

Moroccan J. of Pure and Appl. Anal. (MJPA)A)

Volume 9(3), 2023, Pages 339–353

ISSN: Online 2351-8227 - Print 2605-6364

DOI: [10.2478/mjpaa-2023-0023](https://doi.org/10.2478/mjpaa-2023-0023)

On best proximity point theorems for relatively nonexpansive mappings in locally convex spaces

BRAHIM SAADAOU¹, NOUR-EDDINE EL HARMOUCHI¹, JAMAL MOULINE¹, SAMIH LAZAIZ²

ABSTRACT. This paper explores the concept of normal structure by introducing a generalized form known as $\mathcal{P}$ -proximal normal structure. Focusing on the framework of Hausdorff locally convex spaces, the study establishes optimal proximity outcomes for both cyclic and noncyclic relatively $\mathcal{P}$ -nonexpansive mappings. Furthermore, the paper presents in the realm of probabilistic normed spaces, considered as instances of Hausdorff locally convex spaces, some theorems that address the existence of best proximity points within this context.

Mathematics Subject Classification (2020). 46A03, 47H10, 54E70, 54H25

Key words and phrases. Best proximity point; Hausdorff locally convex space; cyclic mapping; noncyclic mapping; relatively $\mathcal{P}$ -nonexpansive mapping; probabilistic normed space; $\mathcal{P}$ -normal structure.

1. Introduction

The concept of fixed point offers a beneficial and practical approach in resolving nonlinear equation of the form $Tv = v$. These concepts find their roots in the study of a self-mapping within normed spaces, metric spaces or any relevant structures. However, a critical consideration arises when the self-mapping behavior of the operator is not evident. In such cases, the

Received September 30, 2022 - Accepted: August 28, 2023.

© The Author(s) 2023. This article is published with open access by Sidi Mohamed Ben Abdallah University.

¹ Hassan II University of Casablanca, Faculty of Sciences Ben M'Sik, Laboratory of Analyse Modelisation and Simulation, Casablanca, Morocco. e-mail: saadaoui.brahim2015@gmail.com, nouredine.elharmouchi@gmail.com (Corresponding Author), mouline61@gmail.com

² ENSAM Casablanca, Hassan II University, Casablanca, Morocco. e-mail: samih.lazaiz@usmba.ac.ma.

equation might lack a solution, underscoring the importance of understanding the characteristics and properties of the involved mappings.

A closely related concept, that of best proximity points or pairs, serves as a suitable tool to explore and substantiate the existence of an element v that is somewhat near to Tv within the context of locally convex spaces. Indeed, best proximity outcomes are effective in establishing an approximate solution to the nonlinear equation $Tv = v$, yet they may not necessarily provide an optimal approximation.

The central objective of best proximity point outcomes revolves around minimizing a real-valued function, often denoted as $v \rightarrow d(v, Tv)$. It serves as a gauge of the error connected with an approximation of the solution to equation $Tv = v$. Such a solution is referred to as the best proximity point of T .

An influential best proximity point theorem, attributed to Fan [7], asserts that within a Hausdorff locally convex topological vector space $\mathcal{H}$, if K represents a compact and convex nonempty subset, and T stands as a continuous self-mapping over K , then the existence of an element v within K is guaranteed, satisfying $p(v, Tv) = p(Tv, K)$. This theorem has been extended by several authors, including Chaira et al. [1], Chi et al. [3], El harmouchi et al. [5], Espinola [6], Prolla [9], Reich [11], Saadaoui et al. [12] and Sehgal and Singh [13, 14], across a variety of frameworks.

The article aims seeks to contribute to this area of research by establishing and providing some best proximity point theorem for cyclic relatively $\mathcal{P}$ -nonexpansive mapping within the context of locally convex spaces possessing the $\mathcal{P}$ -proximal normal structure. Furthermore, The present paper extends the d -property established by V. S. Raj et al. [4] and A. Eldered et al. [10] to the framework of locally convex spaces in order to characterize strict convexity. Additionally, the article demonstrates that every noncyclic relatively $\mathcal{P}$ -nonexpansive mapping has a best proximity pair. Finally, the normed probabilistic space, provided by Vuong in [17] (see also [3]), will be utilized to develop an application of our main result.

2. Preliminaries

In order to start, we first go over some basic terminology and notations that are helpful in the proofs of our primary findings.

Consider two nonempty subsets $\mathcal{U}$ and $\mathcal{V}$ of a Hausdorff locally convex space $\mathcal{H}$. The subsequent notations are employed:

$$\begin{aligned}\delta_p(\mathcal{U}, \mathcal{V}) &= \sup \{p(v - \vartheta) : v \in \mathcal{U}; \vartheta \in \mathcal{V}\} \\ \delta_p(v, \mathcal{V}) &= \sup \{p(v - \vartheta) : \vartheta \in \mathcal{V}\} \\ \text{dist}_p(\mathcal{U}, \mathcal{V}) &= \inf \{p(v - \vartheta) : v \in \mathcal{U}; \vartheta \in \mathcal{V}\}.\end{aligned}$$

A mapping $T : \mathcal{U} \cup \mathcal{V} \rightarrow \mathcal{U} \cup \mathcal{V}$ is termed relatively $\mathcal{P}$ -nonexpansive if, for any (v, ϑ) in $\mathcal{U} \times \mathcal{V}$ and $p \in \mathcal{P}$, one has

$$p(Tv - T\vartheta) \leq p(v - \vartheta).$$

Observe that when properties are held by both sets $\mathcal{U}$ and $\mathcal{V}$, the pair $(\mathcal{U}, \mathcal{V})$ satisfies the same properties. To exemplify,

- (i) $(\mathcal{U}, \mathcal{V}) \subseteq (\mathcal{U}', \mathcal{V}')$ if and only if $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{V} \subseteq \mathcal{V}'$,
- (ii) $(\mathcal{U}, \mathcal{V})$ is weakly compact if and only if $\mathcal{U}$ and $\mathcal{V}$ are both weakly compact.

Definition 2.1. Within a Hausdorff locally convex space $(\mathcal{H}, \mathcal{P})$, a couple of subsets $(\mathcal{U}, \mathcal{V})$ is termed $\mathcal{P}$ -proximal if, for every $p \in \mathcal{P}$ and for every $(v, \vartheta) \in \mathcal{U} \times \mathcal{V}$, there exists a $(v', \vartheta') \in \mathcal{U} \times \mathcal{V}$ satisfying

$$p(v - \vartheta') = p(v' - \vartheta) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Definition 2.2. Within a Hausdorff locally convex space $(\mathcal{H}, \mathcal{P})$, a convex pair (K_1, K_2) is characterized by possessing a $\mathcal{P}$ -proximal normal structure if any closed, convex, bounded pair $(H_1, H_2) \subseteq (K_1, K_2)$ satisfies

- (i) $\text{dist}_p(H_1, H_2) = \text{dist}_p(K_1, K_2)$,
- (ii) $\delta_p(H_1, H_2) > \text{dist}_p(H_1, H_2)$,

there exists $(v_1, v_2) \in H_1 \times H_2$ such that

$$\delta_p(v_1, H_2) < \delta_p(H_1, H_2) \text{ and } \delta_p(v_2, H_1) < \delta_p(H_1, H_2).$$

Let us introduce the subsequent sets, which will find frequent application in the forthcoming discussion, within the context of a Hausdorff locally convex space $(\mathcal{H}, \mathcal{P})$. Consider two nonempty subsets $\mathcal{U}$ and $\mathcal{V}$ of $(\mathcal{H}, \mathcal{P})$. Define:

$$\mathcal{U}_0^p = \{v \in \mathcal{U} : p(v - \vartheta') = \text{dist}_p(\mathcal{U}, \mathcal{V}) \text{ for some } \vartheta' \in \mathcal{V}\}$$

and

$$\mathcal{V}_0^p = \{\vartheta \in \mathcal{V} : p(v' - \vartheta) = \text{dist}_p(\mathcal{U}, \mathcal{V}) \text{ for some } v' \in \mathcal{U}\}.$$

Clearly, one has $(\mathcal{U}_0^p, \mathcal{V}_0^p) \subseteq (\mathcal{U}, \mathcal{V})$.

