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Basins of attraction of a one-parameter family of root-finding techniques

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ABSTRACT. Initial conditions can have a substantial impact on the behavior of iterative root-finding techniques for nonlinear equations. By allowing complex starting points and complex roots, it is possible to examine the basins of attraction in the complex plane in order to compare the performance of various iterative techniques. In this paper, a one-parameter family of third-order root-finding methods is studied by varying its parameter A within -2.0 and 2.4 and applying it to a polynomial equation of high degree (degree 25). This family includes the Euler–Chebyshev’s ($A = 0$), Halley’s ($A = 1$) and BSC ($A = 2$) techniques. According to the results, the one-parameter family provides the best performance for values near $A = 1$, which equals to the Halley’s method.

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1. Introduction

Creating mathematical representations of real-world phenomena in order to study them and find answers to questions about them is a typical scientific practice. Finding the solutions to these models often requires solving nonlinear equations. As a rule, numerical approaches are required to determine the approximate solutions if exact methods are lacking. There are a

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great variety of different iterative approaches documented in the literature [6], [1], [12], [10], [9], [8], [30], [26], [13], [15], [14].

Authors have taken a number of different ways in order to analyze and compare various techniques to find roots of nonlinear equations. Some measures involve selecting nonlinear equations, picking initial conditions for each equation, counting iterations until convergence (or until some tolerance is met), counting functions evaluated at each step, or measuring CPU time requirements [1], [10], [24], [25], [20].

Nonetheless, as pointed out by Scott et al. [27], even when a starting point is taken at random, it merely represents one option among an infinite number of possibilities. In order to work around these restrictions, basins of attraction are seen as a helpful tool that permit the analysis and graphical observation of the methods' behavior in the complex plane under a wide range of initial conditions [29], [28], [27], [23], [22], [13], [3], [4], [5], [9], [8], [19], [14]. Basins of attraction provide an alternative tool for contrasting the efficacy of various iterative approaches [27], [23], [22], [9], [8]. Basins of attraction for various polynomials with simple and multiple roots and with real and complex coefficients have been presented and discussed by many authors such [27], [23], and [22].

One way to visualize how a method behaves in relation to its inputs is to color-code a section in $\mathbb{C}$ according to the root to which it converges. However, for a certain nonlinear equation, the basins of attraction provide no information regarding the method's rate of convergence. Assigning a different color to the number of iterations needed to obtain a root with a required precision, is another procedure that allows for the visualization of the rate of convergence as a function of the different starting points [5], [15]. Additionally, it is possible to merge the two approaches and adjust the shade of each basin's color to match the appropriate number of iterations [31], or accommodate both representations separately [8].

By selecting a region belonging to $\mathbb{C}$ and coloring each point in accordance with the root to which the iterative technique converges, this paper studies and makes use of the basins of attraction for a one-parameter family of third-order root-finding methods, from which BSC, [10], Halley's [11] and Euler-Chebyshev's [17] methods are well-described in the literature. Specifically, the newest among those, the BSC root-finding approach, was built on top of Adomian's decomposition method [2], Abbasbandy's suggestion to enhance the accuracy of the Newton-Raphson method [1], and Babolian and Biazar's idea to boost the accuracy of Adomian's decomposition technique [7].

The purpose of this paper is to investigate the behavior of that family, varying its parameter in order to accommodate the three aforementioned methods, and to discuss the parameter's optimal value for determining the roots of a high-degree (degree 25) polynomial equation.

The present paper is organized as follows:

First, the one-parameter family of third-order root-finding methods and the research polynomial equation are presented. A mathematical approach is then provided for the basins of attraction of an iterative method. In the following section the basins of attraction are illustrated. Thereafter, the basins of attraction are discussed in terms of complexity, basin boundaries, and convergence regions for each approach. The findings are finally reviewed in the conclusion.

2. Materials and Methods

2.1. The one-parameter family of third-order root-finding methods. Previously, three iterative techniques of third-order with the same structure were compared numerically relying on the so-called basins of attraction [9], [8]. The three iterative methods studied were: the BSC method, Euler–Chebyshev’s method and the Halley’s method.

