seismic recordings are intercorrelated. In the following chapters we will use an approach which uses the concept of bivariate stochastic processes. Gaussian bivariate stochastic processes are completely described by their second order moments, which are called the intercorrelation function in the time domain, and the cross spectral density function in the frequency domain.[8]

2.2.1. Cross covariance function

Similar, with the autocovariance function, the cross-covariance function between $aa(\vec{r}_k, t_2)$ at two different locations in space $\vec{r}_j$ și $\vec{r}_k$and different time moments t_1 și $t_2$is defined as follows: [8]

$$\widetilde{R_{aa}}(\vec{r}_j, t_1; \vec{r}_k, t_2) = E[\{a(\vec{r_j}, t_1) - m_a(\vec{r_j}, t_1)\}\{a(\vec{r_k}, t_2) - m_a(\vec{r_k}, t_2)\} \tag{22}$$
$$= E\{a(\vec{r_j}, t_1)a(\vec{r_k}, t_2)\} - m_a(\vec{r_j}, t_1)m_a(\vec{r_k}, t_2)$$

$$\widetilde{R_{aa}}(\vec{r}_j, t_1; \vec{r}_k, t_2) = \widetilde{R_{jk}}(t_1, t_2) \tag{23}$$

Considering that the means of the processes are zero, we can observe that the cross covariance between the two station is independent of absolute time.

$$\widetilde{R_{jk}}(t_1, t_2) = \widetilde{R_{jk}}(t, t + \tau) = \widetilde{R_{jk}}(\tau)$$

2.2.2. Cross Spectral Density

The cross spectral density function between two accelerograms is defined by its Fourier transform of the cross-covariance function:

$$\widehat{S_{jk}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{R_{jk}}(\tau)e^{-i\omega t}dt \tag{24}$$

The above equation indicates the fact that the cross spectrum is a complex function having the following property $\widehat{S_{jk}}(\omega) = \widehat{S_{jk}}^*(-\omega)$. [6]

Similar to the power spectral density function the cross- covariance function and the cross spectrum are a Weiner-Khinchin transformation.

$$\widehat{R_{jk}}(\tau) = \int_{-\infty}^{+\infty}\widehat{S_{jk}}(\tau)e^{i\omega t}d\omega \tag{25}$$

As an alternative we can express the function of the cross spectrum as follows:

In the following alternative the Fourier Transforms of the recorded motions are considered. The cross spectrum of the two accelerations is considered as follows:

$$A_j(\omega) = \Lambda_j(\omega)\exp\,[i\phi_j(\omega)] \tag{26}$$

$$A_k(\omega) = \Lambda_k(\omega)\exp\,[i\phi_k(\omega)] \tag{27}$$

$$\widehat{S_{jk}}(\omega) = \frac{2\pi}{T}A_j^*(\omega)A_k(\omega) \tag{28}$$

$$\widehat{S_{jk}}(\omega) = \frac{2\pi}{T}\Lambda_j(\omega)\Lambda_k(\omega)\exp\,[i\phi_k(\omega) - i\phi_j(\omega)] \tag{29}$$

The absolute value of the cross spectrum is evaluated as follows:

$$\left|\widehat{S_{jk}}(\omega)\right| = \frac{2\pi}{T}\Lambda_j(\omega)\Lambda_k(\omega) \tag{30}$$

The phase difference is defined as follows:

$$\hat{\varphi}_{jk}(\omega) = tan^{-1}\left[\frac{Im(\widehat{S_{jk}}(\omega))}{Re(\widehat{S_{jk}}(\omega))}\right] \tag{31}$$

$$\hat{\varphi}_{jk}(\omega) = \phi_k(\omega) - \phi_j(\omega) \tag{32}$$

It can be observed that the cross spectrum is controlled by the Fourier amplitudes of each accelerogram and the phase spectrum indicates if the frequency component of the ground motion recordings at a certain station preceded or not the other time series at that frequency.

The statistical properties of the cross spectrum are similar to those of the power spectrum regarding the variance, which cannot be reduced by increasing the number of data, instead it is recommended to use smoothing windows.[8]

2.2.3. Smoothed cross spectral density

The smoothed cross spectral density is obtained using the following equation:

$$\overline{S_{jk}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{R_{jk}}(\tau)w(\tau)e^{-i\omega t}dt \tag{33}$$

$$\overline{S_{jk}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\overline{R_{jk}}(\tau)e^{-i\omega t}dt \tag{34}$$

$$\overline{R_{jk}}(\tau) = \widehat{R_{jk}}(\tau)w(\tau) \tag{35}$$

$$w(\tau)\ is\ the\ smoothing\ window$$

Equivalent the cross spectrum can be evaluated through the convolution expression using the smoothing window whose mathematical properties have been already defined:

$$\overline{S_{jk}}(\omega) = \int_{-\infty}^{+\infty}W(u)\widehat{S_{jk}}(\omega - u)du \tag{36}$$

For discreet frequencies the above equation can be rewritten as:

$$\overline{S_{jk}}^{M}(\omega_n) = \frac{2\pi}{T}\sum_{m=-M}^{+M}W(m\Delta\omega)\Lambda_j(\omega_n + m\Delta\omega)\Lambda_k(\omega_n + m\Delta\omega)\exp\left[i\phi_k(\omega_n + m\Delta\omega) - i\phi_j(\omega_n + m\Delta\omega)\right] \tag{37}$$

