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Nonlinear elliptic boundary value problems with convection term and Hardy potential

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ABSTRACT. In this paper, we deal with a nonlinear elliptic problems that incorporate a Hardy potential and a nonlinear convection term. We establish the existence and regularity of solutions under various assumptions concerning the summability of the source term f .

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1. Introduction

This paper is concerned with the study of a class of nonlinear elliptic equations whose model is

$$\begin{cases} -\operatorname{div} (b(x, v, Dv) + \mathcal{B}(x, v)) = \gamma \frac{|v|^{p-2}v}{|x|^p} + f & \text{in } \mathcal{W}, \\ u = 0 & \text{on } \partial\mathcal{W}. \end{cases} \quad (1.1)$$

Here $\mathcal{W}$ denotes bounded open subset of $\mathbb{R}^N$, $N > 2$ and $1 < p < \infty$.

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The functions $b : \mathcal{W} \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $\mathcal{B} : \mathcal{W} \times \mathbb{R} \longrightarrow \mathbb{R}^N$ are a Carathodory functions that satisfying:

$$b(x, \zeta, \varsigma) \cdot \varsigma \geq \alpha |\varsigma|^p, \quad (1.2)$$

$$|b(x, \zeta, \varsigma)| \leq \beta \left(a(x) + |\zeta|^{p-1} + |\varsigma|^{p-1} \right), \quad (1.3)$$

$$(b(x, \zeta, \varsigma) - b(x, \zeta, \varsigma'))(\varsigma - \varsigma') > 0, \quad (1.4)$$

for a.e. $x \in \mathcal{W}$, for every $(\zeta, \varsigma) \in \mathbb{R} \times \mathbb{R}^N$, α and β are positive real numbers, and $a \in L^{p'}(\mathcal{W})$.

$$|\mathcal{B}(x, \zeta)| \leq c_0(x) |\zeta|^\lambda, \text{ with } c_0 \in L^{\frac{N}{p-1}}(\mathcal{W}), \text{ and } 0 \leq \lambda < p-1. \quad (1.5)$$

Furthermore, the datum f and the parameter γ satisfying:

$$f \in L^t(\mathcal{W}), \text{ with } t > 1, \text{ and } \gamma \geq 0. \quad (1.6)$$

Remark 1.1. Note that the hypothesis (1.3), will be changed in the section 4 to the following condition

$$|b(x, \zeta, \varsigma)| \leq \beta \left(|\zeta|^{p-1} + |\varsigma|^{p-1} \right). \quad (1.7)$$

There have been a large number of research activities on the study of the existence and the regularity of the solutions depending on the summability of f for general Leray-Lions problems either in the case $\gamma = 0$ or $\mathcal{B} \equiv 0$. In particular, when $\gamma = 0$ and $c_0 \equiv 0$, the authors in [5] proved that if $f \in L^t(\mathcal{W})$, there exist a distributional solution $v \in W_0^{1,t^*(p-1)}(\mathcal{W})$ for $t \in]1, \frac{Np}{Np-N+p}[,$ also they established the existence of a weak solution $v \in W_0^{1,p}(\mathcal{W}) \cap L^{\frac{(tp)^*}{p'}}(\mathcal{W})$ for $t \in [\frac{Np}{Np-N+p}, \frac{N}{p}]$. For the case when $\gamma = 0$ and $f \in W^{-1,p'}(\mathcal{W})$ Del Vecchio, T. and Posteraro, M.R. proved in [8], the existence and regularity of weak solution when $\|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})}$ is sufficiently small. Moreover, if $\mathcal{B} \equiv 0$ and $f \in L^t(\mathcal{W})$ with $t > 1$, it is well known, from the paper [2], that the existence and regularity of a solutions of (1.1) depends both on the summability of the datum f and on the size of γ .

Our aim is to generalize the existence and regularity results obtained in [5, 2] to the case when the Hardy potential $\frac{|v|^{p-2}v}{|x|^p}$ and the convection term $-\operatorname{div}(\mathcal{B}(x, v))$ are combining in the same equation. We point out that the presence of both the nonlinear convection term and the one singular at the origin, the so-called Hardy potential seems to prevent the achievement of the priori estimates. To overcome this failure we give an alternative proof to drive the required a priori estimates for problem (1.1). Furthermore, we analyze the interaction between these two terms and their impact on the existence and the regularity of weak and distributional solutions to the problem (1.1). We also take into account the difficult case where $p < 2 - \frac{1}{N}$ and $t = \frac{N}{N(p-1)+1}$. This case is particularly delicate as it is not so usual in the study of elliptic problems because in this limit framework the exponent $t^*(p-1) = 1$, i.e. we are dealing with distributional solutions in the non-reflexive space $W_0^{1,1}(\mathcal{W})$.

In the sequel we will denote by $\mathcal{S}$ the best constant in Sobolev inequality. Moreover, given $k > 0$, the classical truncated functions $\mathcal{T}_k$ and $\mathcal{G}_k$ defined by

$$\mathcal{G}_k(l) = (|l| - k)^+ \operatorname{sgn}(l), \quad l \in \mathbb{R},$$

$$\mathcal{T}_k(l) = l - \mathcal{G}_k(l), \quad l \in \mathbb{R}.$$

Let us also introduce the real function, defined by

$$\begin{aligned} \mathcal{F} :]1, +\infty[&\longrightarrow \mathbb{R} \\ l &\longmapsto \mathcal{F}(l) := \frac{\alpha}{p(p-1)^{p-2}} \frac{N}{lp} (l-1) (N-lp)^{p-1}. \end{aligned} \quad (1.8)$$

We explicitly observe that if $l = (p^*)' = \frac{Np}{Np-N+p}$, then

$$\mathcal{F}(l) = \alpha \frac{p-1}{p} \left(\frac{N-p}{p} \right)^p.$$

Here we state the following inequality (Hardy inequality) which will be used in the proof of our main results.

If $v \in W_0^{1,p}(\mathcal{W})$. Then, we have

$$\int_{\mathcal{W}} \frac{|v|^p}{|x|^p} dx \leq \mathcal{H} \int_{\mathcal{W}} |Dv|^p dx,$$

with $\mathcal{H} = \left(\frac{p}{N-p} \right)^p$ optimal and not achieved constant.

The paper is outlined as follows: Section 2 is dedicated to proof the existence and regularity of weak solution, and in Section 3, we demonstrate the existence and regularity of distributional solution.

2. Weak solution

This section is devoted to establish the following existence theorem. In what follows, C_i with $i \in \mathbb{N}$, will denote a positive constants.

Theorem 2.1. Assume that (1.2) – (1.6) hold. If $f \in L^t(\mathcal{W})$ with $t \geq (p^*)'$ and $\gamma < \frac{\alpha}{p'\mathcal{H}}$, then there exists at least a weak solution $v \in W_0^{1,p}(\mathcal{W})$ for (1.1) in the following sense

$$\begin{cases} \mathcal{B}(x, v) \in (L^{p'}(\mathcal{W}))^N, \frac{|v|^{p-2}v}{|x|^p} \in L^1(\mathcal{W}) \\ \int_{\mathcal{W}} b(x, v, Dv) D\varphi dx + \int_{\mathcal{W}} \mathcal{B}(x, v) D\varphi dx = \gamma \int_{\mathcal{W}} \frac{|v|^{p-2}v}{|x|^p} \varphi dx + \int_{\mathcal{W}} f \varphi dx \end{cases} \quad (2.1)$$

for every $\varphi \in W_0^{1,p}(\mathcal{W}) \cap L^\infty(\mathcal{W})$.

Moreover, if $\gamma < \mathcal{F}(t)$, then

$$v \in L^{\frac{(tp)^*}{p'}}(\mathcal{W}). \quad (2.2)$$

Remark 2.1. We point out that the problem (1.1) is not coercive except for $\gamma < \frac{\alpha}{p'\mathcal{H}}$. Indeed letting $v \in W_0^{1,p}(\mathcal{W})$ be the solution of (1.1) and $A(v) = -\operatorname{div}(b(x, v, Dv) + \mathcal{B}(x, v)) - \gamma \frac{|v|^{p-2}v}{|x|^p}$.

We have

$$\langle A(v), v \rangle \geq \left[\frac{\alpha}{p'} - \gamma \mathcal{H} \right] \|Dv\|_{L^p(\mathcal{W})}^p - C,$$

where

$$C = C(\alpha, p, N, c_0, \gamma) \quad \text{and} \quad \mathcal{H} = \left(\frac{p}{N-p} \right)^p.$$

Then, for every value of m , one has $\gamma < \frac{\alpha}{p' \mathcal{H}}$, the operator A is coercive in $W_0^{1,p}(\mathcal{W})$.

Remark 2.2. Let us recall that, where $c_0 \equiv 0$, the study of the effect of Hardy potential and the optimality of the condition on γ , with respect to the summability of f , has been thoroughly studied in the paper [2]. To provide a more detailed insight, the authors prove that the parameter γ must be less than a critical value denoted as

$$\Lambda(t) = \alpha \frac{N(t-1)(N-pt)^{p-1}}{t^p(p-1)^{p-1}}$$

in order to establish that (1.1) admits a solution $v_\Lambda \in W_0^{1,p}(\mathcal{W}) \cap L^{\frac{(tp)^*}{p'}}(\mathcal{W})$.

Furthermore, the study establishes that if $f \in L^t(\mathcal{W})$, with $t > \frac{N}{p}$, solutions of problem (1.1) are unbounded.