Iteratively solving a nonlinear equation involves looking for fixed points that are zeros of the specified function. However, there are several iterative algorithms with fixed points that are not zeros of the function, known as extraneous fixed points. If these points are attractive, they will be mistakenly identified as roots of the equation to be solved. If they are repulsive, they should not be a problem, however the iterative approach may converge to a root that is distant from the initial point. Halley’s method does not have extraneous fixed points [21]. The extraneous fixed points for BSC and Euler–Chebyshev’s methods are repulsive [21], revealing that the methods do not converge to those values.

According to a number of studies, Halley’s approach is the best available iterative third-order root-finding method [23] [21] [9] [8].

The BSC approach can be expressed mathematically by equation (2.1) [10],

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3 - 2 \cdot f(x_n) f'(x_n) f''(x_n)} \quad (2.1)$$

The Euler–Chebyshev’s method can be expressed mathematically by equation (2.2) [17],

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3}, \quad (2.2)$$

and the Halley’s method can be expressed by equation (2.3) [11],

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{1}{2}f(x_n)f''(x_n)/f'(x_n)}. \quad (2.3)$$

All these three methods belong to the one-parameter family of root-finding techniques (2.4) studied in this research.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3 - A \cdot f(x_n) f'(x_n) f''(x_n)} \quad (2.4)$$

Examining equation (2.3), Halley’s approach may not seem to belong to the same family, but after conducting some easy computations [9], it emerges that, for $A = 1$ in (2.4), the corresponding mathematical equation is identical to equation (2.3), with Halley’s approach being expressed by the recursion (2.5),

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3 - f(x_n) f'(x_n) f''(x_n)} \quad (2.5)$$

Therefore, the family of iterative methods (2.4) includes the BSC method for $A = 2$, Halley’s method for $A = 1$, and the Euler–Chebyshev’s method for $A = 0$ as special cases.

The iterative methods of this family are all third-order techniques. To illustrate this, let's start with the definition of an iterative method's order of convergence.

Definition 2.1. Let x^* be a root of the equation $f(x) = 0$. Let $e_n = x^* - x_n$ be the truncation error in the n th iterate. The order of convergence is given by a number $p \geq 1$, if a constant $c \neq 0$ exists such that

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^p} = c \quad (2.6)$$

The subsequent theorem holds [9]:

Theorem 2.1. Assume that $f \in C^5$. Consider the equation $f(x) = 0$. Let x^* be a root of the equation $f(x) = 0$. Therefore, the order of convergence of the one-parameter family of root-finding methods (2.4), is at least of order 3.

Proof:

Consider the iteration function g as it is represented in equation (2.7).

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2 f''(x)}{2[f'(x)]^3 - A \cdot f(x) f'(x) f''(x)} \quad (2.7)$$

By doing a few simple computations, one achieves to equations (2.8) and (2.9):

$$g'(x^*) = g''(x^*) = 0 \quad (2.8)$$

$$g'''(x^*) = \frac{-2f'''(x^*)f'(x^*) - 3f''(x^*)^2(A-2)}{2f'(x^*)^2} \quad (2.9)$$

It follows from the assumption $f \in C^5$ that $g \in C^3$. Performing the Taylor expansion of $g(x_n)$ around x^* , equation (2.10) is achieved, for $\min(x_n, x^*) < \xi_n < \max(x_n, x^*)$.

$$x_{n+1} - x^* = g(x_n) - g(x^*) = \frac{g'''(\xi_n)}{6} (x_n - x^*)^3 \quad (2.10)$$

For $x_n \neq x^*$:

$$\frac{x_{n+1} - x^*}{(x_n - x^*)^3} = \frac{g'''(\xi_n)}{6} \quad (2.11)$$

Applying limits to equation (2.11) the proof is complete.

End of proof.

In this paper, computations were performed numerically by varying the family parameter from -2.0 to 2.4 , which includes the three particular aforementioned techniques.

The following polynomial (2.12) of high degree, studied in [8] for BSC, Euler-Chebyshev's and Halley's methods, was chosen to perform the present study.

$$f(x) = (2x - (3 + 2x)^{24} + 3)(x - 1) \quad (2.12)$$

This polynomial has three real roots, namely $z = -1.5$, $z = -1$ and $z = 1$.

2.2. Basins of attraction. Given a function f and an initial point $z_0 \in \mathbb{C}$, the iterative process T_f to find a zero α of function f , produces successive points, $z_1 = T_f(z_0), \dots, z_n = T_f(z_{n-1}) = T_f^2(z_{n-2}) = T_f^n(z_0)$, whether or not it will reach a zero α of f .