Lemma 2.1. Let $(\mathcal{U}, \mathcal{V})$ stands for a nonempty, weakly compact and convex pair of a Hausdorff locally convex space $(\mathcal{H}, \mathcal{P})$. Then, the pair $(\mathcal{U}_0^p, \mathcal{V}_0^p)$ is nonempty, weakly compact, convex and $\text{dist}_p(\mathcal{U}_0^p, \mathcal{V}_0^p) = \text{dist}_p(\mathcal{U}, \mathcal{V})$. Moreover, if $(\mathcal{U}, \mathcal{V})$ possesses $\mathcal{P}$ -proximal normal structure, it follows that $(\mathcal{U}_0^p, \mathcal{V}_0^p)$ also exhibits $\mathcal{P}$ -proximal normal structure.

Proof. First, let us prove that $(\mathcal{U}_0^p, \mathcal{V}_0^p)$ is nonempty.

Given that $\text{dist}_p(\mathcal{U}, \mathcal{V}) = \inf \{p(v - \vartheta) : v \in \mathcal{U}; \vartheta \in \mathcal{V}\}$, we establish that for any $\varepsilon > 0$, a collection $\{v_\alpha, \vartheta_\alpha\}_\alpha \subset \mathcal{U} \times \mathcal{V}$ exists, satisfying the ensuing inequality:

$$\text{dist}_p(\mathcal{U}, \mathcal{V}) \leq p(v_\alpha - \vartheta_\alpha) \leq \text{dist}_p(\mathcal{U}, \mathcal{V}) + \varepsilon. \quad (2.1)$$

The weak compactness of $(\mathcal{U}, \mathcal{V})$ involves the existence of subnets $\{v_{\alpha'}\}_{\alpha'}$ and $\{\vartheta_{\alpha'}\}_{\alpha'}$ of the nets $\{v_\alpha\}_\alpha$ and $\{\vartheta_\alpha\}_\alpha$ which converge weakly to $v_0 \in \mathcal{U}$ and $\vartheta_0 \in \mathcal{V}$, respectively.

Furthermore, the weakly lower semi-continuity of each $p \in \mathcal{P}$ implies that the function

$$\begin{aligned} \varphi : \mathcal{U} \times \mathcal{V} &\longrightarrow \mathbb{R}_+ \\ (v, \vartheta) &\longmapsto \varphi(v, \vartheta) = p(v - \vartheta) \end{aligned}$$

is weakly lower semi-continuous. Thus, the inequality (2.1) implies that $(v_0, \vartheta_0) \in \mathcal{U}_0^p \times \mathcal{V}_0^p$. Therefore, $(\mathcal{U}_0, \mathcal{V}_0)$ is nonempty.

For the convexity of the pair $(\mathcal{U}_0^p, \mathcal{V}_0^p)$, let $v_1, v_2 \in \mathcal{U}_0^p$ and $\mu \in [0, 1]$. Thus, there exist $\vartheta_1, \vartheta_2 \in \mathcal{V}$ in order that

$$p(v_1 - \vartheta_1) = p(v_2 - \vartheta_2) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

One has,

$$\begin{aligned} p(\mu v_1 + (1 - \mu)v_2 - \mu\vartheta_1 - (1 - \mu)\vartheta_2) &= p(\mu(v_1 - \vartheta_1) + (1 - \mu)(v_2 - \vartheta_2)) \\ &\leq \mu p(v_1 - \vartheta_1) + (1 - \mu)p(v_2 - \vartheta_2) \\ &\leq \text{dist}_p(\mathcal{U}, \mathcal{V}) \\ &\leq p(\mu v_1 + (1 - \mu)v_2 - \mu\vartheta_1 - (1 - \mu)\vartheta_2) \end{aligned}$$

Thus, $\mu v_1 + (1 - \mu)v_2 \in \mathcal{U}_0^p$. Therefore, $\mathcal{U}_0^p$ is convex. Likewise, we get $\mathcal{V}_0^p$ is convex.

Furthermore, the description of $(\mathcal{U}_0^p, \mathcal{V}_0^p)$ implies that $\text{dist}_p(\mathcal{U}, \mathcal{V}) = \text{dist}_p(\mathcal{U}_0^p, \mathcal{V}_0^p)$.

To establish $\mathcal{U}_0^p$ is weakly compact, consider a net $\{v_\alpha\}_\alpha$ on $\mathcal{U}_0^p$ that converges to v , that is,

$$\lim_\alpha p(v_\alpha - v) = 0.$$

In addition, for each $v_\alpha \in \mathcal{U}_0^p$, there exists $\vartheta_\alpha \in \mathcal{V}_0^p$ so that $p(v_\alpha - \vartheta_\alpha) = \text{dist}_p(\mathcal{U}, \mathcal{V})$. The weak compactness of $\mathcal{V}$ guarantees the existence of a subnet $\{\vartheta_{\alpha'}\}_{\alpha'}$ of $\{\vartheta_\alpha\}_\alpha$ in order that

$$\text{w-lim } \vartheta_{\alpha'} = \vartheta,$$

for some $\vartheta \in \mathcal{V}$. Since $\mathcal{H}$ is a Hausdorff space, one has

$$\lim_{\alpha'} p(v_{\alpha'} - v) = 0.$$

Thus,

$$\text{w-lim } v_{\alpha'} = v$$

and

$$p(v_{\alpha'} - \vartheta_{\alpha'}) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Due to the subsets $\mathcal{U}$ and $\mathcal{V}$ being weakly closed and the weak lower semi-continuity of the function p , one has

$$p(v - \vartheta) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Then, $\mathcal{U}_0^p$ is closed. Since $\mathcal{U}_0^p$ is convex, then it is weakly closed. Thus, $\mathcal{U}_0^p$ is weakly compact. In the same manner, we demonstrate also the weak compactness of $\mathcal{V}_0^p$.

Since $\text{dist}_p(\mathcal{U}, \mathcal{V}) = \text{dist}_p(\mathcal{U}_0^p, \mathcal{V}_0^p)$, conclude that $(\mathcal{U}_0^p, \mathcal{V}_0^p)$ has $\mathcal{P}$ -proximal normal structure.