Summarizing the results of the above theorem and the previous remark, we have the following picture

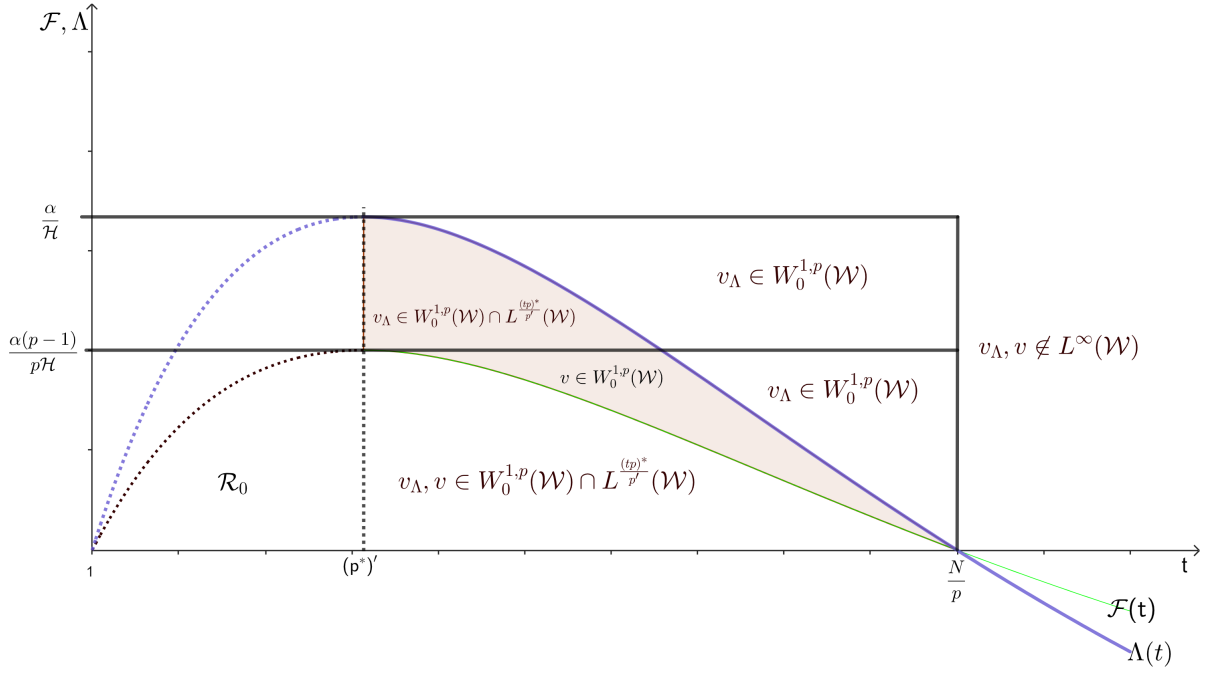


FIGURE 1. Regularity results

It appears clearly from the figure above that the presence of the convection term $-\operatorname{div}(\mathcal{B}(x, v))$ affect on the classical optimal condition on γ obtained in the paper [2], since the theorem 2.1 provide that the estimate (2.2) is not sharp if

$$\Lambda(t) < \gamma < \mathcal{F}(t).$$

Moreover, in the next section we deal with region $\mathcal{R}_0$.

Remark 2.3. We stress that the case where $\gamma = p - 1$ and $f \in L^t(\Omega)$, with $t > 1$, is more delicate since, in this setting, the test function method fails, in some sense, in giving a priori estimates. To Surmount this shortcomings, in an upcoming paper, we achieve the necessary a priori estimates through a contradictory approach. This indirect method, has been already used in [7] to tackle the following linear problem

Proof. We split the proof in forth steps.

First step: Approximate problem.

Let us consider the following approximate problem:

$$\begin{cases} -\operatorname{div}(b(x, v_m, Dv_m) + \mathcal{B}_m(x, v_m)) = \gamma \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{\frac{1}{m} + |x|^p} + f_m & \text{in } \mathcal{W}, \\ v_m = 0 & \text{on } \partial\mathcal{W}, \end{cases} \quad (2.3)$$

where

$$\mathcal{B}_m(x, s) = \mathcal{B}(x, \mathcal{T}_m(s)) \quad \forall s \in \mathbb{R}. \quad (2.4)$$

$$f_m = \mathcal{T}_m(f). \quad (2.5)$$

The existence of $v_m \in W_0^{1,p}(\mathcal{W})$ such that,

$$\int_{\mathcal{W}} b(x, v_m, Dv_m) D\varphi dx + \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\varphi dx = \int_{\mathcal{W}} f_m \varphi dx + \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{\frac{1}{m} + |x|^p} \varphi dx \quad (2.6)$$

for every $\varphi \in W_0^{1,p}(\mathcal{W}) \cap L^\infty(\mathcal{W})$, is guaranteed by the Leray-Lions theorem (see [11]).

Remark 2.4. Note that the presence of $\mathcal{T}_m(v_m)$ in the right-hand side and in $\mathcal{B}_m(x, v_m)$ is needed in order to have solutions for every $\gamma > 0$, using only v_m , will force γ to be smaller than $\frac{\alpha}{p'\mathcal{H}}$ in order that the operator $A(v_m)$ be coercive.

Second step: A priori estimates.

In this step we give the following proposition which gives some a priori estimates on v_m , $\mathcal{T}_k(v_m)$ and $\mathcal{G}_k(v_m)$ in Sobolev spaces.

Proposition 2.1. Suppose that the hypotheses (1.2) – (1.5) hold. If $f \in L^t(\mathcal{W})$ with $t \geq (p^*)'$ and $\gamma < \frac{\alpha}{p'\mathcal{H}}$, then there exists a positive constants L_0, L_1, M_0, M_1 and M_2 such that

$$\int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx \leq L_0 + M_0 k^p \quad \forall k > 0, \quad (2.7)$$

$$\int_{\mathcal{W}} |D\mathcal{T}_k(v_m)|^p dx \leq L_1 + M_1 k + M_2 k^p \quad \forall k > 0. \quad (2.8)$$

Moreover, for any $\gamma < \mathcal{F}(t)$, v_m is bounded in $L^{\frac{(tp)^*}{p'}}(\mathcal{W})$.

Proof. We take $\mathcal{G}_k(v_m)$ as test function in (2.6) we get:

$$\begin{aligned} \int_{\mathcal{W}} b(x, v_m, Dv_m) D\mathcal{G}_k(v_m) dx + \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\mathcal{G}_k(v_m) dx &= \int_{\mathcal{W}} f_m \mathcal{G}_k(v_m) dx \\ &+ \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} \mathcal{G}_k(v_m) dx. \end{aligned}$$

Using (1.2), (1.5), Hlder, Sobolev, Hardy and Young inequalities it follows that

$$\begin{aligned} \alpha \int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx &\leq \frac{C_\alpha}{p'} \int_{\mathcal{W}} |c_0|^{p'} |v_m|^{\lambda p'} dx + \frac{\alpha}{3p} \int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx \\ &+ \gamma k^{p-1} \mathcal{H}^{\frac{1}{p}} \left(\int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx \right)^{\frac{1}{p}} + \gamma \mathcal{H} \int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx \\ &+ \mathcal{S} \left(\int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx \right)^{\frac{1}{p}} \|f_m\|_{L^t(\mathcal{W})} (\text{meas}(\mathcal{W}))^{\frac{1}{(p^*)'} - \frac{1}{t}}, \\ &\leq C_1 + C_2 k^p + \frac{\alpha}{p} \int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx + \gamma \mathcal{H} \int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx, \end{aligned}$$

therefore, we get

$$\left(\frac{\alpha}{p'} - \gamma \mathcal{H} \right) \int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx \leq C_1 + C_2 k^p,$$

since we have $\gamma < \frac{\alpha}{p'\mathcal{H}}$, thus we get (2.7).

We take $\mathcal{T}_k(v_\varepsilon)$ as test function in (2.6) we get:

$$\begin{aligned} \int_{\mathcal{W}} b(x, v_\varepsilon, Dv_\varepsilon) D\mathcal{T}_k(v_\varepsilon) dx + \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\mathcal{T}_k(v_m) dx &= \int_{\mathcal{W}} f_m \mathcal{T}_k(v_m) dx \\ &+ \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} \mathcal{T}_k(v_m) dx. \end{aligned}$$

Thanks to (1.2), (1.5), Hlder, Sobolev, Hardy and Young inequalities we obtain

$$\begin{aligned} \alpha \int_{\mathcal{W}} |D\mathcal{T}_k(v_m)|^p dx &\leq \int_{\mathcal{W}} c_0(x) |v_m|^\lambda |D\mathcal{T}_k(v_m)| dx + \gamma \int_{\mathcal{W}} \frac{|v_m|^{p-1}}{|x|^p + \frac{1}{m}} |\mathcal{T}_k(v_m)| dx + k \|f_m\|_{L^1(\mathcal{W})}, \\ &\leq C_3 + \frac{\alpha}{p} \int_{\mathcal{W}} |D\mathcal{T}_k(v_m)|^p dx + \gamma \mathcal{H} \int_{\mathcal{W}} |D\mathcal{G}_k(v_m)|^p dx + k \|f_m\|_{L^1(\mathcal{W})}, \end{aligned}$$

thanks to (2.7), we get (2.8).