A point α is an attractive fixed point for T_f if $\lim_{n \rightarrow \infty} T_f^n(z_0) = \alpha$.

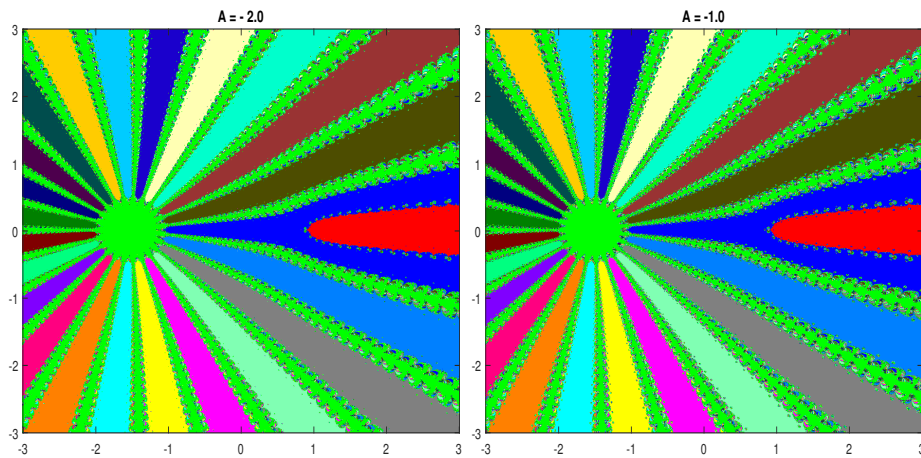
The set of all points belonging to $\mathbb{C}$ that converge asymptotically to α is known as the basin of attraction of the zero α , that is, $A(\alpha) = \{z \in \mathbb{C} : T_f^n(z) \rightarrow \alpha, n \rightarrow \infty\}$ [16]. Clearly, these iterative procedures are complex-plane discrete dynamical systems.

Geometrically, the basins of attraction's borders can take on a variety of forms, with the roots' basins of attraction potentially exhibiting a fractal-like structure due to their intricate intertwined interconnections. In general, basins of attraction consist of a primary set and a large number of minor subsets that are frequently infinitely dispersed [18].

This method of analysis is flexible enough to accommodate both real and complex variables starting points. Each starting point $z_0 \in \mathbb{C}$, based on the root to which z_0 converges, is highlighted with a distinct color. Changing the value of the parameter A from -2.0 to 2.4 causes Figures 1 to 6 to display different basins of attraction.

3. Results

Figures 1 through 6 illustrate the basins of attraction for various values of parameter A from the family of iterative methods (2.4).



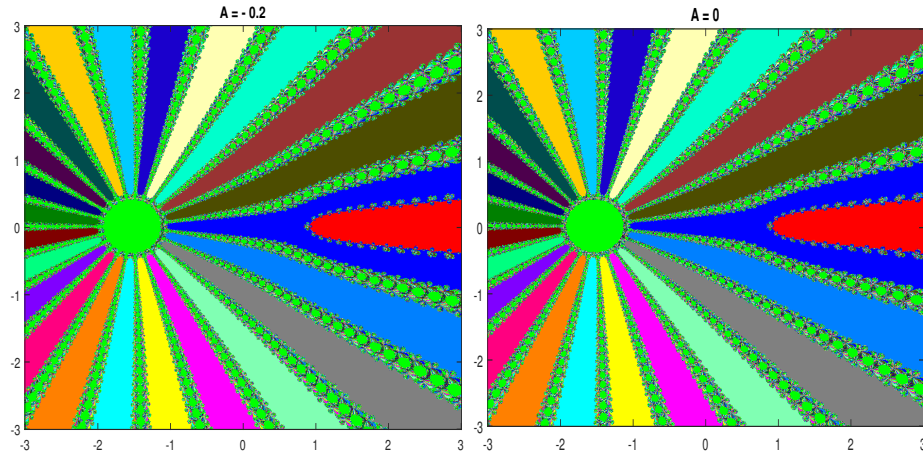


FIGURE 1. Varying the value of the parameter A from -2.0 to 0.0 , with the last value corresponding to the Euler-Chebyshev's technique.