The amplitude of the smoothed cross spectrum is given by:

$$\left|\overline{S_{jk}}^{M}(\omega_n)\right| = \frac{2\pi}{T}\left|\sum_{m=-M}^{+M}W(m\Delta\omega)\Lambda_j(\omega_n + m\Delta\omega)\Lambda_k(\omega_n + m\Delta\omega)\exp\left[i\phi_k(\omega_n + m\Delta\omega) - i\phi_j(\omega_n + m\Delta\omega)\right]\right| \tag{38}$$

The phase spectrum is defined by:

$$\bar{\varphi}_{jk}(\omega) = tan^{-1}\left[\frac{Im(\overline{S_{jk}}^{M}(\omega_n))}{Re(\overline{S_{jk}}^{M}(\omega_n))}\right] \tag{39}$$

It can be observed that in this last equation there is not any phase difference like in the statistical estimator without the smoothing window.

There exists a significative bias level in the estimation of the power spectral density and at the cross spectral density which is due to the fact that the autocovariance function is symmetrical at time shift equal to zero whereas the cross-covariance function is symmetrical at a time shift different from zero.

A way to solve this problem is to align the data which means to offset the two ground motion recordings with a time shift of $|\tau_0|$, at which the cross-covariance function will have a maximum. [8]

3. The coherency function

The cross-spectrum estimators included in the above chapter describe de dependence of two ground motion recordings located at the surface of the ground in free field

The coherency function presented herein is obtained through the smoothed cross spectrum from two ground motion accelerograms at stations ,,j", respective ,.k" divided by the square root of the corresponding smoothed power spectral densities.

$$\bar{\gamma}_{jk}^{M}(\omega) = \frac{\bar{S}_{jk}^{M}(\omega)}{\sqrt{\bar{S}_{jj}^{M}(\omega)\bar{S}_{kk}^{M}(\omega)}} \tag{40}$$

M is the number of points of the smoothing window.

This function is a complex function whose absolute value is called coherency and take values from zero to unity. [8]

$$\left|\bar{\gamma}_{jk}^{M}(\omega)\right|^2 = \frac{\left|\bar{S}_{jk}^{M}(\omega)\right|^2}{\bar{S}_{jj}^{M}(\omega)\bar{S}_{kk}^{M}(\omega)} \tag{41}$$

The phase of the coherency function is given by the smoothed phase spectrum

$$\bar{\varphi}^{M}{}_{jk}(\omega) = tan^{-1}\left[\frac{Im(\bar{\gamma}_{jk}^{M}(\omega))}{Re(\bar{\gamma}_{jk}^{M}(\omega))}\right] \tag{42}$$

$$\bar{\varphi}^{M}{}_{jk}(\omega) = tan^{-1}\left[\frac{Im(\bar{S}_{jk}^{M}(\omega))}{Re(\bar{S}_{jk}^{M}(\omega))}\right] \tag{43}$$

Alternatively, the coherency function can be expressed as follows:

$$\bar{\gamma}_{jk}^{M}(\omega) = \left|\bar{\gamma}_{jk}^{M}(\omega)\right| \exp[i\bar{\varphi}^{M}{}_{jk}(\omega)] \tag{44}$$

4. Study case

In the following example there will be illustrated the calculations for a coherency function for two recorded ground motions from Lotung Taiwan, SMART-1 Array.

In this example there will be considered the two stations C00 and I01 for the event 40 from 20.051986, the ground motions recorded for the N-S direction.