For any integer $m \geq 1$, choosing the function $g_m(v_m) = |\mathcal{T}_m(v_m)|^s \mathcal{T}_m(v_m)$ as test function in (2.6), with $s = \frac{t(Np - (N - p)) - Np}{N - tp}$, leads to

$$\begin{aligned} \int_{\mathcal{W}} b(x, v_m, Dv_m) Dg_m(v_m) dx + \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) Dg_m(v_m) dx &= \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} g_m(v_m) dx \\ &+ \int_{\mathcal{W}} f_m g_m(v_m) dx. \end{aligned}$$

By reason of (1.2) and (1.6), we get

$$\begin{aligned} \alpha(s+1) \int_{\mathcal{W}} |D\mathcal{T}_m(v_m)|^p |\mathcal{T}_m(v_m)|^s dx &\leq (s+1) \int_{\mathcal{W}} c_0(x) |\mathcal{T}_m(v_m)|^\lambda |D\mathcal{T}_m(v_m)| |\mathcal{T}_m(v_m)|^s dx \\ &+ \int_{\mathcal{W}} |f_m| |\mathcal{T}_m(v_m)|^{s+1} dx + \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{s+p}}{|x|^p} dx, \end{aligned}$$

using Young, Hardy inequalities and the relation

$$|D|\mathcal{T}_m(v_m)|^{s_1}|^p = s_1^p |D\mathcal{T}_m(v_m)|^p |\mathcal{T}_m(v_m)|^{p(s_1-1)},$$

with $s_1 = \frac{s}{p} + 1$, we obtain

$$\begin{aligned} \int_{\mathcal{W}} |D|\mathcal{T}_m(v_m)|^{s_1}|^p dx &\leq C_4 \int_{\mathcal{W}} |c_0(x)|^{p'} |\mathcal{T}_m(v_m)|^{\lambda p'} |\mathcal{T}_m(v_m)|^s dx + C_5 \int_{\mathcal{W}} |D\mathcal{T}_m(v_m)|^p |\mathcal{T}_m(v_m)|^s dx \\ &+ \|f_m\|_{L^t(\mathcal{W})} \left(\int_{\mathcal{W}} |\mathcal{T}_m(v_m)|^{(s+1)t'} dx \right)^{\frac{1}{t'}} + \gamma \mathcal{H} \int_{\mathcal{W}} |D|\mathcal{T}_m(v_m)|^{s_1}|^p dx, \\ &\leq C_6 \left(\int_{\mathcal{W}} |\mathcal{T}_m(v_m)|^{s_1} |x|^{\frac{s_1 p}{p^*}} dx \right)^{\frac{s_1 p}{s_1 p^*}} + C_7 \int_{\mathcal{W}} |D|\mathcal{T}_m(v_m)|^{s_1}|^p dx \\ &+ \|f_m\|_{L^t(\mathcal{W})} \left(\int_{\mathcal{W}} |\mathcal{T}_m(v_m)|^{(s+1)t'} dx \right)^{\frac{1}{t'}} + \gamma \mathcal{H} \int_{\mathcal{W}} |D|\mathcal{T}_m(v_m)|^{s_1}|^p dx, \end{aligned}$$

by Sobolev inequality, we have

$$\left[C_8 - \gamma \mathcal{H} p' \right] \left(\int_{\mathcal{W}} |\mathcal{T}_m(v_m)|^{s_1} |x|^{\frac{s_1 p}{p^*}} dx \right)^{\frac{p}{p^*}} \leq C_9 + C_{10} \left(\int_{\mathcal{W}} |\mathcal{T}_m(v_m)|^{(s+1)t'} dx \right)^{\frac{1}{t'}},$$

where

$$s_\lambda = \frac{\lambda}{p-1} + \frac{s}{p}.$$

Considering that $\gamma < \mathcal{F}(t)$, we can derive the following inequality

$$\left(\int_{\mathcal{W}} |\mathcal{T}_m(v_m)|^{s_1 p^*} dx \right)^{\frac{p}{p^*}} \leq C_{11} + C_{12} \left(\int_{\mathcal{W}} |\mathcal{T}_m(v_m)|^{(s+1)t'} dx \right)^{\frac{1}{t'}}.$$

By the definition of s we have

$$s_1 p^* = (s+1)t' = \frac{(tp)^*}{p'}.$$

Hence, being $\frac{p}{p^*} > \frac{1}{t'}$, we conclude, based on the previous inequality, that

$$\|\mathcal{T}_m(v_m)\|_{L^{\frac{(tp)^*}{p'}}(\mathcal{W})} \leq C_{13}. \quad (2.9)$$

In summary, this concludes the proof of the proposition.

Remark 2.5. We observe that, since $u_m = T_k(u_m) + G_k(u_m)$, the estimates (2.7) and (2.8) give that

$$\forall n > 0, \forall \varepsilon > 0, \exists k_\varepsilon > 0 \text{ such that } \text{meas}\{|v_m| > k\} \leq \varepsilon \quad k > k_\varepsilon. \quad (2.10)$$

Third step: The strong convergence of v_m in $W_0^{1,p}(\mathcal{W})$.

In this step we prove that the convergence in (2.16) is actually strong. To this aim, choosing $\mathcal{T}_k(v_m) - \mathcal{T}_k(v)$ as test function in (2.6) we obtain

$$\begin{aligned} & \int_{\mathcal{W}} b(x, v_m, Dv_m) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx + \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx \\ &= \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} (\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx + \int_{\mathcal{W}} f_m(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx. \end{aligned}$$

Since, we have

$$\begin{aligned} \int_{\mathcal{W}} b(x, v_m, Dv_m) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx &= \int_{\mathcal{W}} b(x, \mathcal{T}_k(v_m), D\mathcal{T}_k(v_m)) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx \\ &\quad - \int_{\mathcal{W}} b(x, v_m, Dv_m) D\mathcal{T}_k(v) \chi_{\mathcal{D}_m(k)} dx. \end{aligned}$$

And

$$\begin{aligned} \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx &= \int_{\mathcal{W}} \mathcal{B}_m(x, \mathcal{T}_k(v_m)) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx \\ &\quad - \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\mathcal{T}_k(v) \chi_{\mathcal{D}_m(k)} dx, \end{aligned}$$

where $\mathcal{D}_m(k) = \{x \in \mathcal{W}, |v_m(x)| > k\}$.

Thus, we obtain

$$\begin{aligned} & \int_{\mathcal{W}} (b(x, \mathcal{T}_k(v_m), D\mathcal{T}_k(v_m)) - b(x, \mathcal{T}_k(v_m), D\mathcal{T}_k(v))) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx \\ &= \int_{\mathcal{W}} b(x, v_m, Dv_m) D\mathcal{T}_k(v) \chi_{\mathcal{D}_m(k)} dx - \int_{\mathcal{W}} \mathcal{B}_m(x, \mathcal{T}_k(v_m)) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx \\ &+ \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\mathcal{T}_k(v) \chi_{\mathcal{D}_m(k)} dx + \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} (\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx \end{aligned}$$

$$+ \int_{\mathcal{W}} f_m(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) \, dx.$$

Putting

$$I_m = K_m^1 + K_m^2 + J_m^1 + J_m^2 + R_m^1 + R_m^2,$$

where

$$\begin{aligned} I_m &= \int_{\mathcal{W}} (b(x, \mathcal{T}_k(v_m), D\mathcal{T}_k(v_m)) - b(x, \mathcal{T}_k(v_m), D\mathcal{T}_k(v))) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) \, dx, \\ K_m^1 &= - \int_{\mathcal{W}} \mathcal{B}_m(x, \mathcal{T}_k(v_m)) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) \, dx, \\ K_m^2 &= \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\mathcal{T}_k(v) \chi_{\mathcal{D}_m(k)} \, dx, \\ R_m^1 &= \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} (\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) \, dx, \\ R_m^2 &= \int_{\mathcal{W}} f_m(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) \, dx, \\ J_m^1 &= \int_{\mathcal{W}} b(x, \mathcal{T}_k(v_m), D\mathcal{T}_k(v)) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) \, dx, \\ J_m^2 &= \int_{\mathcal{W}} b(x, v_m, Dv_m) D\mathcal{T}_k(v) \chi_{\mathcal{D}_m(k)} \, dx. \end{aligned}$$

Using the fact that $D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) \rightharpoonup 0$ in $(L^p(\mathcal{W}))^N$ and (2.19) we are able to pass to the limit in K_m^1 as n tends to $+\infty$, it follows that

$$\lim_{m \rightarrow +\infty} K_m^1 = 0.$$

Note that, $\forall k > 0$, we have $D\mathcal{T}_k(v) \chi_{\mathcal{D}_m(k)} \rightarrow 0$ strongly in $(L^p(\mathcal{W}))^N$, then from (2.16) we deduce that

$$\lim_{m \rightarrow +\infty} K_m^2 = 0.$$

Since, we have

$$b(x, \mathcal{T}_k(v_m), D\mathcal{T}_k(v)) \rightarrow b(x, \mathcal{T}_k(v), D\mathcal{T}_k(v)) \text{ strongly in } (L^{p'}(\mathcal{W}))^N,$$

and

$$D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) \rightharpoonup 0 \text{ weakly in } (L^p(\mathcal{W}))^N.$$