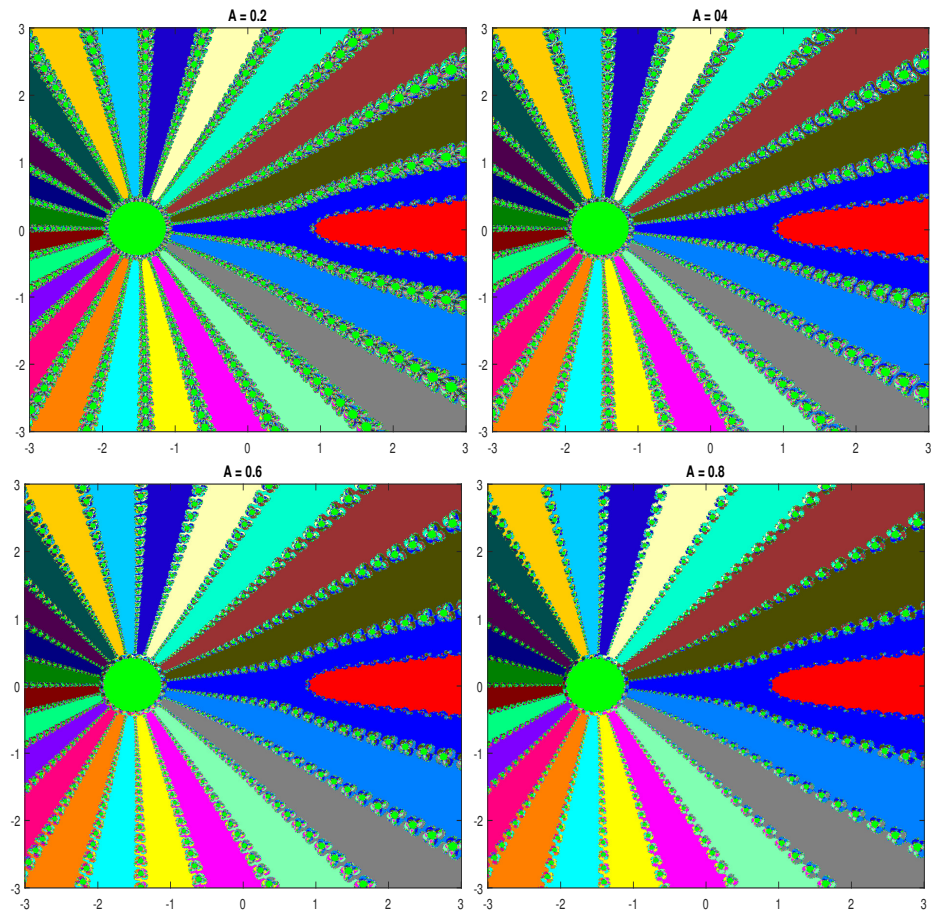


FIGURE 2. Varying the value of the parameter A from 0.2 to 0.8 , values that fall between those corresponding to the Euler-Chebyshev's and Halley's methods.