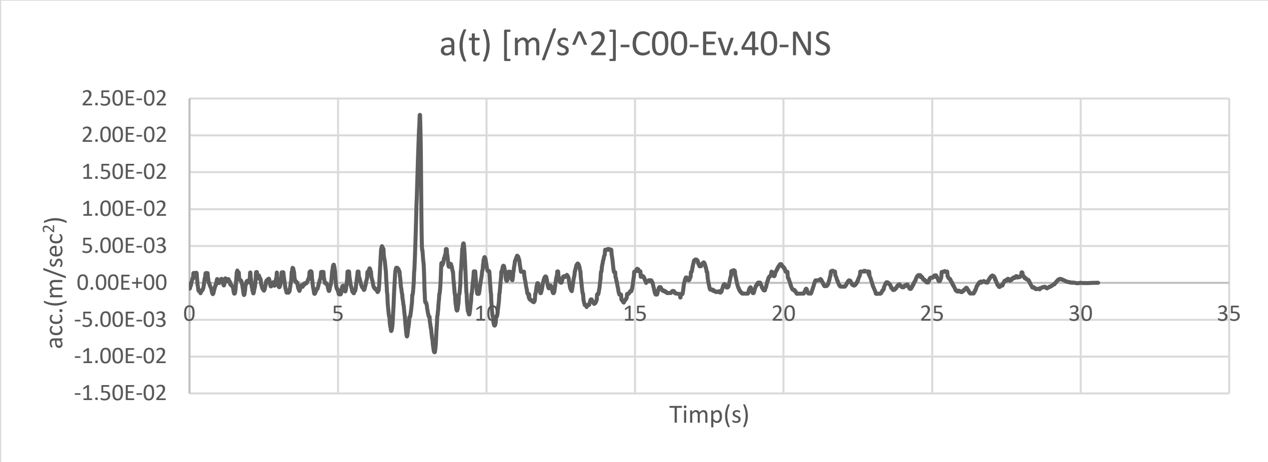

Fig. 2 - Accelerogram for C00 station

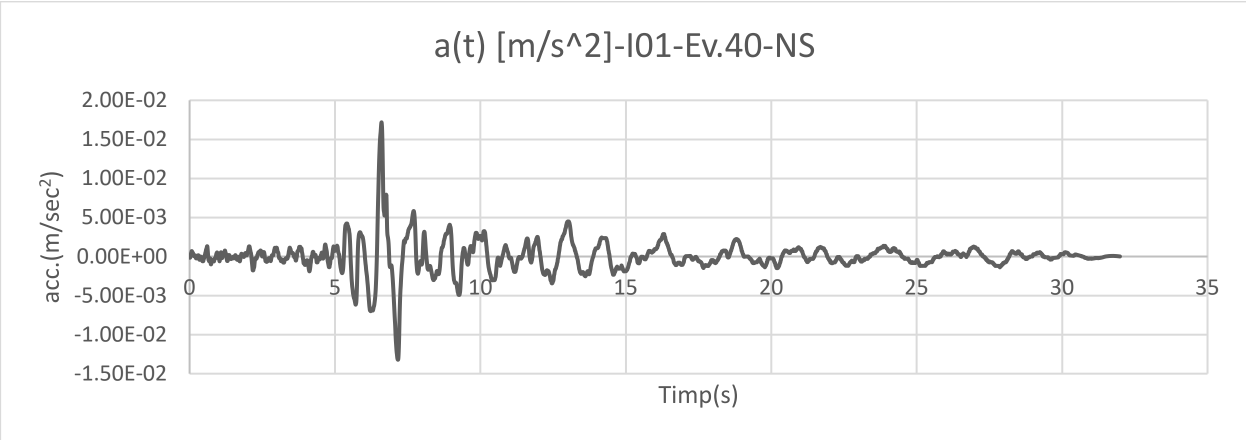

Fig. 3 - Accelerogram for I01 station.

First step of the calculation is to evaluate the cross-covariance function of the two recorded ground motions.

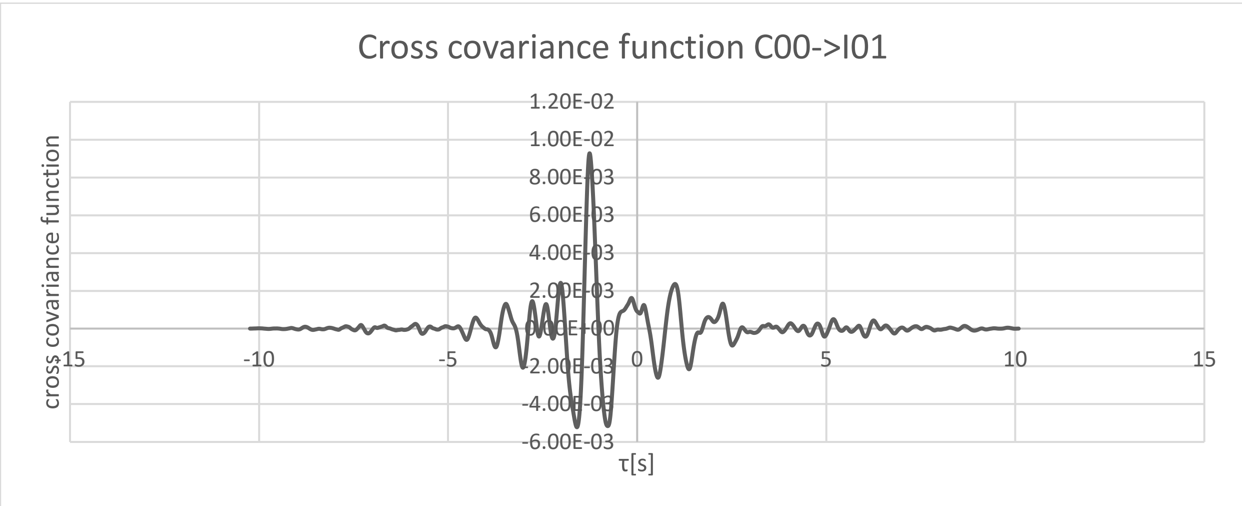

Fig. 4 - Cross covariance function between C00->I01

It can be observed as expected that there is a time shift between c00 and I01. In order to reduce the bias of the two recordings of the ground motion need to be aligned. The time for offsetting the first station C00 is the time necessary for the cross covariance to be centered at time shit equals to zero.