Then, we prove that

$$\lim_{m \rightarrow +\infty} J_m^1 = 0.$$

Due to the strong convergence (2.21) and the *weak* $*$ convergence of $\mathcal{T}_k(v_m) - \mathcal{T}_k(v)$ to 0 in $L^\infty(\mathcal{W})$ we can pass to the limit in R_m^1 and deduce that

$$\lim_{m \rightarrow +\infty} R_m^1 = 0.$$

Since we have $\mathcal{T}_k(v_m) - \mathcal{T}_k(v) \rightharpoonup 0$ *weak* $*$ in $L^\infty(\mathcal{W})$ and by recalling the definition (2.5), then we can reach the limit in R_m^2 as $m \rightarrow +\infty$ and get to the conclusion that

$$\lim_{m \rightarrow +\infty} R_m^2 = 0.$$

Note that we have for any $k > 0$

$$D\mathcal{T}_k(v) \chi_{\mathcal{D}_m(k)} \rightarrow 0 \text{ strongly in } (L^p(\mathcal{W}))^N,$$

and due to (2.16) we get that

$$b(x, v_m, Dv_m) \rightharpoonup \sigma \text{ weakly in } (L^{p'}(\mathcal{W}))^N,$$

such convergences permitting us to take the limit in J_m^2 as $m \rightarrow +\infty$, it follows that

$$\lim_{m \rightarrow +\infty} J_m^2 = 0.$$

Therefore, by (1.4), it follows that

$$\lim_{m \rightarrow +\infty} \int_{\mathcal{W}} (b(x, \mathcal{T}_k(v_m), D\mathcal{T}_k(v_m)) - b(x, \mathcal{T}_k(v), D\mathcal{T}_k(v))) D(\mathcal{T}_k(v_m) - \mathcal{T}_k(v)) dx = 0, \quad (2.11)$$

from which we deduce, like in [3], that for any $k > 0$

$$\mathcal{T}_k(v_m) \rightarrow \mathcal{T}_k(v) \text{ strongly in } W_0^{1,p}(\mathcal{W}). \quad (2.12)$$

Now we proceed to demonstrate that $|Dv_m|^p \rightarrow |Dv|^p$ strongly in $L^1(\mathcal{W})$. To employ Vitali's Theorem once more, it is necessary to establish the equi-integrability of $|Dv_m|^p$. Consider a measurable set $\mathcal{E} \subset \mathcal{W}$, we have

$$\int_{\mathcal{E}} |Dv_m|^p dx \leq \int_{\mathcal{E}} |D\mathcal{T}_k(v_m)|^p dx + \int_{\mathcal{D}_m(k) \cap \mathcal{E}} |Dv_k|^p dx.$$

Due to (2.12), we can infer that $\int_{\mathcal{E}} |D\mathcal{T}_k(v_m)|^p dx$ becomes uniformly small for $meas(\mathcal{E})$ small enough. Moreover, thanks to the remark 2.5, we get

$$\lim_{k \rightarrow +\infty} \int_{\mathcal{D}_m(k)} |Dv_m|^p dx = 0 \text{ uniformly in } m. \quad (2.13)$$

which implies

$$\int_{\mathcal{D}_m(k) \cap \mathcal{E}} |Dv_m|^p dx \leq \int_{\mathcal{D}_m(k)} |Dv_m|^p dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence, Vitali's Theorem entails that

$$|Dv_m|^p \rightarrow |Dv|^p \text{ strongly in } L^1(\mathcal{W}). \quad (2.14)$$

Therefore, leveraging the uniform convexity of $L^p(\mathcal{W})$ we get

$$v_m \rightarrow v \text{ strongly in } W_0^{1,p}(\mathcal{W}). \quad (2.15)$$

Fourth step: Passing to the limit.

The estimates (2.7) and (2.8) implies that the sequence v_m is bounded in $W_0^{1,p}(\mathcal{W})$, it follows from classical theory (see for instance [1]) that there exists a function $v \in W_0^{1,p}(\mathcal{W})$ such that, up to a subsequence still denoted by v_m one has

$$v_m \rightharpoonup v \text{ weakly in } W_0^{1,p}(\mathcal{W}), \quad (2.16)$$

$$v_m \rightarrow v \text{ strongly in } L^p(\mathcal{W}) \text{ and a.e. in } \mathcal{W}, \quad (2.17)$$

$$\mathcal{T}_k(v_m) \rightharpoonup \mathcal{T}_k(v) \text{ weakly in } W_0^{1,p}(\mathcal{W}), \quad (2.18)$$

$$\mathcal{T}_k(v_m) \rightarrow \mathcal{T}_k(v) \text{ strongly in } L^p(\mathcal{W}) \text{ and a.e. in } \mathcal{W}, \quad (2.19)$$

$$\forall k > 0, \quad b(x, \mathcal{T}_k(v_m), D\mathcal{T}_k(v_m)) \rightharpoonup \sigma_k \text{ weakly in } (L^{p'}(\mathcal{W}))^N. \quad (2.20)$$

Let us show the following interesting convergences of the sequences $\frac{|v_m|^{p-1}}{|x|^p}$ and $\mathcal{B}_m(x, v_m)$.

Assuming $\mathcal{E}$ is a measurable subset $\mathcal{E}$ of $\mathcal{W}$ such that $0 \in \mathcal{E}$ and $meas(\mathcal{E})$ is small enough, we have

$$\begin{aligned} \int_{\mathcal{E}} \frac{|v_m|^{p-1}}{|x|^p} dx &\leq \left(\int_{\mathcal{E}} \frac{|v_m|^p}{|x|^p} dx \right)^{\frac{1}{p'}} \left(\int_{\mathcal{E}} \frac{dx}{|x|^p} \right)^{\frac{1}{p}}, \\ &\leq \left(\mathcal{H} \int_{\mathcal{W}} |Dv_m|^p dx \right)^{\frac{1}{p'}} \left(\int_{\mathcal{E}} \frac{dx}{|x|^p} \right)^{\frac{1}{p}}. \end{aligned}$$

Utilizing the estimates (2.7) and (2.8), as well as the property of absolute continuity of the Lebesgue integral, we can assert this for any $\varepsilon_1 > 0$

$$\int_{\mathcal{E}} \frac{|v_m|^{p-1}}{|x|^p} dx \leq C_{14} \left(\int_{\mathcal{E}} \frac{dx}{|x|^p} \right)^{\frac{1}{p}} \leq \varepsilon_1.$$

Consequently, we get

$$\int_{\mathcal{E}} \frac{|v_m|^{p-1}}{|x|^p} dx \leq \varepsilon.$$

Thus, by Vitali's theorem it follows that

$$\frac{|v_m|^{p-1}}{|x|^p} \rightarrow \frac{|v|^{p-1}}{|x|^p} \text{ strongly in } L^1(\mathcal{W}). \quad (2.21)$$

Since $\mathcal{B}_m$ is continuous in the last argument we have

$$\mathcal{B}_m(x, v_m) \rightarrow \mathcal{B}(x, v) \quad \text{a.e. in } \mathcal{W},$$

from (1.5), Hlder, Sobolev inequalities and the boundedness of the sequence Dv_m in $W_0^{1,p}(\mathcal{W})$ we infer that

$$\begin{aligned} \int_{\mathcal{W}} |\mathcal{B}_m(x, v_m)|^{p'} dx &\leq \int_{\mathcal{W}} |c_0(x)|^{p'} |v_m|^{\lambda p'} dx, \\ &\leq \mathcal{S}^{\lambda p'} (meas(\mathcal{W}))^{\frac{p'(p-1-\lambda)}{p^*}} \|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})}^{p'} \|Dv_m\|_{L^p(\mathcal{W})}^{\lambda p'}, \\ &\leq C_{15}. \end{aligned}$$

Thus, From the above inequality and by Lebesgue's dominated convergence Theorem, we obtain

$$\mathcal{B}_m(x, v_m) \rightarrow \mathcal{B}(x, v) \text{ strongly in } (L^{p'}(\mathcal{W}))^N. \quad (2.22)$$

According to the convergence (2.15) and (1.3), we achieve that

$$\lim_{m \rightarrow +\infty} \int_{\mathcal{W}} b(x, v_m, Dv_m) D\varphi dx = \int_{\mathcal{W}} b(x, v, Dv) D\varphi dx$$

$$\forall \varphi \in W_0^{1,p}(\mathcal{W}) \cap L^\infty(\mathcal{W}).$$

Using (2.22) and (1.4), we get

$$\lim_{m \rightarrow +\infty} \int_{\mathcal{W}} \mathcal{B}(x, v_m) D\varphi \, dx = \int_{\mathcal{W}} \mathcal{B}(x, v) D\varphi \, dx$$

$$\forall \varphi \in W_0^{1,p}(\mathcal{W}) \cap L^\infty(\mathcal{W}).$$

By the strong convergence (2.21) we drive that

$$\lim_{m \rightarrow +\infty} \int_{\mathcal{W}} \frac{|v_m|^{p-2} v_m}{|x|^p + \frac{1}{m}} \varphi \, dx = \int_{\mathcal{W}} \frac{|v|^{p-2} v}{|x|^p} \varphi \, dx$$

$$\forall \varphi \in W_0^{1,p}(\mathcal{W}) \cap L^\infty(\mathcal{W}).$$

In summary, considering all the aforementioned convergences, we can pass to the limit in (2.6), leading us to the conclusion that the limit v satisfies:

$$\int_{\mathcal{W}} b(x, v, Dv) D\varphi \, dx + \int_{\mathcal{W}} \mathcal{B}(x, v) D\varphi \, dx = \gamma \int_{\mathcal{W}} \frac{|v|^{p-2} v}{|x|^p} \varphi \, dx + \int_{\mathcal{W}} f\varphi \, dx \quad (2.23)$$

$$\forall \varphi \in W_0^{1,p}(\mathcal{W}) \cap L^\infty(\mathcal{W}).$$

Furthermore, if $m \rightarrow +\infty$ and $\gamma < \mathcal{F}(t)$, the Fatou Lemma implies that (2.2) is hold from (2.9).

3. Distributional solution

Before proving the main results of this section we must first introduce some preliminary lemmas. The next lemma guarantees the boundedness of $\mathcal{T}_k(v_m)$.