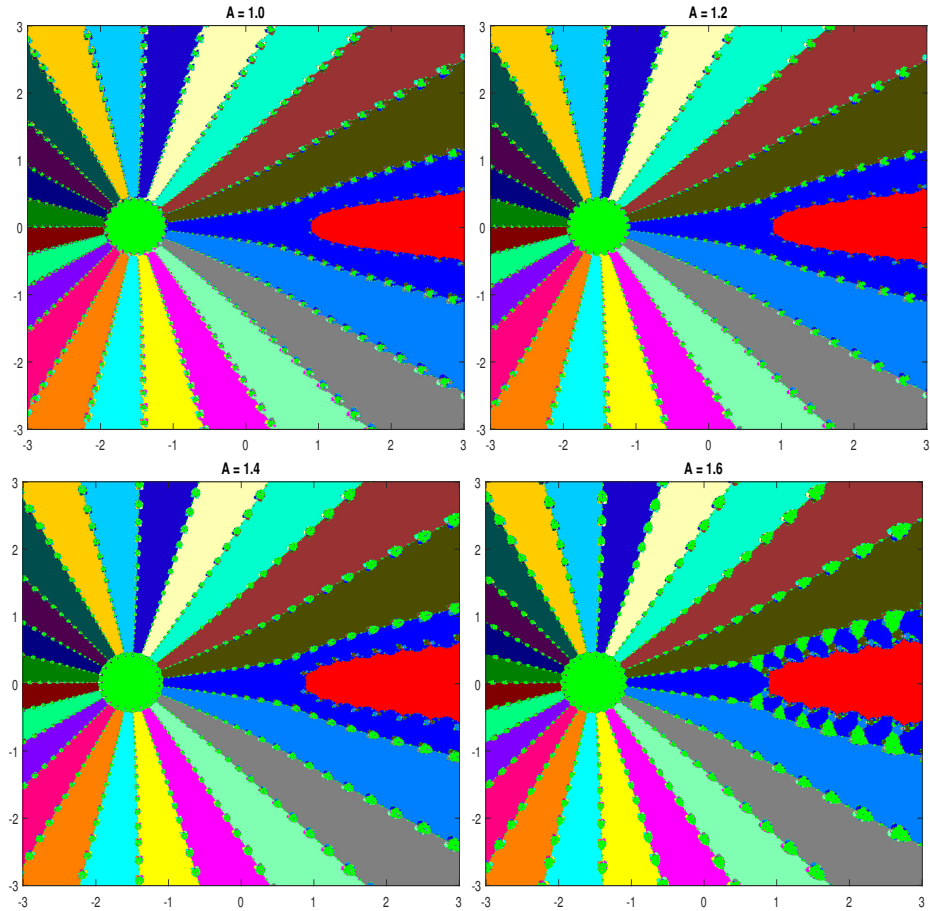
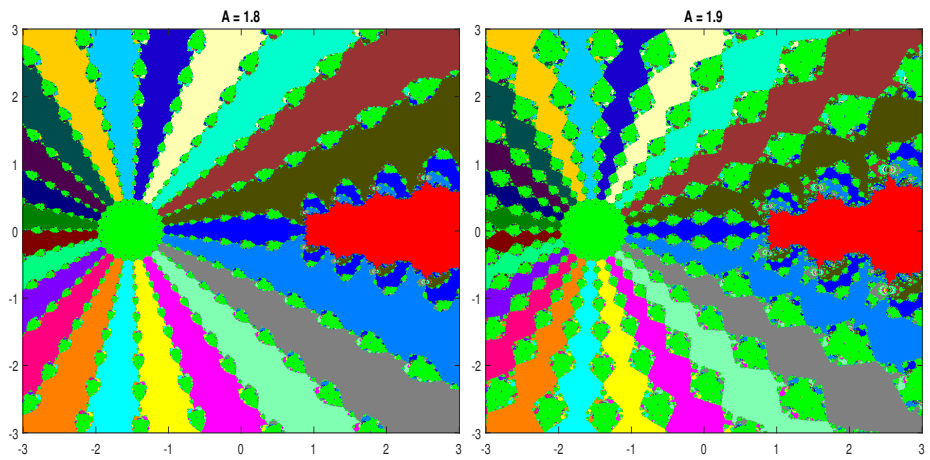


FIGURE 3. Varying the value of the parameter A from 1.0 to 1.6, that is, from the Halley's technique and approaching the value corresponding to the BSC method.



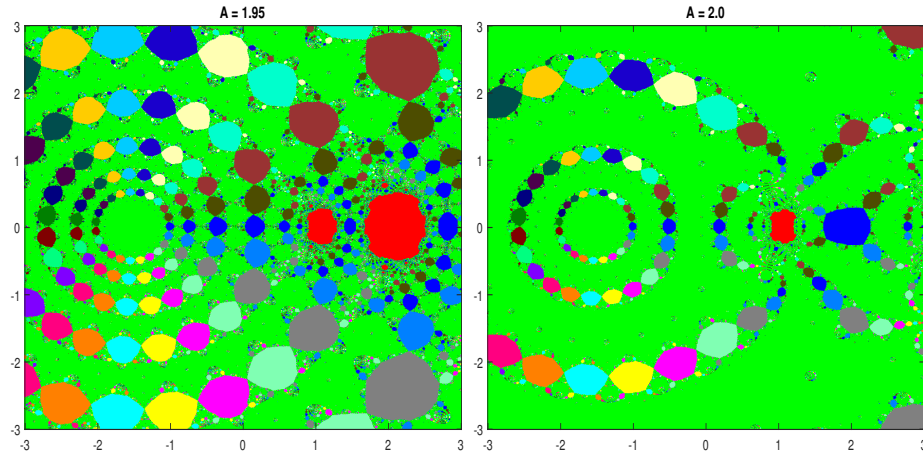


FIGURE 4. Varying the value of the parameter A from 1.8 to 2.0, with the last value corresponding to the BSC technique.