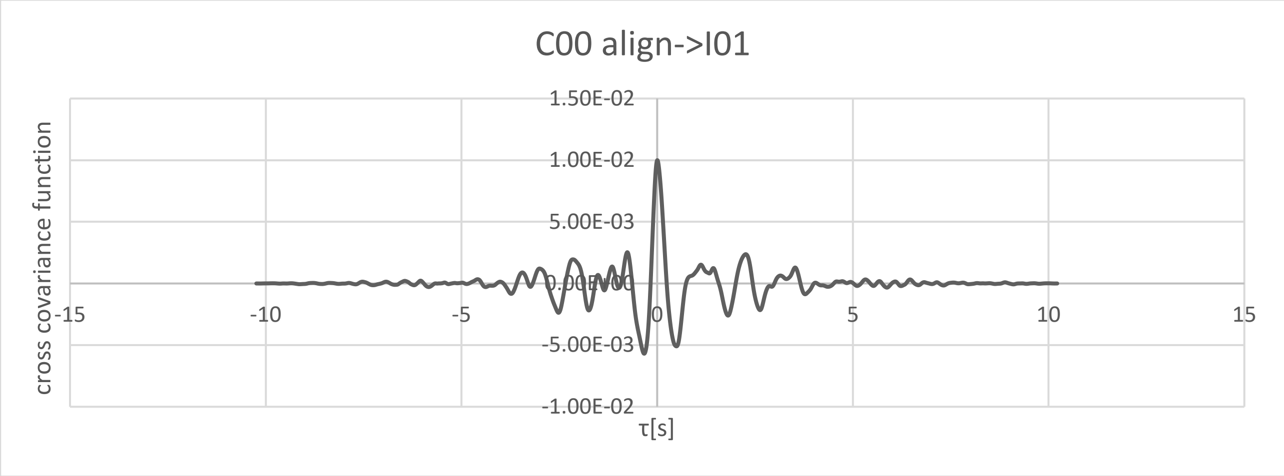

Fig. 5 - The cross-covariance function between C00-aligned station and I01.

As expected, the aligned C00- station and the I01 station have where their cross covariance has a maximum value the time shift is equal to zero seconds.

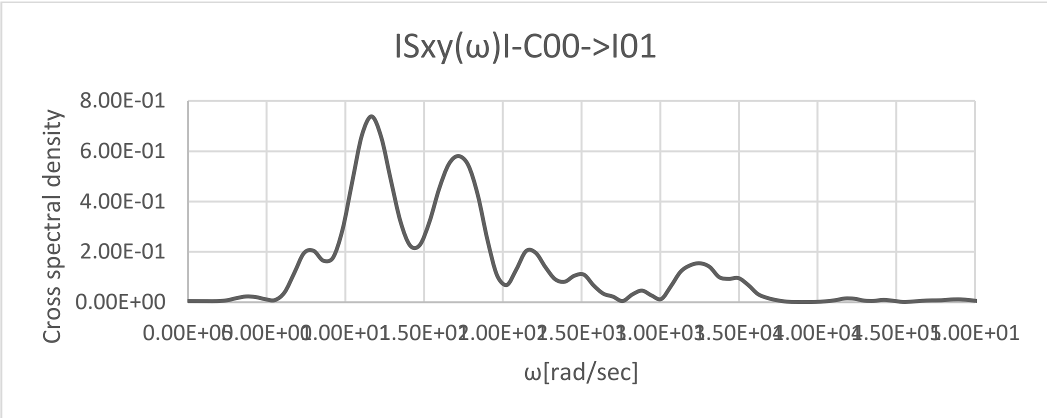

Fig. 6 - The absolute value of the cross spectral density between station c00 aligned with the I01 station.