Lemma 3.1. *Let us assume (1.2), (1.5) and let v_m be the sequence of approximating solutions of (2.3). If $v_m \in W_0^{1,q}(\mathcal{W})$ with $q = \max \left\{ 1, \frac{N(p-1)}{N-1} \right\}$. Then, for any $k > 0$, $\mathcal{T}_k(v_m)$ is bounded in $W_0^{1,p}(\mathcal{W})$.*

Proof. Choosing $\mathcal{T}_k(v_m)$ as test function in (2.6) we obtain

$$\begin{aligned} \int_{\mathcal{W}} b(x, v_m, Dv_m) D\mathcal{T}_k(v_m) \, dx + \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\mathcal{T}_k(v_m) \, dx &= \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} \mathcal{T}_k(v_m) \, dx \\ &+ \int_{\mathcal{W}} f_m \mathcal{T}_k(v_m) \, dx. \end{aligned}$$

By virtue of (1.2), (1.5), Hlder, Sobolev, Hardy and Young inequalities we obtain

$$\begin{aligned} \alpha \int_{\mathcal{W}} |D\mathcal{T}_k(v_m)|^p \, dx &\leq \int_{\mathcal{W}} c_0(x) |v_m|^\lambda |D\mathcal{T}_k(v_m)| \, dx + \gamma \int_{\mathcal{W}} \frac{|v_m|^{p-1}}{|x|^p + \frac{1}{m}} |\mathcal{T}_k(v_m)| \, dx + k \|f_m\|_{L^1(\mathcal{W})}, \\ &\leq C_{16} + \frac{\alpha}{p} \int_{\mathcal{W}} |D\mathcal{T}_k(v_m)|^p \, dx + \gamma k \left(\int_{\mathcal{W}} |v_m|^{\frac{Nq}{N-q}} \, dx \right)^{\frac{1}{r}} \left(\int_{\mathcal{W}} \frac{dx}{|x|^{pr'}} \right)^{\frac{1}{r}} + k \|f_m\|_{L^1(\mathcal{W})}, \end{aligned}$$

$$\text{with } r = \frac{Nq}{(p-1)(N-q)} \text{ and } r' = \frac{r}{r-1}.$$

Thus, we have

$$\int_{\mathcal{W}} |D\mathcal{T}_k(v_m)|^p \, dx \leq C_{17} + C_{18}k, \quad \text{for any } k > 0. \quad (3.1)$$

Next, we will prove the following lemma.

Lemma 3.2. *If we assume that the sequence v_m is bounded in $W_0^{1,q}(\mathcal{W})$. Then, up to a subsequence still denoted by v_m it holds*

$$\frac{|v_m|^{p-1}}{|x|^p} \rightarrow \frac{|v|^{p-1}}{|x|^p} \quad \text{strongly in } L^1(\mathcal{W}). \quad (3.2)$$

Proof. Since v_m is bounded in $W_0^{1,q}(\mathcal{W})$, then up to a subsequence still denoted by v_m on has

$$v_m \rightarrow v \quad \text{a.e. in } \mathcal{W}.$$

For a measurable subset $\mathcal{E}$ of $\mathcal{W}$ such that $0 \in \mathcal{E}$ and $\text{meas}(\mathcal{E})$ is small enough, we apply Hlder inequality with the exponents $\varrho = \frac{q^*}{p-1}$ and $\varrho' = \frac{\varrho}{\varrho-1}$ we get

$$\begin{aligned} \int_{\mathcal{E}} \frac{|v_m|^{p-1}}{|x|^p} dx &\leq \left(\int_{\mathcal{E}} |v_m|^{\frac{Nq}{N-q}} dx \right)^{\frac{1}{\varrho}} \left(\int_{\mathcal{E}} \frac{dx}{|x|^{p\varrho'}} \right)^{\frac{1}{\varrho'}}, \\ &\leq C_{19} \left(\int_{\mathcal{E}} \frac{dx}{|x|^{p\varrho'}} \right)^{\frac{1}{\varrho'}}, \end{aligned}$$

which, by Vitali's Theorem and the absolute continuity of the Lebesgue integral, gives (3.2).

The following last lemma ensures the almost everywhere convergence of the sequence Dv_m under appropriate compactness assumptions on v_m .

Lemma 3.3. *Assume (1.2), (1.4)-(1.7). If $v_m \in W_0^{1,q}(\mathcal{W})$ then, up to a subsequence, Dv_m converges almost everywhere in $\mathcal{W}$ to Dv .*

Proof. Let $0 < j < k$, using $\mathcal{T}_j(v_m - \mathcal{T}_k(v))$ as test function in (2.6). It yields

$$\begin{aligned} &\int_{\mathcal{W}} b(x, v_m, Dv_m) D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx + \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx \\ &= \int_{\mathcal{W}} f_m \mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx + \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} \mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_{\mathcal{W}} (b(x, v_m, Dv_m) - b(x, v_m, D\mathcal{T}_k(v))) D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx \\ &\leq j \int_{\mathcal{W}} |f_m| dx + j\gamma \int_{\mathcal{W}} \frac{|v_m|^{p-1}}{|x|^p} dx - \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx \\ &\quad - \int_{\mathcal{W}} b(x, v_m, D\mathcal{T}_k(v)) D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx. \end{aligned} \quad (3.3)$$

Let's analyze the previous terms one by one. First of all, thanks to the properties of f_m we have

$$\lim_{j \rightarrow 0} \lim_{m \rightarrow +\infty} j \int_{\mathcal{W}} |f_m| dx = \lim_{j \rightarrow 0} j \int_{\mathcal{W}} |f| dx = 0.$$

Noticing that the sequence $b(x, v_m, D\mathcal{T}_k(v))$ is bounded in $(L^{p'}(\mathcal{W}))^N$ and recalling the lemma 3.1 we can pass to the limit with $m \rightarrow +\infty$ to obtain that

$$\lim_{m \rightarrow +\infty} \int_{\mathcal{W}} b(x, v_m, D\mathcal{T}_k(v)) D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx = \int_{\mathcal{W}} b(x, v, D\mathcal{T}_k(v)) D\mathcal{T}_j(\mathcal{G}_k(v)) dx,$$

hence

$$\lim_{j \rightarrow 0} \lim_{m \rightarrow +\infty} \int_{\mathcal{W}} b(x, v_m, D\mathcal{T}_k(v)) D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx = 0.$$

Moreover, by means of the lemma 3.2, we have

$$\lim_{j \rightarrow 0} \lim_{m \rightarrow +\infty} j\gamma \int_{\mathcal{W}} \frac{|v_m|^{p-1}}{|x|^p} dx = 0.$$

By the growth condition (1.6) we have

$$|\mathcal{B}_m(x, \mathcal{T}_{j+k}(v_m))| \leq (j+k)^\lambda c_0(x),$$

therefore, by Lebesgue's convergence theorem, we obtain

$$\mathcal{B}_m(x, \mathcal{T}_{j+k}(v_m)) \rightarrow \mathcal{B}(x, \mathcal{T}_{j+k}(v)) \quad \text{strongly in } (L^{p'}(\mathcal{W}))^N,$$

for $m > j+k$, we have

$$\lim_{m \rightarrow +\infty} \int_{\mathcal{W}} \mathcal{B}_m(x, \mathcal{T}_{j+k}(v_m)) D\mathcal{T}_j(\mathcal{T}_{j+k}(v_m) - \mathcal{T}_k(v)) dx = \int_{\mathcal{W}} \mathcal{B}(x, \mathcal{T}_{j+k}(v)) D\mathcal{T}_j(\mathcal{T}_{j+k}(v) - \mathcal{T}_k(v)) dx,$$

since, for any $j \leq 1$, we have

$$D\mathcal{T}_j(\mathcal{T}_{j+k}(v) - \mathcal{T}_k(v)) \rightarrow 0 \quad \text{a.e. as } j \rightarrow 0,$$

and

$$|D\mathcal{T}_j(\mathcal{T}_{j+k}(v) - \mathcal{T}_k(v))| \leq |D\mathcal{T}_1(\mathcal{T}_{k+1}(u) - \mathcal{T}_k(v))| \in L^p(\mathcal{W}).$$

then, by Lebesgue's convergence theorem it follows that

$$\lim_{j \rightarrow 0} \int_{\mathcal{W}} \mathcal{B}(x, \mathcal{T}_{j+k}(v)) D\mathcal{T}_j(\mathcal{T}_{j+k}(v) - \mathcal{T}_k(v)) dx = 0,$$

which implies that

$$\lim_{j \rightarrow 0} \lim_{m \rightarrow +\infty} \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx = 0.$$

In such a way, from (3.3), we deduce that

$$\lim_{j \rightarrow 0} \lim_{m \rightarrow +\infty} \int_{\mathcal{W}} (b(x, v_m, Dv_m) - b(x, v_m, D\mathcal{T}_k(v))) D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) dx = 0. \quad (3.4)$$

Let us consider the sets

$$\mathcal{C}_k = \{x \in \mathcal{W} : |v(x)| \leq k\}, \quad \mathcal{A}_k = \{x \in \mathcal{W} : |v(x)| > k\}.$$