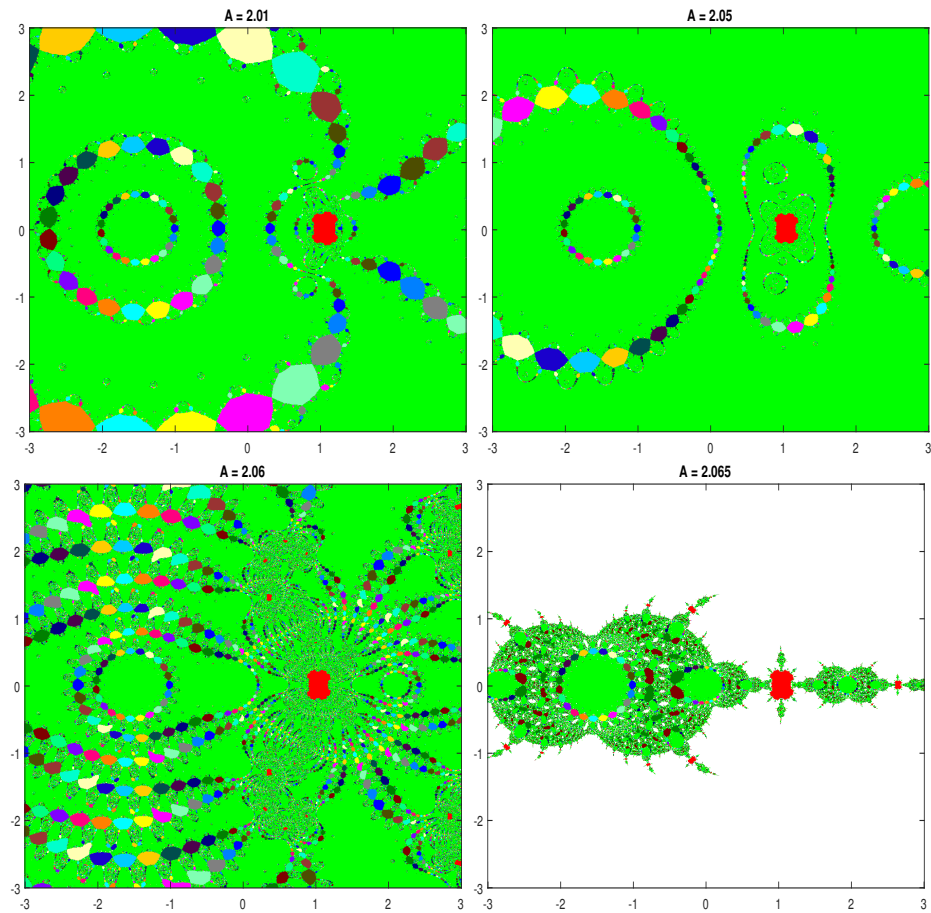


FIGURE 5. Tiny increases in parameter A variation from 2.01 to 2.065, above the value corresponding to the BSC technique.