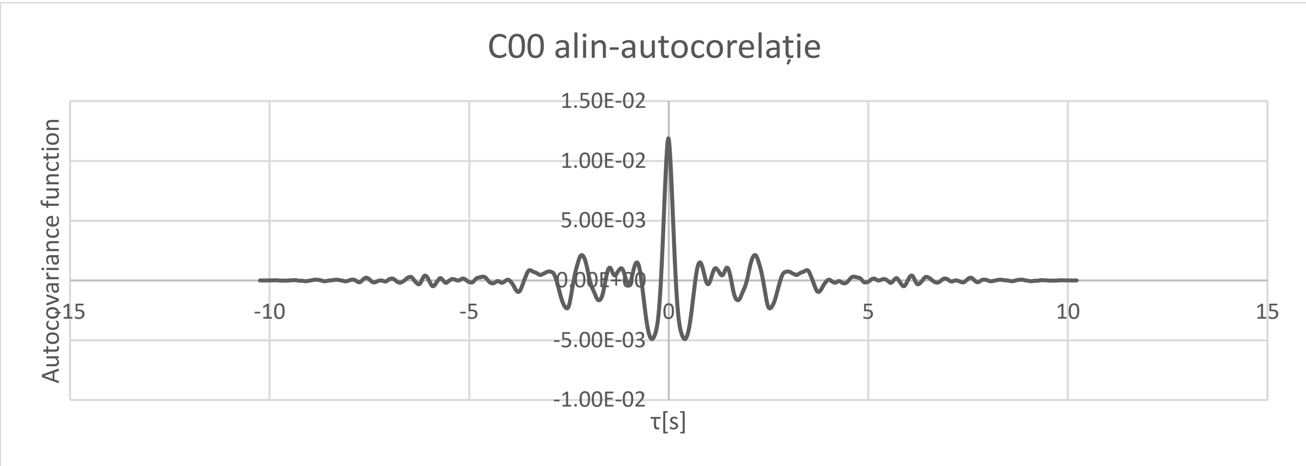

Fig. 7 - Autocovariance function for the aligned C00 station

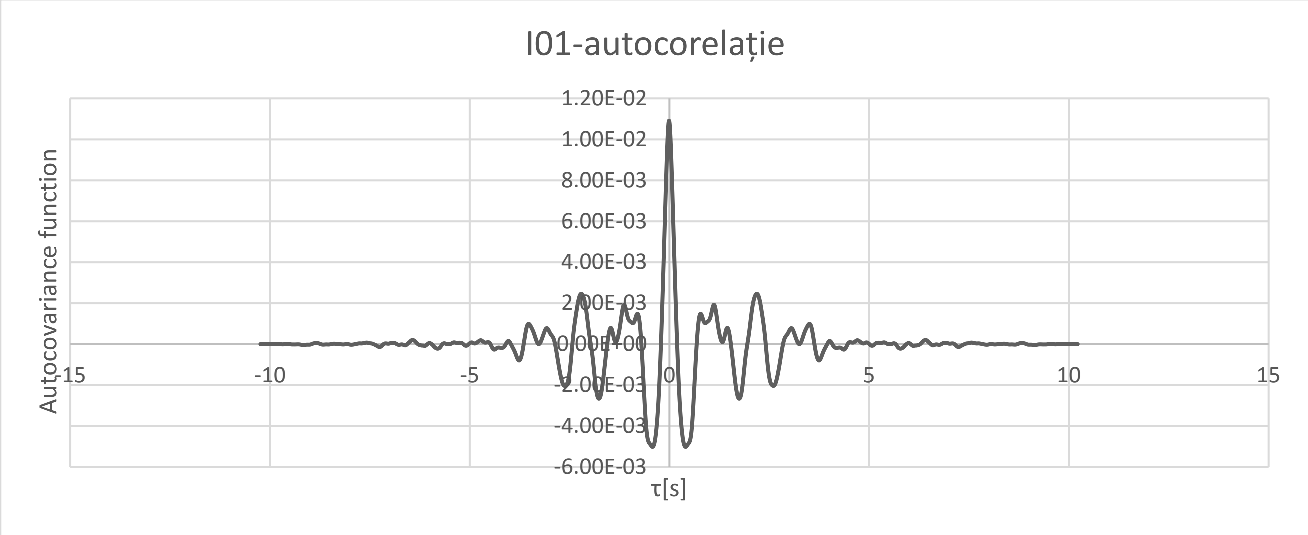

Fig. 8 - Autocovariance function for the aligned I01 station

The power spectral densities for the two stations are smoothed with a Hamming window.

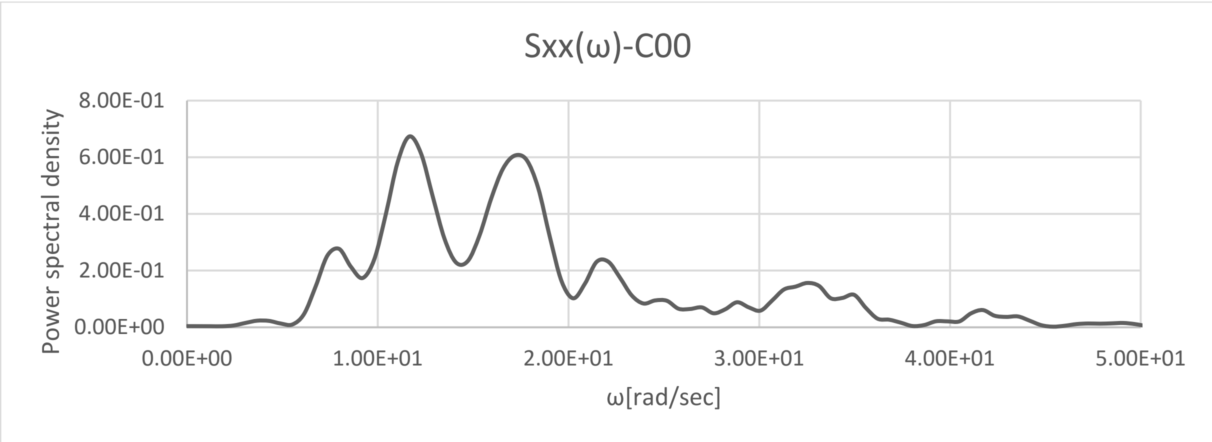

Fig. 9 - Power spectral density for the C00 aligned station.