Definition 2.3. Consider a locally convex space denoted as $(\mathcal{H}, \mathcal{P})$, let $\mathcal{U}$ and $\mathcal{V}$ two nonempty subsets of $\mathcal{H}$. We define a mapping $T : \mathcal{U} \cup \mathcal{V} \rightarrow \mathcal{U} \cup \mathcal{V}$ as relatively $\mathcal{P}$ -nonexpansive if it satisfies

$$p(Tv - T\vartheta) \leq p(v - \vartheta),$$

for any $v \in \mathcal{U}$, $\vartheta \in \mathcal{V}$ and $p \in \mathcal{P}$. Moreover,

- (i) If T has the property that $T(\mathcal{U}) \subseteq \mathcal{V}$ and $T(\mathcal{V}) \subseteq \mathcal{U}$, it is categorized as a cyclic relatively $\mathcal{P}$ -nonexpansive mapping;

- (ii) If T has the property that $T(\mathcal{U}) \subseteq \mathcal{U}$ and $T(\mathcal{V}) \subseteq \mathcal{V}$, it is categorized as a noncyclic relatively $\mathcal{P}$ -nonexpansive mapping.

Lemma 2.2. Consider a locally convex space $(\mathcal{H}, \mathcal{P})$ and let $(\mathcal{U}_0^p, \mathcal{V}_0^p)$ represent the $\mathcal{P}$ -proximal pair associated with the original pair $(\mathcal{U}, \mathcal{V})$. If T is a cyclic (or noncyclic) relatively $\mathcal{P}$ -nonexpansive mapping on $\mathcal{U} \cup \mathcal{V}$, then T also retains the cyclic (or noncyclic) relatively $\mathcal{P}$ -nonexpansive property on $\mathcal{U}_0^p \cup \mathcal{V}_0^p$.

Proof. We will specifically provide the proof for the scenario of a cyclic relatively $\mathcal{P}$ -nonexpansive mapping. For noncyclic case we follow a similar approach.

Referring to the characterization of $\mathcal{U}_0^p$ and $\mathcal{V}_0^p$, when considering an element $v \in \mathcal{U}_0^p$ it implies an element $\vartheta \in \mathcal{V}_0^p$ exists so that, for any $p \in \mathcal{P}$,

$$p(v - \vartheta) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Given that T is cyclic relatively $\mathcal{P}$ -nonexpansive, then $p(Tv - T\vartheta) \leq \text{dist}_p(\mathcal{U}, \mathcal{V})$ for any $p \in \mathcal{P}$. Thus, $Tv \in \mathcal{V}_0^p$. Therefore, $T(\mathcal{U}_0^p) \subseteq \mathcal{V}_0^p$. Similarly, $T(\mathcal{V}_0^p) \subseteq \mathcal{U}_0^p$.

3. Main result

Theorem 3.1. Let $(\mathcal{U}, \mathcal{V})$ be a nonempty, weakly compact and convex pair of a Hausdorff locally convex space $(\mathcal{H}, \mathcal{P})$. Assume that $(\mathcal{U}, \mathcal{V})$ has $\mathcal{P}$ -proximal normal structure. Let $T : \mathcal{U} \cup \mathcal{V} \rightarrow \mathcal{U} \cup \mathcal{V}$ be a cyclic relatively $\mathcal{P}$ -nonexpansive mapping. Then, for any $p \in \mathcal{P}$, there exists $(v, \vartheta) \in \mathcal{U} \times \mathcal{V}$ such that

$$p(v - Tv) = p(\vartheta - T\vartheta) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Proof. Observe that providing a proof is unnecessary if $\mathcal{U} \cap \mathcal{V}$ is nonempty (see Theorem 1 in [17]). Suppose that $\text{dist}_p(\mathcal{U}, \mathcal{V}) > 0$, for all $p \in \mathcal{P}$.

Denote by Γ the set comprising all nonempty subsets F^p of $\mathcal{U}_0^p \cup \mathcal{V}_0^p$ that fulfill the conditions $F^p \cap \mathcal{U}_0^p$ and $F^p \cap \mathcal{V}_0^p$ are nonempty, closed, convex such that

$$T(F^p \cap \mathcal{U}_0^p) \subseteq F^p \cap \mathcal{V}_0^p \quad \text{and} \quad T(F^p \cap \mathcal{V}_0^p) \subseteq F^p \cap \mathcal{U}_0^p \quad \text{and} \quad \text{dist}_p(F^p \cap \mathcal{U}_0^p, F^p \cap \mathcal{V}_0^p) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Since $\mathcal{U}_0^p \cup \mathcal{V}_0^p$ is in Γ , one has Γ is nonempty.

Let $\{F_\alpha^p\}_{\alpha \in J}$ be a descending chain in Γ and

$$F_0^p = \bigcap_{\alpha \in J} F_\alpha^p.$$

Hence,

$$F_0^p \cap \mathcal{U}_0^p = \bigcap_{\alpha \in J} (F_\alpha^p \cap \mathcal{U}_0^p).$$

Since $\mathcal{U}_0^p$ is weakly compact, one has $F_0^p \cap \mathcal{U}_0^p$ is nonempty, closed and convex. Likewise, $F_0^p \cap \mathcal{V}_0^p$ is nonempty, closed and convex. Moreover,

$$T(F_0^p \cap \mathcal{U}_0^p) = \bigcap_{\alpha \in J} T(F_\alpha^p \cap \mathcal{U}_0^p) \subseteq F_0^p \cap \mathcal{V}_0^p;$$

and similarly we have $T(F_0^p \cap \mathcal{V}_0^p) \subseteq F_0^p \cap \mathcal{U}_0^p$.

To confirm that $F_0^p \in \Gamma$, we only need to verify that

$$\text{dist}_p(F_0^p \cap \mathcal{U}_0^p, F_0^p \cap \mathcal{V}_0^p) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Let $\alpha \in J$, it is possible to select $u_\alpha^p \in F_0^p \cap \mathcal{U}_0^p$ and $v_\alpha^p \in F_0^p \cap \mathcal{V}_0^p$ such that $p(v_\alpha^p - u_\alpha^p) = \text{dist}_p(\mathcal{U}, \mathcal{V})$, for any $p \in \mathcal{P}$. Furthermore, weakly convergent subnets $\{v_{\alpha'}^p\}$ and $\{u_{\alpha'}^p\}$ can be selected from $\{v_\alpha^p\}$ and $\{u_\alpha^p\}$, respectively, such that

$$\text{w-lim } v_{\alpha'}^p = v \quad \text{and} \quad \text{w-lim } u_{\alpha'}^p = u.$$

Since $F_0^p \cap \mathcal{U}_0$ and $F_0^p \cap \mathcal{V}_0$ are closed convex subsets of $\mathcal{H}$, $v \in F_0^p \cap \mathcal{U}_0$ and $u \in F_0^p \cap \mathcal{V}_0$. According to weak lower semi-continuity of p , one has

$$p(v - u) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Hence, $\text{dist}_p(F_0^p \cap \mathcal{U}_0^p, F_0^p \cap \mathcal{V}_0^p) = \text{dist}_p(\mathcal{U}, \mathcal{V})$.