Here and in the sequel, for any $i \in \mathbb{N}$ and $k > 0$ we define $\omega_i(k)$ to represent certain quantities such that

$$\lim_{k \rightarrow +\infty} \omega_i(k) = 0.$$

We define

$$\begin{aligned} I_m &= \int_{\mathcal{W}} \{[b(x, v_m, Dv_m) - b(x, v_m, Dv)] D(v_m - v)\}^\theta, \\ &= \int_{\mathcal{A}_k} \{[b(x, v_m, Dv_m) - b(x, v_m, Dv)] D(v_m - v)\}^\theta \\ &\quad + \int_{\mathcal{C}_k} \{[b(x, v_m, Dv_m) - b(x, v_m, Dv)] D(v_m - v)\}^\theta, \end{aligned}$$

and by Hlder inequality (with exponents $\frac{q}{p\theta}$ and $\frac{q}{q-p\theta}$), (1.7) and the boundedness of the sequence v_m in $W_0^{1,q}(\mathcal{W})$ we get

$$\begin{aligned} I_m &\leq \int_{\mathcal{W}} \{[b(x, v_m, Dv_m) - b(x, v_m, D\mathcal{T}_k(v))] D(v_m - \mathcal{T}_k(v))\}^\theta \\ &\quad + C_{20} \left(\int_{\mathcal{W}} (|Dv_m|^q + |v_m|^q + |Dv|^q + |v|^q) dx \right)^{\frac{p\theta}{q}} \text{meas}\{\mathcal{A}_k\}^{\frac{q-p\theta}{q}}, \\ &\leq \int_{\mathcal{W}} \{[b(x, v_m, Dv_m) - b(x, v_m, D\mathcal{T}_k(v))] D(v_m - \mathcal{T}_k(v))\}^\theta + C_{18}\omega_1(k). \end{aligned}$$

Now, we have to study the behavior of the first integral in the previous inequality, it can be rewritten as

$$\begin{aligned} &\int_{\mathcal{W}} \{[b(x, v_m, Dv_m) - b(x, v_m, D\mathcal{T}_k(v))] D(v_m - \mathcal{T}_k(v))\}^\theta \\ &= \int_{\mathcal{W}} \{[b(x, v_m, Dv_m) - b(x, v_m, D\mathcal{T}_k(v))] D\mathcal{T}_j(v_m - \mathcal{T}_k(v))\}^\theta \\ &\quad + \int_{\{|v_m - \mathcal{T}_k(v)| > j\}} \{[b(x, v_m, Dv_m) - b(x, v_m, D\mathcal{T}_k(v))] D(v_m - \mathcal{T}_k(v))\}^\theta, \end{aligned}$$

by means of Hlder inequality with exponents $(\frac{1}{\theta}, \frac{1}{1-\theta})$ and $(\frac{q}{p\theta}, \frac{q}{q-p\theta})$, we get

$$\begin{aligned} &\int_{\mathcal{W}} \{[b(x, v_m, Dv_m) - b(x, v_m, D\mathcal{T}_k(v))] D(v_m - \mathcal{T}_k(v))\}^\theta \\ &\leq C_{21} \left(\int_{\mathcal{W}} [b(x, v_m, Dv_m) - b(x, v_m, D\mathcal{T}_k(v))] D\mathcal{T}_j(v_m - \mathcal{T}_k(v)) \right)^\theta \\ &\quad + C_{22} \text{meas}\{|v_m - \mathcal{T}_k(v)| > j\}^{\frac{q-p\theta}{q}}. \end{aligned}$$

Since

$$\limsup_{m \rightarrow +\infty} \text{meas}\{|v_m - \mathcal{T}_k(v)| > j\}^{\frac{q-p\theta}{q}} \leq \text{meas}\{|v - \mathcal{T}_k(v)| > j\}^{\frac{q-p\theta}{q}} = \omega_2(k).$$

Then, through (3.4), we deduce that

$$\lim_{m \rightarrow +\infty} I_m = 0.$$

Therefore

$$Dv_m(x) \rightarrow Dv(x) \quad \text{a.e. in } \mathcal{W}. \quad (3.5)$$

3.1. Study $W_0^{1,t^*(p-1)}(\mathcal{W})$ — solution. We will dedicate all this subsection to study existence and summability result for the problem (1.1) when f belongs to $L^t(\mathcal{W})$ with $\max\left\{1, \frac{N}{N(p-1)+1}\right\} < t < (p^*)'$.

Theorem 3.1. *Let us assume (1.2) – (1.6) and $\gamma < \mathcal{F}(t)$. Then, there exists a distributional solution $u \in W_0^{1,t^*(p-1)}(\mathcal{W})$ of the problem (1.1) such that:*

$$\begin{cases} \mathcal{B}(x, v) \in (L^1(\mathcal{W}))^N, \frac{|v|^{p-2}v}{|x|^p} \in L^1(\mathcal{W}), \\ \int_{\mathcal{W}} b(x, v, Dv) D\varphi dx + \int_{\mathcal{W}} \mathcal{B}(x, v) D\varphi dx = \gamma \int_{\mathcal{W}} \frac{|v|^{p-2}v}{|x|^p} \varphi dx + \int_{\mathcal{W}} f \varphi dx \end{cases}$$

for every $\varphi \in C_c^\infty(\mathcal{W})$.

Remark 3.1. An existence result of the distributional solution $v_\Lambda \in W_0^{1,t^*(p-1)}(\mathcal{W})$, when $c_0 \equiv 0$, was obtained in the papers [2, 9], respectively in the cases $p = 2$ and $p > 1$.

We can draw the following picture for distributional solution.

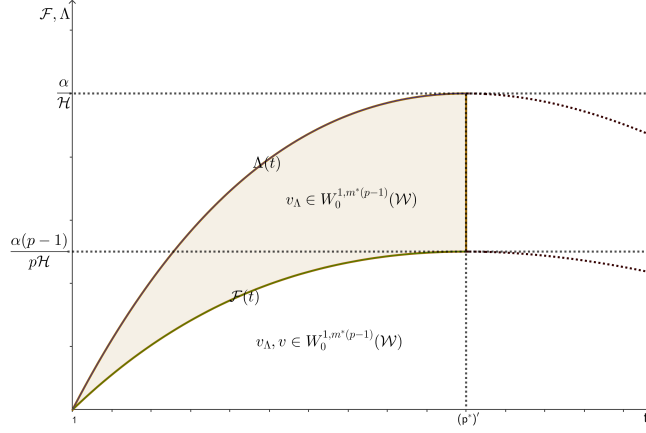


FIGURE 2. Regularity results: $p \geq 2 - \frac{1}{N}$

Once again the colored part in the graph represent the region in which we lost the usual regularity because of the presence of the convection term.

Proof. We establish the existence of distributional solution, in two steps, for (1.1).

Step 1 : v_m is bounded in $W_0^{1,t^*(p-1)}(\mathcal{W})$.

For $\delta > 0$ fixed, $l \in \mathbb{R}$, and $s = \frac{Nt(p-1) + p(t-N)}{N - tp}$, we define the function

$$\psi_\delta(l) = [(\delta + |l|)^{s+1} - \delta^{s+1}] \operatorname{sgn}(l).$$

Using $\psi_\delta(v_m)$ as test function in (2.6) leads to

$$\begin{aligned} \int_{\mathcal{W}} b(x, v_m, Dv_m) D\psi_\delta(v_m) dx + \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\psi_\delta(v_m) dx &= \int_{\mathcal{W}} f_m \psi_\delta(v_m) dx \\ &+ \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} \psi_\delta(v_m) dx. \end{aligned}$$

By dint of (1.2), (1.5) and for $s_1 = \frac{s}{p} + 1$, we get

$$\begin{aligned} \int_{\mathcal{W}} |D[(\delta + |v_m|)^{s_1} - \delta^{s_1}]|^p dx &\leq C_{23} \int_{\mathcal{W}} c_0(x) |v_m|^\lambda (\delta + |v_m|)^s |Dv_m| dx \\ &+ \gamma \mathcal{H} \int_{\mathcal{W}} |D[(\delta + |v_m|)^{s_1} - \delta^{s_1}]|^p dx \\ &+ \gamma \int_{\mathcal{W}} \frac{1}{|x|^p} \left[|v_m|^{p-1} (\delta + |v_m|)^{s+1} - \delta^{s+1} \right] \\ &- \gamma \int_{\mathcal{W}} \frac{1}{|x|^p} \left[((\delta + |v_m|)^{s_1} - \delta^{s_1})^p \right] dx \end{aligned}$$

$$\begin{aligned}
& + \|f_m\|_{L^t(\mathcal{W})} \left(\int_{\mathcal{W}} |(\delta + |v_m|)^{s+1} - \delta^{s+1}|^{t'} dx \right)^{\frac{1}{t'}}, \\
& \leq C_{24} \left(\int_{\mathcal{W}} |(\delta + |v_m|)^{s_1}|^{p^*} dx \right)^{\frac{s_1 p}{s_1 p^*}} + C_{25} \int_{\mathcal{W}} |D[(\delta + |v_m|)^{s_1} - \delta^{s_1}]|^p dx \\
& + \gamma \mathcal{H} \int_{\mathcal{W}} |D[(\delta + |v_m|)^{s_1} - \delta^{s_1}]|^p dx - \gamma \int_{\mathcal{W}} \frac{1}{|x|^p} \left[((\delta + |v_m|)^{s_1} - \delta^{s_1})^p \right] dx \\
& + \gamma \int_{\mathcal{W}} \frac{1}{|x|^p} \left[|v_m|^{p-1} |(\delta + |v_m|)^{s+1} - \delta^{s+1}| \right] dx \\
& + \|f_m\|_{L^t(\mathcal{W})} \left(\int_{\mathcal{W}} |(\delta + |v_m|)^{s+1} - \delta^{s+1}|^{t'} dx \right)^{\frac{1}{t'}}.
\end{aligned}$$