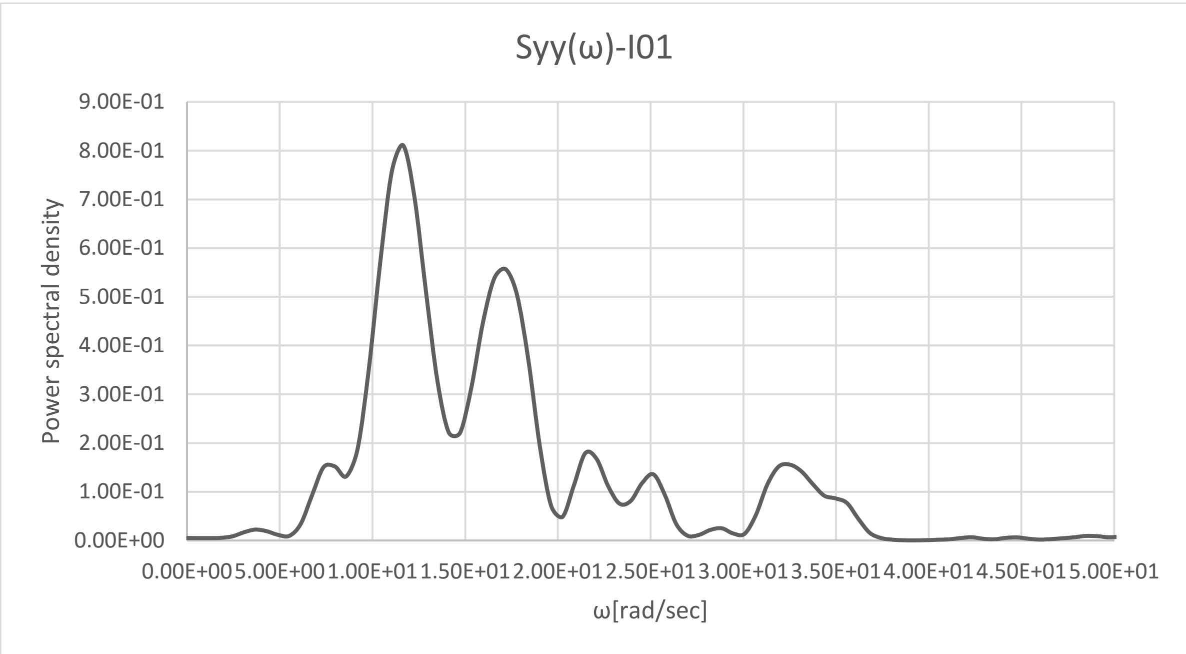

Fig. 10 - Power spectral density for the I01 station.