Thus, in conformity with Zorn's lemma, Γ contains a minimum element, represented by K , since the intersection of every descending chain in Γ is nonempty into Γ .

Let $K_1 = K \cap \mathcal{U}_0$ and $K_2 = K \cap \mathcal{V}_0$. Suppose that

$$\delta_p(K_1, K_2) > \text{dist}_p(K_1, K_2).$$

Otherwise,

$$p(v - Tv) = \text{dist}_p(K_1, K_2) = \text{dist}_p(\mathcal{U}, \mathcal{V}),$$

for any $v \in K_1$. As K is minimal, so (K_1, K_2) is a proximal pair in $(\mathcal{U}_0, \mathcal{V}_0)$. By $\mathcal{P}$ -proximal normal structure, there exists $(\vartheta_1, \vartheta_2)$ in $K_1 \times K_2$ and $\beta \in (0, 1)$ such that

$$\delta_p(\vartheta_1, K_2) \leq \beta \delta_p(K_1, K_2), \quad \delta_p(\vartheta_2, K_1) \leq \beta \delta_p(K_1, K_2)$$

and there exists $(\vartheta'_1, \vartheta'_2) \in K_1 \times K_2$ such that

$$p(\vartheta_1 - \vartheta'_2) = p(\vartheta_2 - \vartheta'_1) = \text{dist}_p(K_1, K_2).$$

Thus, for any $z \in K_2$,

$$\begin{aligned} p\left(\frac{\vartheta_1 + \vartheta'_2}{2} - z\right) &\leq p\left(\frac{\vartheta_1 - z}{2}\right) + p\left(\frac{\vartheta'_2 - z}{2}\right) \\ &\leq \beta \frac{\delta_p(K_1, K_2)}{2} + \frac{\delta_p(K_1, K_2)}{2} \\ &= \alpha \delta_p(K_1, K_2), \end{aligned}$$

where $\alpha = \frac{1+\beta}{2} \in]0, 1[$.

Let $v_1 = \frac{\vartheta_1 + \vartheta'_1}{2}$ and $v_2 = \frac{\vartheta_2 + \vartheta'_2}{2}$, then

$$\delta_p(v_1, K_2) \leq \alpha \delta_p(K_1, K_2), \quad \delta_p(v_2, K_1) \leq \alpha \delta_p(K_1, K_2) \quad \text{and} \quad p(v_1 - v_2) = \text{dist}(K_1, K_2).$$

Put

$$L_1 = \{v \in K_1 : \delta_p(v, K_2) \leq \alpha \delta_p(K_1, K_2)\} \quad \text{and} \quad L_2 = \{v \in K_2 : \delta_p(v, K_1) \leq \alpha \delta_p(K_1, K_2)\}.$$

Subsequently, for each $i \in \{1; 2\}$, L_i constitutes a closed and convex nonempty subset within K_i . Since $v_1 \in L_1$ and $v_2 \in L_2$, one has

$$\text{dist}_p(L_1, L_2) = \text{dist}_p(K_1, K_2) (= \text{dist}_p(\mathcal{U}, \mathcal{V})).$$

Now, let $v \in L_1, z \in K_2$. Due to T is nonexpansive, one can be inferred that

$$p(Tv - Tz) \leq p(v - z) \leq \alpha \delta_p(K_1, K_2).$$

Hence,

$$T(K_2) \subseteq \mathfrak{B}_p(Tv, \alpha \delta_p(K_1, K_2)) \cap K_1 = K'_1$$

where $\mathfrak{B}(c, r)$ represents the ball centered at point c and having a radius of r .

Evidently, K'_1 is a subset characterized by being closed and convex. If $\vartheta \in K_2$ such that $p(v - \vartheta) = \text{dist}_p(\mathcal{U}, \mathcal{V})$, then $p(Tv - T\vartheta) = \text{dist}_p(K_1, K_2)$. Since $T\vartheta \in K'_1$, one has $\text{dist}_p(K'_1, K_2) = \text{dist}_p(\mathcal{U}, \mathcal{V})$.

Therefore, $K'_1 \cup K_2 \in \Gamma$. Minimality of K conclude that $K'_1 = K_1$. Hence, $K_1 \subseteq \mathfrak{B}(Tv; \alpha \delta(K_1, K_2))$. Since $v \in L_1$ was arbitrary chosen, this confirms that $T(L_1) \subseteq L_2$. Likewise, one has $T(L_2) \subseteq L_1$. Therefore, $L_1 \cup L_2 \in \Gamma$. However, $\delta_p(L_1, L_2) \leq \alpha \delta_p(K_1, K_2)$, which contradicts with K is minimal.

If $\mathcal{U} = \mathcal{V}$, as a corollary, one has

Corollary 3.1. [17, Theorem 1.] *Consider a nonempty weakly compact and convex subset C of a Hausdorff locally convex space $(\mathcal{H}, \mathcal{P})$ which has $\mathcal{P}$ -normal structure, and $T : C \rightarrow C$ stands for a $\mathcal{P}$ -nonexpansive mapping. Then, T has a fixed point.*

Let us illustrate using the subsequent example.

Example 3.1. Let the real space $X = \mathbb{R}^{N^*} = \{v = (v_n)_{n \geq 1} : v_n \in \mathbb{R}\}$ and $(p_n)_{n \geq 1}$ represents a collection of semi-norms generating the topology of X such that $p_n(v) = \max_{1 \leq i \leq n} |v_i|$, for every $v = (v_i)_{i \geq 1} \in X$.

Suppose that $\{e_n\}$ is the canonical basis of X and let

$$\mathcal{U} = \{\lambda e_1 + (1 - \lambda)e_2 : \lambda \in [0, 1]\} \text{ and } \mathcal{V} = \{\lambda e_3 + (1 - \lambda)e_4 : \lambda \in [0, 1]\}.$$

Since $(\mathcal{U}, \mathcal{V})$ is closed convex and bounded, then it is weakly compact.

As $\frac{1}{2} \leq \max(\lambda; 1 - \lambda)$, one has $p_n\left(\frac{e_1 + e_2}{2} - \frac{e_3 + e_4}{2}\right) = \frac{1}{2} = \text{dist}_{p_n}(\mathcal{U}, \mathcal{V})$. However, $(\mathcal{U}, \mathcal{V})$ does not have a $\mathcal{P}$ -proximal convex sub-pair containing more than one element. Thus, $(\mathcal{U}, \mathcal{V})$ has $\mathcal{P}$ -proximal normal structure.