Hence

$$(C_{26} + \mathcal{W}(\delta)) \left(\int_{\mathcal{W}} |(\delta + |v_m|)^{s_1}|^{p^*} dx \right)^{\frac{p}{p^*}} \leq C_{28} + C_{27} \left(\int_{\mathcal{W}} |(\delta + |v_m|)^{s+1} - \delta^{s+1}|^{t'} dx \right)^{\frac{1}{t'}},$$

where $\mathcal{W}(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Now, for any $m > 0$, we let δ goes to 0 and thanks to (3.1), we get

$$\left(\int_{\mathcal{W}} |v_m|^{s_1 p^*} dx \right)^{\frac{p}{p^*}} \leq C_{29} + C_{30} \left(\int_{\mathcal{W}} |v_m|^{(s+1)t'} dx \right)^{\frac{1}{t'}}, \quad (3.6)$$

According to the definition of s , we have

$$s_1 p^* = (s+1)t' = \frac{(tp)^*}{p'}.$$

Given that

$$\frac{p}{p^*} > \frac{1}{t'},$$

we conclude, with reference to (3.6), that

$$\|v_m\|_{L^{\frac{(tp)^*}{p'}}(\mathcal{W})} \leq C_{31}. \quad (3.7)$$

Following the argument contained in [4] we will drive that v_m is bounded in $W_0^{1,t^*(p-1)}(\mathcal{W})$. Indeed, fix $\delta = R$ and use (3.7), we get

$$\int_{\mathcal{W}} |Dv_m|^p (R + |v_m|)^s dx \leq C_{32}.$$

Applying Hlder inequality with exponents $\frac{p'}{m^*}$ and $\frac{p'}{p' - m^*}$, we obtain

$$\begin{aligned}
\int_{\mathcal{W}} |Dv_m|^{t^*(p-1)} dx &= \int_{\mathcal{W}} \frac{|Dv_m|^{t^*(p-1)}}{(R + |v_m|)^{\frac{sm^*(1-p)}{p}}} (R + |v_m|)^{\frac{sm^*(1-p)}{p}} dx, \\
&\leq C_{33} \left(\int_{\mathcal{W}} (R + |v_m|)^{\frac{sm^*}{m^* - p'}} dx \right)^{\frac{p' - m^*}{p'}},
\end{aligned}$$

and thus that

$$\int_{\mathcal{W}} |Dv_m|^{t^*(p-1)} dx \leq C_{34} \left(\int_{\mathcal{W}} (R + |v_m|)^{\frac{(tp)^*}{p'}} dx \right)^{\frac{p'-m^*}{p'}},$$

which implies the boundedness of the sequence $\{v_m\}$ in $W_0^{1,t^*(p-1)}(\mathcal{W})$. Therefore, as $t^*(p-1) > 1$, there exists u such that, up to a subsequence,

$$v_m \rightharpoonup v \text{ weakly in } W_0^{1,t^*(p-1)}(\mathcal{W}), \quad (3.8)$$

$$v_m \rightarrow v \text{ strongly in } L^{t^*(p-1)}(\mathcal{W}) \text{ and a.e. in } \mathcal{W}, \quad (3.9)$$

Step 2: Passing to the limit.

To pass to the limit in (2.6), it's crucial to note that weak convergence of v_m is not sufficient because of the non-linearity present in the principal part of the operator. However, by Lemma 3.3, we guarantee that the sequence Dv_m converges almost everywhere in $\mathcal{W}$. Consequently, with the help of (1.3), we can deduce that

$$b(x, v_m, Dv_m) \rightarrow b(x, v, Dv) \text{ strongly in } (L^r(\mathcal{W}))^N, \quad 1 \leq r < t^*. \quad (3.10)$$

Consider a measurable subset $\mathcal{E}$ of $\mathcal{W}$ such that $0 \in \mathcal{E}$ and $\text{meas}(\mathcal{E})$ is small enough. According to the growth assumption (1.5) and Hlder inequality we have

$$\int_{\mathcal{E}} |\mathcal{B}_m(x, v_m)| dx \leq \int_{\mathcal{W}} c_0(x) |v_m|^\lambda dx \leq \left(\int_{\mathcal{E}} |c_0(x)|^{\zeta'} dx \right)^{\frac{1}{\zeta'}} \left(\int_{\mathcal{E}} |v_m|^{\lambda \zeta} dx \right)^{\frac{1}{\zeta}},$$

$$\text{with } \zeta = \frac{Nt(p-1)}{\lambda(N-tp)} \text{ and } \zeta' = \frac{Nt(p-1)}{Nt(p-1) - \lambda(N-tp)}.$$

Remark that

$$\frac{Nt(p-1)}{\lambda(N-tp)} > 1 \Leftrightarrow t > \frac{N}{N+p},$$

and

$$\lambda < p-1 \Leftrightarrow \frac{Nt(p-1)}{Nt(p-1) - \lambda(N-tp)} < \frac{N}{p-1}.$$

If this is the case, by (3.7) and Hlder inequality, we deduce that

$$\begin{aligned} \int_{\mathcal{E}} |\mathcal{B}_m(x, v_m)| dx &\leq (\text{meas}(\mathcal{E}))^{\frac{1}{\zeta'} - \frac{p-1}{N}} \|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})} \left(\int_{\mathcal{W}} |v_m|^{\frac{(tp)^*}{p'}} dx \right)^{\frac{1}{\zeta}}, \\ &\leq C_{34} (\text{meas}(\mathcal{E}))^{\frac{1}{\zeta'} - \frac{p-1}{N}} \|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})}. \end{aligned}$$

Thus, the sequence $\mathcal{B}_m(x, v_m)$ is equi-integrable. Therefore, by (3.9) and Vitali's Theorem we have

$$\mathcal{B}_m(x, v_m) \rightarrow \mathcal{B}(x, v) \text{ strongly in } (L^1(\mathcal{W}))^N. \quad (3.11)$$

By means of Hlder inequality and (3.7), we deduce that the sequence $\frac{|v_m|^{p-1}}{|x|^p}$ is equi-integrable, which implies with (3.9), that we can apply Vitali's theorem to drive that

$$\frac{|v_m|^{p-1}}{|x|^p} \rightarrow \frac{|v|^{p-1}}{|x|^p} \text{ strongly in } L^1(\mathcal{W}). \quad (3.12)$$

As a result of the convergences (3.10), (3.11) and (3.12) we pass to the limit in the weak formulation of the approximate problem (2.6) to deduce that the limit u satisfies

$$\int_{\mathcal{W}} b(x, v, Dv) D\varphi dx + \int_{\mathcal{W}} \mathcal{B}(x, v) D\varphi dx = \gamma \int_{\mathcal{W}} \frac{|v|^{p-2}v}{|x|^p} \varphi dx + \int_{\mathcal{W}} f \varphi dx$$

for every $\varphi \in C_c^\infty(\mathcal{W})$.

3.2. Study $W_0^{1,1}(\mathcal{W})$ – solution. In this subsection we deal with the borderline case i.e.:

$$1 < p < 2 - \frac{1}{N} \text{ and } t^*(p-1) = 1.$$

Theorem 3.2. *Let $f \in L^{\bar{t}}(\mathcal{W})$, $\bar{t} = \frac{N}{N(p-1)+1}$ and $\gamma < \mathcal{F}(\bar{t})$. Then, there exists a distributional solution $u \in W_0^{1,1}(\mathcal{W})$ of the problem (1.1) such that*

$$\left\{ \begin{array}{l} \mathcal{B}(x, v) \in (L^1(\mathcal{W}))^N, \frac{|v|^{p-2}v}{|x|^p} \in L^1(\mathcal{W}), \\ \int_{\mathcal{W}} b(x, v, Dv) D\varphi dx + \int_{\mathcal{W}} \mathcal{B}(x, v) D\varphi dx = \gamma \int_{\mathcal{W}} \frac{|v|^{p-2}v}{|x|^p} \varphi dx + \int_{\mathcal{W}} f \varphi dx \end{array} \right.$$

for every $\varphi \in C_c^\infty(\mathcal{W})$.

Summarizing the classical results proved in [2, 6, 9] and result obtained in the above theorem, we have the following picture

$$\leq \left[\int_{\mathcal{W}} \frac{|Dv_m|^p}{|v_m|^{\frac{N(p-1)}{N-1}}} dx \right]^{\frac{1}{p}} \left[\int_{\mathcal{W}} |v_m|^{\frac{N}{N-1}} dx \right]^{\frac{1}{p'}}.$$

As a consequence of (3.13), we obtain

$$\|v_m\|_{W_0^{1,1}(\mathcal{W})} \leq C_{37}, \quad (3.14)$$

in kind, there exists a subsequence v_m such that

$$v_m \rightarrow v \quad \text{a.e. in } \mathcal{W}, \quad (3.15)$$

$$v_m \rightarrow v \quad \text{strongly in } L^r(\mathcal{W}), \quad 1 \leq r < \frac{N}{N-1}. \quad (3.16)$$

Step 2: The sequence Dv_m converge strongly in $L^1(\mathcal{W})$.