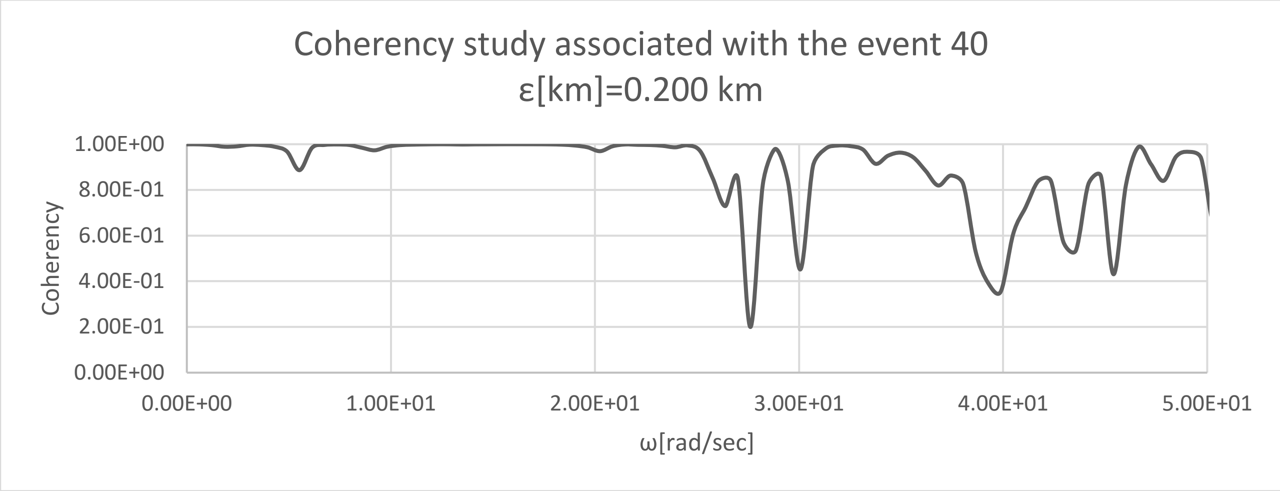

Fig. 11 - Coherency for the two stations analyzed, C00 and I01

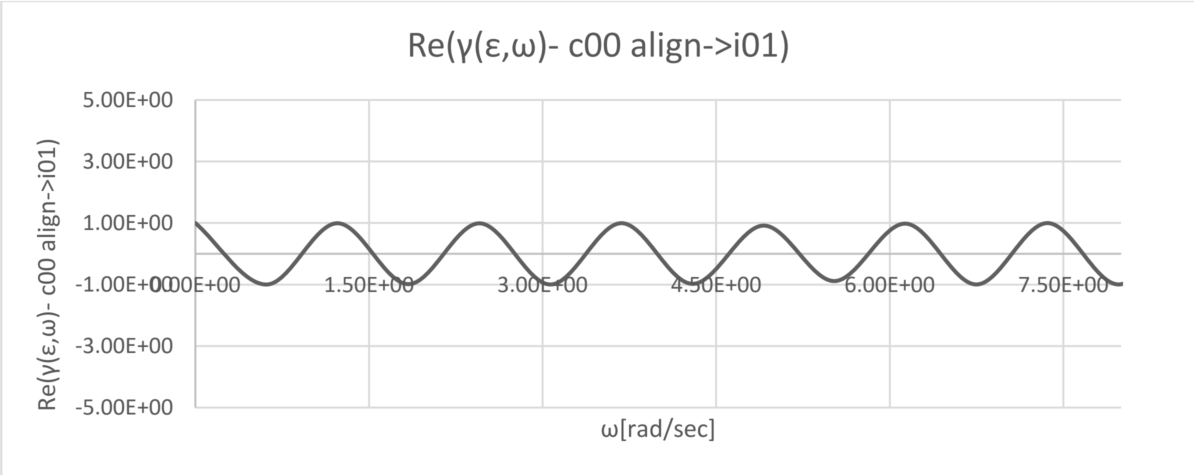

Fig. 12 - Real part of the coherency function for the pair C00 and I01

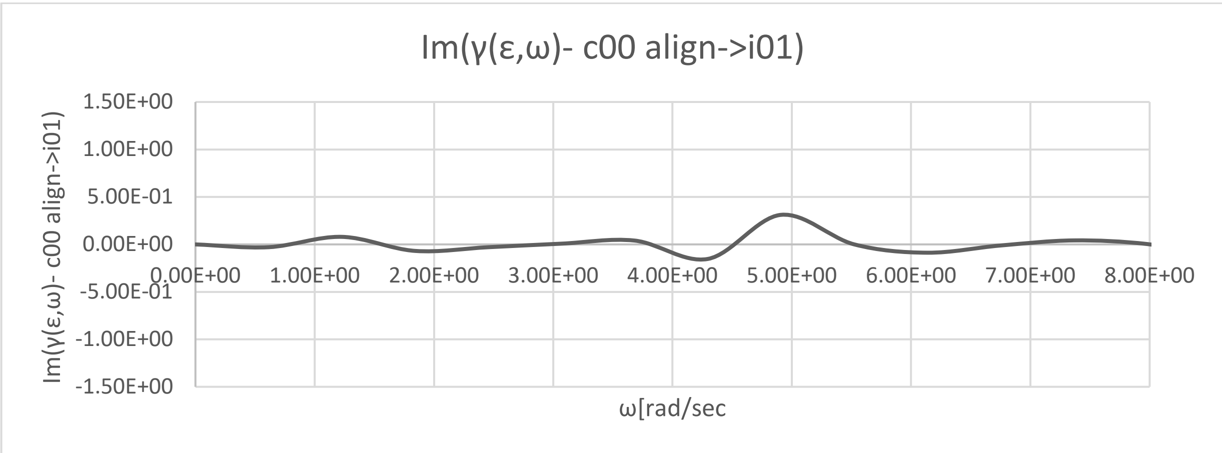

Fig. 13 - Imaginary part of the coherency function for the pair C00 and I01

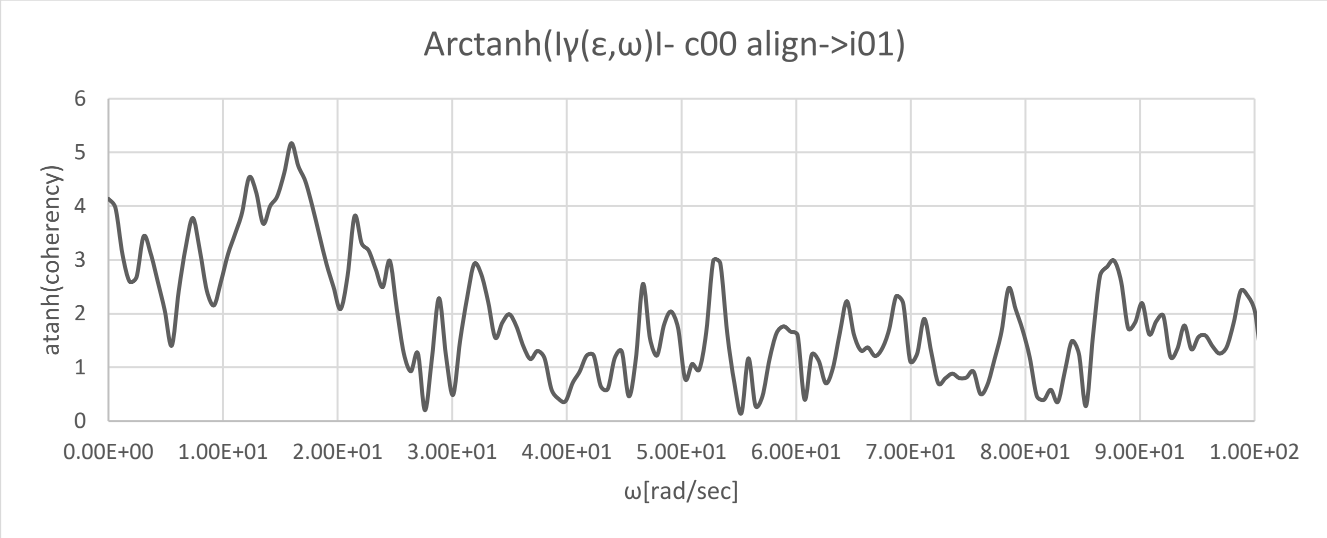

Fig. 14 - Arctanh (coherency) for the pair C00 and I01

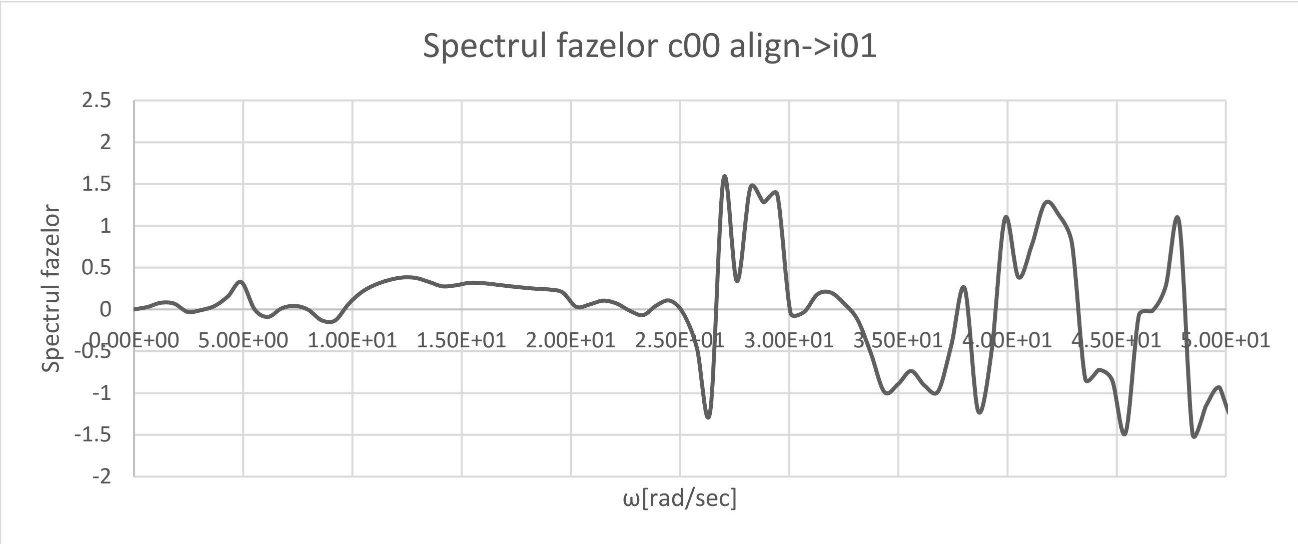

Fig. 15 - Phase spectrum for the pair C00 and I01

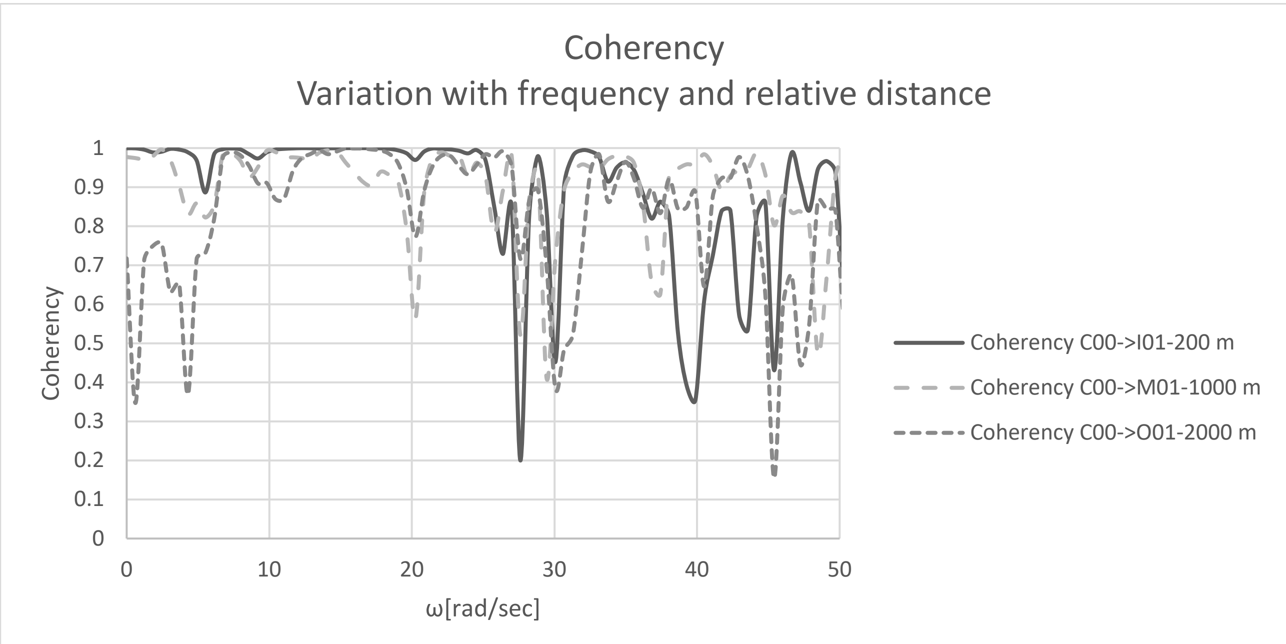

Fig. 16- The coherency for the three pairs of stations C00-I01, C00-M01, C00-O01

5. Conclusions

The coherency $\left|\bar{\gamma}_{jk}^{M}(\omega)\right|$, represents a measure of the similarity between recorded ground motions and indicates the degree the time series are related by a linear transfer function.

If a random process is obtained through a linear combination of another stochastic process hence the coherency is equal to unity, whereas the processes are uncorrelated the coherency is equal to zero.

It is expected that the coherency to tend to unity, when the distances between the stations are small and when they are similar in frequency content.

On the other hand, at high frequencies and large distances the time series become very uncorrelated and theoretically would tend to zero.

The value of the coherency is situated between these two extreme cases, and will vary with distance and frequency.

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