Consider $T : \mathcal{U} \cup \mathcal{V} \rightarrow \mathcal{U} \cup \mathcal{V}$ described by

$$T(\lambda e_1 + (1 - \lambda)e_2) = \lambda e_3 + (1 - \lambda)e_4$$

and

$$T(\lambda e_3 + (1 - \lambda)e_4) = \lambda e_2 + (1 - \lambda)e_1.$$

One has

$$\begin{aligned} p_n[T(\lambda e_1 + (1 - \lambda)e_2) - T(\lambda e_3 + (1 - \lambda)e_4)] &= p_n[\lambda e_3 + (1 - \lambda)e_4 - (\lambda e_2 + (1 - \lambda)e_1)] \\ &= p_n[\lambda e_1 + (1 - \lambda)e_2 - (\lambda e_3 + (1 - \lambda)e_4)]. \end{aligned}$$

Thus, T is nonexpansive. As a consequence, $(u, v) \in \mathcal{U} \times \mathcal{V}$ exists, so that

$$p_n(v - Tv) = p_n(\vartheta - T\vartheta) = \text{dist}_{p_n}(\mathcal{U}, \mathcal{V}),$$

and

$$v = \frac{e_1 + e_2}{2} \quad ; \quad \vartheta = \frac{e_3 + e_4}{2}.$$

Recollect the subsequent definition.

Definition 3.1. [17, Definition 3] Let $(\mathcal{H}, \mathcal{P})$ be a Hausdorff locally convex space. $\mathcal{H}$ is designated as strictly convex if

$$p(v) = 1, \quad p(\vartheta) = 1 \quad \text{and} \quad v \neq \vartheta \quad \text{implies that} \quad p\left(\frac{v + \vartheta}{2}\right) < 1,$$

for any $v, \vartheta \in \mathcal{H}$ and $p \in \mathcal{P}$.

Example 3.2. Assume Λ is a nonempty set, and for each $\alpha \in \Lambda$, let X_α represent a strictly convex normed space equipped with the norm $\|\cdot\|_\alpha$.

Consider the Cartesian product $X = \prod_{\alpha \in \Lambda} X_\alpha$ and for each $\alpha \in \Lambda$ we define $p_\alpha(v) = \|v_\alpha\|_\alpha$, for all

$v = \{v_\alpha\}_{\alpha \in \Lambda} \subset X$. It is obvious that the set $\{p_\alpha : \alpha \in \Lambda\}$ define a family of seminorms on X . Therefore, it generates a locally convex topology, which is also the product topology, on X .

Next, let $v = \{v_\alpha\}_{\alpha \in \Lambda}$ and $\vartheta = \{\vartheta_\alpha\}_{\alpha \in \Lambda}$ in X such that $p_\alpha(v) = p_\alpha(\vartheta) = 1$ (i.e. $\|v_\alpha\|_\alpha = \|\vartheta_\alpha\|_\alpha = 1$). Assume that $p_\alpha\left(\frac{v + \vartheta}{2}\right) = 1$. Then, $\|v_\alpha + \vartheta_\alpha\|_\alpha = 2 = \|v_\alpha\|_\alpha + \|\vartheta_\alpha\|_\alpha$.

As X_α is a strictly convex normed space, there exists some $\lambda_\alpha > 0$ such that $v_\alpha = \lambda_\alpha \vartheta_\alpha$. Since $\|v_\alpha\|_\alpha = \|\vartheta_\alpha\|_\alpha = 1$, one has $v_\alpha = \vartheta_\alpha$ for each $\alpha \in \Lambda$. Hence, $v = \vartheta$. Therefore, X is strictly convex locally convex space.

Lemma 3.1. Let $(\mathcal{H}, \mathcal{P})$ be a strictly convex Hausdorff locally convex space and $v \neq \vartheta$ in $\mathcal{H}$ such that $p(v) \neq 0$ or $p(\vartheta) \neq 0$. Then,

$$p\left(\frac{v + \vartheta}{2}\right) < \max\{p(v), p(\vartheta)\}.$$

Proof. By contradiction, assume that $v \neq \vartheta$ such that

$$p\left(\frac{v + \vartheta}{2}\right) \geq \max\{p(v), p(\vartheta)\} \geq \frac{p(v) + p(\vartheta)}{2}.$$

Thus,

$$p\left(\frac{v + \vartheta}{2}\right) = \frac{p(v) + p(\vartheta)}{2} = \max\{p(v), p(\vartheta)\}.$$

Therefore,

$$p(v) = p(\vartheta) = p\left(\frac{v + \vartheta}{2}\right).$$

Set $v' = \frac{v}{p(v)}$ and $\vartheta' = \frac{\vartheta}{p(\vartheta)}$.

Just like in the verification of Proposition 1 from [17], one has $u' = v'$ i.e.

$$\frac{v}{p(v)} = \frac{\vartheta}{p(\vartheta)}.$$

Then, $v = \vartheta$ this contradicts the initial assumption that $v \neq \vartheta$.

Definition 3.2. A pair $(\mathcal{U}, \mathcal{V})$ of nonempty subsets within a locally convex space $\mathcal{H}$ is deemed to possess the d -property if, and only if,

$$\begin{cases} p(v_1 - \vartheta_1) = \text{dist}_p(\mathcal{U}, \mathcal{V}) \\ p(v_2 - \vartheta_2) = \text{dist}_p(\mathcal{U}, \mathcal{V}) \end{cases} \quad \text{implies that} \quad p(v_1 - v_2) = p(\vartheta_1 - \vartheta_2),$$

whenever $v_1, v_2 \in \mathcal{U}$ and $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $p \in \mathcal{P}$.

Definition 3.3. The d -property is possessed by a locally convex space $\mathcal{H}$ if, for every pair of nonempty, closed, and convex subsets $(\mathcal{U}, \mathcal{V})$ within $\mathcal{H}$, the d -property holds.

As indicated in [1], a proof establishes that in the case where $\mathcal{H}$ is a normed linear space, the equivalence between strict convexity of $\mathcal{H}$ and the d -property of $\mathcal{H}$ holds.

The theorem presented below extends Theorem 3.1 in [1].

Theorem 3.2. A locally convex space $\mathcal{H}$ is strictly convex if and only if $\mathcal{H}$ has the d -property.

Proof. In the proof, we focus exclusively on demonstrating the implication that strict convexity leads to the d -property. An analogous reasoning indicates that the converse also holds. Consider $\mathcal{H}$ as a strictly convex space, and let $\mathcal{U}$ and $\mathcal{V}$ denote nonempty, closed, and convex subsets of $\mathcal{H}$. Presume that $v_1, v_2 \in \mathcal{U}$ and $\vartheta_1, \vartheta_2 \in \mathcal{V}$ so that

$$p(v_1 - \vartheta_1) = p(v_2 - \vartheta_2) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Set $\theta = v_1 - \vartheta_1$ and $\omega = v_2 - \vartheta_2$. If $\theta = \omega$, then $v_1 - v_2 = \vartheta_1 - \vartheta_2$. Hence,

$$p(v_1 - v_2) = p(\vartheta_1 - \vartheta_2).$$

Therefore, $(\mathcal{U}, \mathcal{V})$ has d -property.

If $\theta \neq \omega$ such that $(p(\theta), p(\omega)) \neq (0, 0)$, then

$$\begin{aligned} \text{dist}_p(\mathcal{U}, \mathcal{V}) &\leq p\left(\frac{v_1 + v_2}{2} - \frac{\vartheta_1 + \vartheta_2}{2}\right) = p\left(\frac{\theta + \omega}{2}\right) \\ &< \max\{p(\theta), p(\omega)\} = \text{dist}_p(\mathcal{U}, \mathcal{V}), \text{ by Lemma 3.1,} \end{aligned}$$

this presents a contradiction.. Therefore, $\theta = \omega$.