The use of

$$\psi_k(v_m) = \left[|v_m|^{\frac{Np_0}{N-1}} - k^{\frac{Np_0}{N-1}} \right]^+ \text{sgn}(v_m)$$

as a test function in (2.6) with $p_0 = 2 - p - \frac{1}{N}$, gives

$$\begin{aligned} \int_{\mathcal{W}} b(x, v_m, Dv_m) D\psi_k(v_m) dx + \int_{\mathcal{W}} \mathcal{B}_m(x, v_m) D\psi_k(v_m) dx &= \gamma \int_{\mathcal{W}} \frac{|\mathcal{T}_m(v_m)|^{p-2} \mathcal{T}_m(v_m)}{|x|^p + \frac{1}{m}} \psi_k(v_m) dx \\ &+ \int_{\mathcal{W}} f_m \psi_k(v_m) dx. \end{aligned}$$

By means of the hypothesis (1.2) – (1.6), Hlder's and Young's inequalities, we obtain

$$\begin{aligned} \alpha c_1 \int_{\mathcal{Q}_{m,k}} \left| D|v_m|^{\frac{N}{p^*(N-1)}} \right|^p dx &\leq c_2 \left(\int_{\mathcal{Q}_{m,k}} |c_0(x)|^{\frac{N}{p-1}} dx \right)^{\frac{p}{N}} \left(\int_{\mathcal{Q}_{m,k}} |v_m|^{\lambda p' + \frac{N(1-p)}{N-1}} dx \right)^{\frac{N-p}{p}} \\ &+ c_3 \int_{\mathcal{Q}_{m,k}} \left| D|v_m|^{\frac{N}{p^*(N-1)}} \right|^p dx + \gamma \int_{\mathcal{Q}_{m,k}} \frac{\left| |v_m|^{\frac{N}{p^*(N-1)}} \right|^p}{|x|^p} dx \\ &+ \|f_m\|_{L^t(\mathcal{W})} \left(\int_{\mathcal{Q}_{m,k}} |v_m|^{\frac{Np_0 t'}{N-1}} dx \right)^{\frac{1}{t'}} \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{\alpha N p^p (N-1)^{p-1} p_0}{(N-p)^p}, \\ c_2 &= \frac{N p_0 (p-1)}{p(N-1) \alpha^{\frac{p'}{p}}}, \\ c_3 &= \frac{\alpha c_1}{p}. \end{aligned}$$

Being $\lambda p' + \frac{N(1-p)}{N-1} < \frac{N}{N-1}$ and thanks to Sobolev and Hardy inequalities, we get that

$$\left[\frac{\alpha c_1}{p'} - \gamma \mathcal{H} \right] \left(\int_{\mathcal{Q}_{m,k}} |v_m|^{\frac{N}{N-1}} dx \right)^{\frac{p}{p^*}} \leq C_{37} + C_{38} \left(\int_{\mathcal{Q}_{m,k}} |v_m|^{\frac{N}{N-1}} dx \right)^{\frac{1}{m'}} \quad (3.17)$$

Let us observe that

$$\gamma < \mathcal{F}(\bar{t}) \Leftrightarrow \frac{\alpha c_1}{p'} - \gamma \mathcal{H} > 0.$$

If this is the case, by (3.14) we get from (3.17) that

$$\left(\int_{\mathcal{Q}_{m,k}} |v_m|^{\frac{N}{N-1}} dx \right)^{\frac{(p-1)(N-1)}{N}} \leq C(p, N, c_0, f_m, \text{meas}(\mathcal{Q}_{m,k})). \quad (3.18)$$

In a manner similar to Step 1 of the demonstration of theorem 3.1, we obtain

$$\begin{aligned} \int_{\mathcal{Q}_{m,k}} |Dv_m| dx &= \int_{\mathcal{Q}_{m,k}} \frac{|Dv_m|}{|v_m|^{\frac{N}{p'(N-1)}}} |v_m|^{\frac{N}{p'(N-1)}} dx, \\ &\leq \left[\int_{\mathcal{Q}_{m,k}} \frac{|Dv_m|^p}{|v_m|^{\frac{N(p-1)}{N-1}}} dx \right]^{\frac{1}{p}} \left[\int_{\mathcal{Q}_{m,k}} |v_m|^{\frac{N}{N-1}} dx \right]^{\frac{1}{p'}}, \\ &\leq C(p, N, c_0, f_m, \mathcal{Q}_{m,k}). \end{aligned}$$

On account of the lemma 3.1 and to the previous inequality, for every measurable subset $\mathcal{E} \subset \mathcal{W}$ with negligible measure, we have

$$\begin{aligned} \int_{\mathcal{E}} |Dv_m| dx &\leq \int_{\mathcal{E}} |D\mathcal{T}_k(v_m)| dx + \int_{\mathcal{Q}_{m,k}} |Dv_m| dx \\ &\leq \left[\int_{\mathcal{W}} |D\mathcal{T}_k(v_m)|^p dx \right]^{\frac{1}{p}} \text{meas}(\mathcal{E})^{1-\frac{1}{p}} + C(p, N, c_0, f_m, \mathcal{Q}_{m,k}). \end{aligned}$$

As follows, the sequence $\{Dv_m\}$ is equi-integrable. According to Vitali's Theorem we obtain that

$$Dv_m \rightarrow Dv \quad \text{in } (L^1(\mathcal{W}))^N.$$

Step 3: Passing to the limit.

Whereas the lemma 3.2 and (3.15) we readily have

$$\begin{aligned} b(x, v_m(x), Dv_m(x)) &\rightarrow b(x, v(x), Dv(x)) \quad \text{a.e. in } \mathcal{W}, \\ \mathcal{B}_m(x, v_m) &\rightarrow \mathcal{B}(x, v) \quad \text{a.e. in } \mathcal{W}, \\ \frac{|v_m|^{p-2}v_m}{|x|^p + \frac{1}{m}} &\rightarrow \frac{|v|^{p-2}v}{|x|^p} \quad \text{a.e. in } \mathcal{W} \end{aligned}$$

nonetheless, for any measurable subset $\mathcal{E} \subset \mathcal{W}$ where $\text{meas}(\mathcal{E})$ is small enough, we have

$$\begin{aligned} \int_{\mathcal{E}} |b(x, v_m, Dv_m)| dx &\leq \int_{\mathcal{E}} \beta \left(|v_m|^{p-1} + |Dv_m|^{p-1} \right) dx, \\ &\leq \beta \int_{\mathcal{E}} |v_m|^{p-1} dx + \beta \int_{\mathcal{E}} |Dv_m|^{p-1} dx, \\ &\leq \beta \left[\left(\int_{\mathcal{W}} |v_m| dx \right)^{p-1} + \left(\int_{\mathcal{W}} |Dv_m| dx \right)^{p-1} \right] \text{meas}(\mathcal{E})^{2-p}, \\ &\leq C_{39} \text{meas}(\mathcal{E})^{2-p}, \end{aligned}$$

the Vitali Theorem yields

$$b(x, v_m, Dv_m) \rightarrow b(x, v, Dv) \quad \text{strongly in } (L^1(\mathcal{W}))^N. \quad (3.19)$$

In light of (1.5), Hlder inequality and since $\frac{1}{2-p} \leq \frac{N}{p-1}$, we have

$$\begin{aligned} \int_{\mathcal{E}} |\mathcal{B}_m(x, v_m)| \, dx &\leq \int_{\mathcal{E}} c_0(x) |v_m|^\lambda \, dx, \\ &\leq \left(\int_{\mathcal{E}} |c_0(x)|^{\frac{1}{2-p}} \, dx \right)^{2-p} \left(\int_{\mathcal{E}} |v_m|^{\frac{\lambda}{p-1}} \, dx \right)^{p-1}, \\ &\leq \left(\int_{\mathcal{W}} |c_0(x)|^{\frac{N}{p-1}} \, dx \right)^{\frac{p-1}{N}} \left(\int_{\mathcal{W}} |v_m| \, dx \right)^\lambda (\text{meas}(\mathcal{E}))^{p-1-\lambda}. \end{aligned}$$

Thusly, the sequence $\{\mathcal{B}_m(x, v_m)\}$ is equi-integrable. By Vitali's Theorem we obtain

$$\mathcal{B}_m(x, v_m) \rightarrow \mathcal{B}(x, v) \quad \text{strongly in } (L^1(\mathcal{W}))^N. \quad (3.20)$$

Moreover, the sequence $\left\{ \frac{|v_m|^{p-1}}{|x|^p} \right\}$ is equi-integrable. Indeed, being $p < 2 - \frac{1}{N}$, using (3.14) and Hlder inequality, we get that

$$\begin{aligned} \int_{\mathcal{E}} \frac{|v_m|^{p-1}}{|x|^p} \, dx &\leq \left(\int_{\mathcal{E}} |v_m|^{\frac{N}{N-1}} \, dx \right)^{\frac{(p-1)(N-1)}{N}} \left(\int_{\mathcal{E}} \frac{dx}{|x|^{\frac{pN}{2N-pN+p-1}}} \right)^{\frac{2N-pN+p-1}{N}}, \\ &\leq C \left(\int_{\mathcal{E}} \frac{dx}{|x|^{\frac{pN}{2N-pN+p-1}}} \right)^{\frac{2N-pN+p-1}{N}}, \end{aligned}$$

hence, we can apply the absolute continuity of the Lebesgue integral, obtaining that $\forall \varepsilon > 0$

$$\int_{\mathcal{E}} \frac{|v_m|^{p-1}}{|x|^p} \, dx \leq \varepsilon,$$

along these lines, by Vitali's theorem it follows that

$$\frac{|v_m|^{p-1}}{|x|^p} \rightarrow \frac{|v|^{p-1}}{|x|^p} \quad \text{strongly in } L^1(\mathcal{W}). \quad (3.21)$$

At least, being v_m satisfy (3.19), (3.20) and (3.21), it is possible to pass to the limit in the weak formulation (2.6) and we prove that $v \in W_0^{1,1}(\mathcal{W})$ is a distributional solution of (1.1).

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