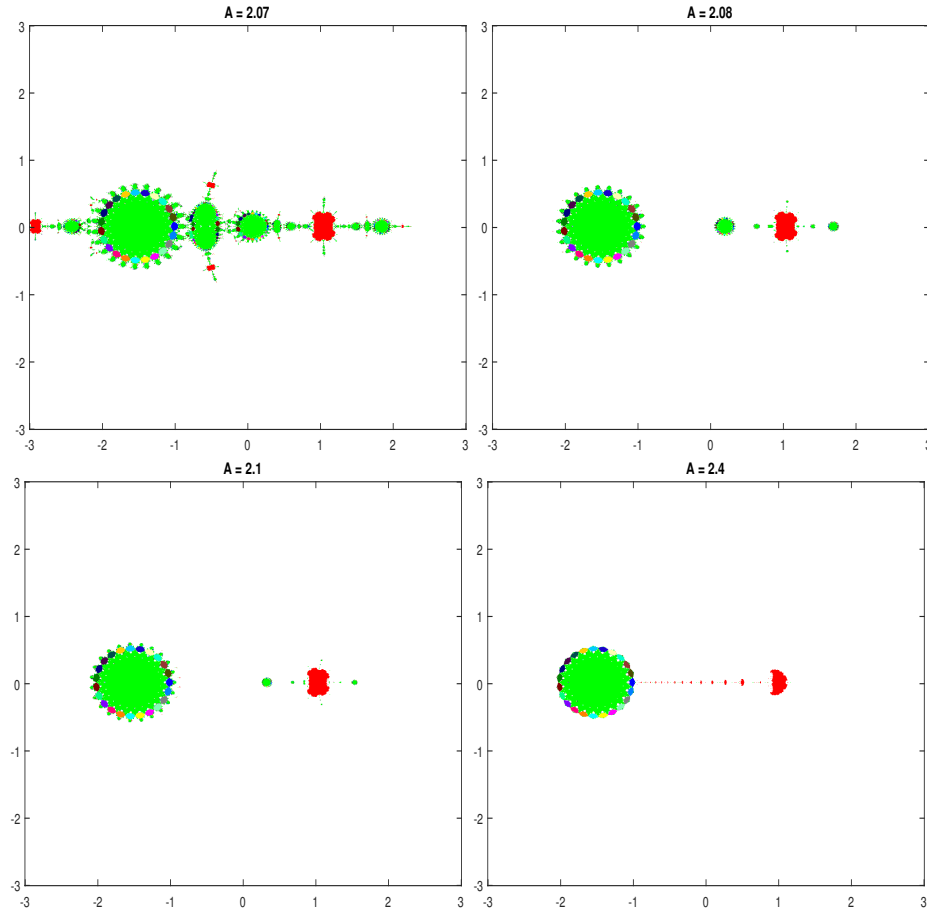


FIGURE 6. Varying the value of the parameter A from 2.07 to 2.2.

4. Discussion

By varying the parameter from $A = -2$ to $A = 0$ (which corresponds to the Euler-Chebyshev's method), the basins of attraction are very similar in the complex region considered, i.e., large variations are not observed in comparison to the Euler-Chebyshev's method, which has the advantage of employing a simpler expression since $A = 0$. Small differences are noticed at the level of the boundaries of the basins of attraction, where their complexity appears to increase slightly as the parameter approaches zero, i.e. approaches the Euler-Chebyshev's technique (Figure 1).

As the parameter is increased above $A=0$, the complexity of the boundaries eventually start to diminish gently (Figure 2). When the value $A = 1$ is reached, which corresponds to the Halley's technique, this border complexity appears to reach its lowest value, i.e., simpler boundaries (Figure 3).

From a small distance above this point, the boundaries begin to deteriorate and the basins of attraction redefine their shapes and locations to values close to $A=2$ (corresponding to the BSC method), where the basins of attraction display complex interwoven basins of attraction and

many small subsets dispersed throughout the complex plane, with the basin of the real root $x = -1.5$ dominating in the considered region (Figures 3 and 4).

Increasing the value of A up to $A = 2.06$, this tendency is strengthened, with numerous small basins of attraction dispersed throughout the complex region considered, and shortly thereafter, for values of A around 2.065, a large region of non-convergence is created, which subsequently increases (Figure 5 and 6).

The optimal value of A seems to be close to $A = 1$ or Halley's technique. Besides the fact that this method has been identified as the most interesting in terms of the behavior of its basins of attraction among the three specific cases already studied in the literature (for $A = 0$, $A = 1$ and $A = 2$), it is now possible to recognize, for at least some polynomials of high degree, and since the optimal value of the parameter A appears to be somewhere in the vicinity of 1, that it is likely to remain the most effective of the examined family of iterative techniques.

5. Conclusion

The basins of attraction of root-finding methods, belonging to a one-parameter family of third-order, were compared by applying the techniques to the zeros of a high-degree polynomial, while varying the parameter A of the family.

Negative values of A appear useless when contrasted to the simpler formula of the Euler-Chebyshev's technique ($A = 0$).

Values above $A = 2$ appear to be useless as well, as the basins become more complex with small basins far from the corresponding roots, and non-convergent regions beginning to emerge.

The results confirm earlier findings by other researchers that Halley's method is superior in terms of complexity, shape and borders of its basins of attraction. However, previous research comparing iterative methods of this family only compared BSC, Euler-Chebyshev's, and Halley's methods. This paper acknowledges that, at least for some polynomials of high degree, among the investigated family of iterative techniques (2.4), the methods whose parameter A value is close to 1, value corresponding to Halley's method, appear to be the best choices. Hence, Halley's method seems to remain the most appropriate iterative method.

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