If $\theta \neq \omega$ and $p(\theta) = p(\omega) = 0$, then

$$\begin{aligned} p(v_1 - v_2) &\leq p(v_1 - \vartheta_1) + p(\vartheta_1 - \vartheta_2) + p(\vartheta_2 - v_2) \\ &\leq p(\theta) + p(\vartheta_1 - \vartheta_2) + p(\omega) \\ &\leq p(\vartheta_1 - \vartheta_2). \end{aligned}$$

Likewise, we can demonstrate that $p(\vartheta_1 - \vartheta_2) \leq p(v_1 - v_2)$. Hence, $p(v_1 - v_2) = p(\vartheta_1 - \vartheta_2)$. Therefore, $\mathcal{H}$ exhibits d -property.

The following outcome establishes a result on best proximity pairs for noncyclic case within the framework of strict convex Hausdorff locally convex spaces.

Theorem 3.3. Assume that $(\mathcal{H}, \mathcal{P})$ is a strictly convex Hausdorff locally convex space, and $(\mathcal{U}, \mathcal{V})$ constitute a nonempty pair of weakly compact convex sets with a $\mathcal{P}$ -proximal normal structure. Consider a noncyclic relatively $\mathcal{P}$ -nonexpansive mapping $T : \mathcal{U} \cup \mathcal{V} \rightarrow \mathcal{U} \cup \mathcal{V}$. Then, for any $p \in \mathcal{P}$, there exists $(v, \vartheta) \in \mathcal{U} \times \mathcal{V}$ that satisfies

$$Tv = v ; T\vartheta = \vartheta \quad \text{and} \quad p(v - \vartheta) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Proof. Via Lemma 2.2, one has $T : \mathcal{U}_0^p \rightarrow \mathcal{U}_0^p$ and $T : \mathcal{V}_0^p \rightarrow \mathcal{V}_0^p$. Denote by Γ the set comprising all nonempty subsets F^p of $\mathcal{U}_0^p \cup \mathcal{V}_0^p$ that fulfill the conditions $F^p \cap \mathcal{U}_0^p$ and $F^p \cap \mathcal{V}_0^p$ are nonempty, closed, convex such that

$$T(F^p \cap \mathcal{U}_0^p) \subseteq F^p \cap \mathcal{U}_0^p, T(F^p \cap \mathcal{V}_0^p) \subseteq F^p \cap \mathcal{V}_0^p \text{ and } \text{dist}_p(F^p \cap \mathcal{U}_0^p, F^p \cap \mathcal{V}_0^p) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Following that, we demonstrate that a minimal element K within Γ exists, as depicted in Theorem 3.1 proof.

Set $K_1 = K \cap \mathcal{U}_0$ and $K_2 = K \cap \mathcal{V}_0$. Suppose that $\delta_p(K_1, K_2) > \text{dist}_p(K_1, K_2)$. Otherwise, one of the sets is a singleton and the result follows from d -property. Indeed, suppose that $K_1 = \{v\}$. It is obvious that $Tv = v$. If ϑ is in K_2 such that

$$p(v - \vartheta) = \text{dist}_p(K_1, K_2)$$

and since T is nonexpansive, one has

$$p(Tv - T\vartheta) = p(v - T\vartheta) \leq p(v - \vartheta) = \text{dist}_p(K_1, K_2).$$

By Theorem 3.2, $\mathcal{H}$ has d -property. Thus, it must be the case that $T\vartheta = \vartheta$. As a consequence of K is minimal, (K_1, K_2) is a proximal pair in $(\mathcal{U}_0, \mathcal{V}_0)$. Moreover, since $(\mathcal{U}_0, \mathcal{V}_0)$ has $\mathcal{P}$ -proximal normal structure, define L_1 and L_2 as in the verification of Theorem 3.1. Put

$$L_1 = \{v \in K_1 : \delta_p(v, K_2) \leq \alpha \delta_p(K_1, K_2)\},$$

and

$$L_2 = \{\vartheta \in K_2 : \delta_p(\vartheta, K_1) \leq \alpha \delta_p(K_1, K_2)\}$$

Let $v \in L_1, z \in K_2$. Then, utilizing the nonexpansiveness of T one has

$$p(Tv - Tz) \leq p(v - z) \leq \alpha \delta_p(K_1, K_2).$$

Whence,

$$T(K_2) \subseteq \mathcal{V}_p(Tv, \alpha \delta_p(K_1, K_2)) \cap K_2.$$

If $\vartheta \in K_2$ such that

$$p(v - \vartheta) = \text{dist}_p(\mathcal{U}, \mathcal{V}).$$

Then, $p(Tv - T\vartheta) = \text{dist}_p(K_1, K_2)$.

Since $T\vartheta \in K_2$, minimality of K implies that $K_2 \subseteq \mathcal{V}(Tv; \alpha \delta_p(K_1, K_2))$. Thus,

$$\delta_p(Tv, K_2) \leq \alpha \delta_p(K_1, K_2).$$

Since $v \in L_1$ was arbitrary chosen, we derive that $T(L_1) \subseteq L_1$. Likewise, $T(L_2) \subseteq L_2$. Thus, minimality of K yields to a contradiction.

4. Application

In 1963, Šerstnev [16] introduced probabilistic normed spaces as a significant family of probabilistic metric spaces. These spaces have demonstrated their applicability in various contexts, including quantum particle physics, besides a fundamental role in nonlinear analysis and its applications, and probabilistic functional analysis (see [15, 2, 3]).

The underlying motivation for this section was to investigate the outcomes of the best proximity pair within the framework of locally convex probabilistic normed spaces, thereby heightening the interest of this particular class of spaces.

Definition 4.1. [16] *If $\mathcal{H}$ is a linear space and $\mathfrak{F} = \{\mathcal{F}_v : v \in \mathcal{H}\}$ is a family of distribution functions satisfying:*

- (1) $\mathcal{F}_v(0) = 0$, for every $v \in \mathcal{H}$;
- (2) $\mathcal{F}_v(\xi) = 1$, for every $\xi > 0$, if and only if $v = 0$;
- (3) $\mathcal{F}_{\alpha v}(\xi) = \mathcal{F}_v\left(\frac{\xi}{|\alpha|}\right)$, for all $\xi > 0$, for all $\alpha \in \mathbb{C}$ such that $\alpha \neq 0$, and for all $v \in \mathcal{H}$.
- (4) $\mathcal{F}_{v+\vartheta}(\xi + \gamma) \geq \min\{\mathcal{F}_v(\xi), \mathcal{F}_\vartheta(\gamma)\}$, for all $\xi, \gamma > 0$ and for all $v, \vartheta \in \mathcal{H}$.

Then, the triple $(\mathcal{H}, \mathfrak{F}, \min)$ is termed as probabilistic normed space.

The neighborhoods system of $0 \in \mathcal{H}$ establish a topology of $\mathcal{H}$ given by

$$U(0, \varepsilon, \lambda) = \{u \in \mathcal{H} : \mathcal{F}_u(\varepsilon) > 1 - \lambda, \varepsilon > 0, 0 < \lambda < 1\},$$

and this structure introduces a locally convex Hausdorff topology, referred to as the (ε, λ) -topology, which is generated by the family of semi-norms $\{p_\lambda\}_{\lambda \in (0,1)}$ given by

$$p_\lambda(v) = \sup\{\xi \in \mathbb{R}^+ : \mathcal{F}_v(\xi) \leq 1 - \lambda\},$$

further details can be found by referring to [17].

In particular, one has

$$\mathcal{F}_v(p_\lambda(v)) \leq 1 - \lambda, \forall v \in \mathcal{H}, \forall \lambda \in (0, 1)$$

and

$$p_\lambda(v) < \varepsilon \text{ if and only if } \mathcal{F}_v(\varepsilon) > 1 - \lambda, \forall v \in \mathcal{H}, \forall \lambda \in (0, 1).$$

Recall that Δ^+ represent the set of all distribution functions satisfying the condition (1) of the definition 4.1 (consult [16] to delve into further details).

Setting

$$\begin{aligned} \mathcal{F} : \mathcal{H} &\longrightarrow \Delta^+ \\ v &\longmapsto \mathcal{F}_v \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}_v : \overline{\mathbb{R}}_+ &\longrightarrow [0, 1] \\ \xi &\longmapsto \mathcal{F}_v(\xi) \end{aligned}$$

Example 4.1. *Let $(\mathbb{R}, |\cdot|)$ be the set of real numbers endowed with the usual norm.*

Set $\mathfrak{F} = \{\varepsilon_{|v|} : v \in \mathbb{R}\}$ where $\varepsilon_a(\xi) = \begin{cases} 0 & \text{if } \xi \leq a \\ 1 & \text{if } \xi > a \end{cases}$, for all $a \in \overline{\mathbb{R}}_+$. One has $\{\varepsilon_a\}_{a \geq 0} \subset \Delta^+$.

Let a continuous triangle norm M such that, $M(\theta, \vartheta) = \min\{\theta, \vartheta\}$, for every $(\theta, \vartheta) \in [0, 1]^2$. Consider the triangle function τ_M generated by the function M as follows:

$$\tau_M(\mathcal{F}_v, \mathcal{F}_\vartheta)(\xi) = \sup \{M(\mathcal{F}_v(\gamma), \mathcal{F}_\vartheta(\zeta)) : \gamma + \zeta = \xi\}.$$

$(\mathbb{R}, \mathfrak{F}, \tau_M)$ is a probabilistic normed space if $\{\varepsilon_{|v|}\}_{v \in \mathbb{R}}$ satisfies the conditions (2)-(4) of the definition 4.1. In fact, it is quite easy to prove that condition (2) and (3) holds. To prove (4), notice first that, for every $a, b \geq 0$, one has $\tau_M(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}$ and $\varepsilon_a(\xi) \geq \varepsilon_b(\xi)$ whenever $a \leq b$. Next, let $\xi, \gamma \in \mathbb{R}_+$ and $v, \vartheta \in \mathbb{R}$. One has,

$$\begin{aligned} \varepsilon_{|v+\vartheta|}(\xi + \gamma) &\geq \varepsilon_{|v|+|\vartheta|}(\xi + \gamma) = \tau_M(\varepsilon_{|v|}, \varepsilon_{|\vartheta|})(\xi + \gamma) \\ &= \sup \{M(\varepsilon_{|v|}(\xi'), \varepsilon_{|\vartheta|}(\gamma')) : \xi' + \gamma' = \xi + \gamma\} \\ &\geq M(\varepsilon_{|v|}(\xi), \varepsilon_{|\vartheta|}(\gamma)) = \min \{\varepsilon_{|v|}(\xi), \varepsilon_{|\vartheta|}(\gamma)\}. \end{aligned}$$

Therefore, (4) holds.

Moreover, it is trivial to confirm that $\{v \in \mathbb{R} : \varepsilon_{|v|}(\xi) \leq r\}$ is a weakly closed subset in $\mathbb{R}$, for all $r \in \mathbb{R}$. Therefore, for $\xi \geq 0$,

$$\begin{aligned} \varepsilon_{|\cdot|}(\xi) : \mathbb{R} &\longrightarrow [0, 1] \\ v &\longmapsto \varepsilon_{|v|}(\xi) \end{aligned}$$

are weakly lower semicontinuous.

The subsequent definition outlines the concept of probabilistic relatively nonexpansive mappings.

Definition 4.2. Let $\mathcal{U}$ and $\mathcal{V}$ be two subsets of a probabilistic normed space $(\mathcal{H}, \mathfrak{F}, \min)$. A mapping $T : \mathcal{U} \cup \mathcal{V} \rightarrow \mathcal{U} \cup \mathcal{V}$ is characterized as probabilistic relatively nonexpansive if

$$\mathcal{F}_{Tv-T\vartheta}(\xi) \geq \mathcal{F}_{v-\vartheta}(\xi),$$

for all $v \in \mathcal{U}$, $\vartheta \in \mathcal{V}$ and $\xi \in \mathbb{R}$.

Definition 4.3. A pair $(\mathcal{U}, \mathcal{V})$ of nonempty within a probabilistic normed space $(\mathcal{H}, \mathfrak{F}, \min)$ is deemed to possess probabilistic uniformly normal structure if, for any closed, convex and bounded pair $(H, K) \subseteq (\mathcal{U}, \mathcal{V})$ not reduced to a single point, there exists $(v_0, \vartheta_0) \in H \times K$ and $0 < k < 1$ such that

$$\inf_{\vartheta \in K} \mathcal{F}_{v_0-\vartheta}(k\xi) \geq \inf_{(v, \vartheta) \in H \times K} \mathcal{F}_{v-\vartheta}(\xi) \quad \text{and} \quad \inf_{v \in H} \mathcal{F}_{v-\vartheta_0}(k\xi) \geq \inf_{(v, \vartheta) \in H \times K} \mathcal{F}_{v-\vartheta}(\xi), \quad \text{for all } \xi \in \mathbb{R}^+.$$

A generalization of lemma 1 of [17] is presented in the subsequent lemma. To demonstrate, we follow the same approach employed in verifying Lemma 1 of [17].

Lemma 4.1. Every probabilistic relatively nonexpansive mapping T in a probabilistic normed space $(\mathcal{H}, \mathfrak{F}, \min)$ is relatively $\mathcal{P}$ -nonexpansive in the corresponding locally convex space $(\mathcal{H}, \{p_\lambda\}_{\lambda \in (0,1)})$.

Proof. Assume that $\mathcal{F}_{Tv-T\vartheta}(t) \geq \mathcal{F}_{v-\vartheta}(t)$, for all $v \in \mathcal{U}$ and $\vartheta \in \mathcal{V}$. One can prove that

$$\{\xi \in \mathbb{R}^+ : \mathcal{F}_{Tv-T\vartheta}(\xi) \leq 1 - \lambda\} \subseteq \{\xi \in \mathbb{R}^+ : \mathcal{F}_{v-\vartheta}(\xi) \leq 1 - \lambda\},$$

for all $v \in \mathcal{U}$ and $\vartheta \in \mathcal{V}$. Indeed, let $\xi \geq 0$ so that $\mathcal{F}_{Tv-T\vartheta}(\xi) \leq 1 - \lambda$. As T is a probabilistic relatively nonexpansive mapping, one has

$$\mathcal{F}_{v-\vartheta}(\xi) \leq \mathcal{F}_{Tv-T\vartheta}(\xi) \leq 1 - \lambda.$$

Then, $p_\lambda(Tv - T\vartheta) \leq p_\lambda(v - \vartheta)$, for all $v \in \mathcal{U}$ and $\vartheta \in \mathcal{V}$.

Lemma 4.2. *Suppose that $(\mathcal{H}, \mathfrak{F}, \min)$ is a probabilistic normed space that satisfies the subsequent criterion:*

for any fixed $\xi \in \mathbb{R}^+$, $\mathcal{F}(\xi) : \mathcal{H} \rightarrow [0, 1]$ is weak lower semi-continuous in $\mathcal{H}$.

Then, each weakly compact pair $(\mathcal{U}, \mathcal{V})$ of $\mathcal{H}$ that possesses probabilistic uniformly normal structure has $\mathcal{P}$ -proximal normal structure within the corresponding locally convex space $(\mathcal{H}, \{p_\lambda\}_{\lambda \in (0,1)})$.

Proof. Consider a pair $(\mathcal{U}, \mathcal{V})$ consisting of nonempty, weakly compact, and convex subsets of $\mathcal{H}$ that possess the probabilistic uniformly normal structure.

Let (H, K) represent a pair that is bounded, closed, and convex $(\mathcal{U}, \mathcal{V})$ such that:

- (i) $\text{dist}_{p_\lambda}(H, K) = \text{dist}_{p_\lambda}(\mathcal{U}, \mathcal{V})$;
- (ii) $\delta_{p_\lambda}(H, K) > \text{dist}_{p_\lambda}(H, K)$.

First, let us prove that for any convex and weakly compact subset $C \subseteq \mathcal{H}$ and $0 < \lambda < 1$ one has

$$\sup_{v \in C} \sup \{ \xi \in \mathbb{R}^+ : \mathcal{F}_v(\xi) \leq 1 - \lambda \} = \sup \{ \xi \in \mathbb{R}^+ : \inf_{v \in C} \mathcal{F}_v(\xi) \leq 1 - \lambda \}. \quad (4.1)$$

Indeed, as $C \subseteq \mathcal{H}$ is weakly compact, denote $f(v) = \inf_{v \in C} \mathcal{F}_v(\xi)$, this leads to

$$\{ \xi \in \mathbb{R}^+ : \mathcal{F}_v(\xi) \leq 1 - \lambda \} \subseteq \{ \xi \in \mathbb{R}^+ : \mathcal{F}(\xi) \leq 1 - \lambda \},$$

for every $0 < \lambda < 1$. Then,

$$\sup_{v \in C} \sup \{ \xi \in \mathbb{R}^+ : \mathcal{F}_v(\xi) \leq 1 - \lambda \} \leq \sup \{ \xi \in \mathbb{R}^+ : \inf_{v \in C} \mathcal{F}_v(\xi) \leq 1 - \lambda \}.$$

Assume that

$$\beta = \sup_{v \in C} \sup \{ \xi \in \mathbb{R}^+ : \mathcal{F}_v(\xi) \leq 1 - \lambda \} < \sup \{ \xi \in \mathbb{R}^+ : \inf_{v \in C} \mathcal{F}_v(\xi) \leq 1 - \lambda \} = \alpha.$$

We get

$$1 - \lambda \geq \mathcal{F}_v(\alpha) > 1 - \lambda,$$

which is a contradiction.

The following inequality

$$\inf_{\vartheta \in K} \mathcal{F}_{v_0-y}(k\xi) \geq \inf_{(v,\vartheta) \in H \times K} \mathcal{F}_{v-\vartheta}(\xi)$$

leads to

$$\{ \xi \in \mathbb{R}^+ : \inf_{\vartheta \in K} \mathcal{F}_{v_0-\vartheta}(k\xi) \leq 1 - \lambda \} \subseteq \{ \xi \in \mathbb{R}^+ : \inf_{(v,\vartheta) \in H \times K} \mathcal{F}_{v-\vartheta}(\xi) \leq 1 - \lambda \}.$$

Hence,

$$\frac{1}{k} \{ \xi \in \mathbb{R}^+ : \inf_{\vartheta \in K} \mathcal{F}_{v_0-\vartheta}(\xi) \leq 1 - \lambda \} \subseteq \{ \xi \in \mathbb{R}^+ : \inf_{(v,\vartheta) \in H \times K} \mathcal{F}_{v-\vartheta}(\xi) \leq 1 - \lambda \}.$$

Thus,

$$\sup\{\xi \in \mathbb{R}^+ : \inf_{\vartheta \in K} \mathcal{F}_{v_0-\vartheta}(\xi) \leq 1 - \lambda\} \leq k \sup\{\xi \in \mathbb{R}^+ : \inf_{(v,\vartheta) \in H \times K} \mathcal{F}_{v-\vartheta}(\xi) \leq 1 - \lambda\}. \quad (4.2)$$

By (4.1) and (4.2), we deduce

$$\sup_{\vartheta \in K} \sup\{\xi \in \mathbb{R}^+ : \mathcal{F}_{v_0-\vartheta}(\xi) \leq 1 - \lambda\} \leq k \sup_{(v,\vartheta) \in H \times K} \sup\{\xi \in \mathbb{R}^+ : \mathcal{F}_{v-\vartheta}(\xi) \leq 1 - \lambda\}.$$

Therefore,

$$\delta_{p_\lambda}(v_0, K) \leq k \delta_{p_\lambda}(H, K).$$

Likewise, we prove that

$$\delta_{p_\lambda}(\vartheta_0, H) \leq k \delta_{p_\lambda}(H, K).$$

Therefore, this concludes the proof.

Theorem 4.1. *Given a nonempty weakly compact and convex pair $(\mathcal{U}, \mathcal{V})$ within a probabilistic normed space $(\mathcal{H}, \mathfrak{F}, \min)$, which possess the probabilistic uniformly normal structure, and for any fixed $\xi \in \mathbb{R}^+$, $\mathcal{F}(\xi) : \mathcal{H} \rightarrow [0, 1]$ is weakly lower semi-continuous. Consider a cyclic probabilistic relatively nonexpansive mapping denoted $T : \mathcal{U} \cup \mathcal{V} \rightarrow \mathcal{U} \cup \mathcal{V}$. Thus, there exists $(v_1, \vartheta_1) \in \mathcal{U} \times \mathcal{V}$ such that*

$$\begin{aligned} \sup\{\xi \in \mathbb{R}^+ : \mathcal{F}_{v_1-Tv_1}(\xi) \leq 1 - \lambda\} &= \sup\{\xi \in \mathbb{R}^+ : \mathcal{F}_{\vartheta_1-T\vartheta_1}(\xi) \leq 1 - \lambda\} \\ &= \inf_{(v,\vartheta) \in \mathcal{U} \times \mathcal{V}} \sup\{\xi \in \mathbb{R}^+ : \mathcal{F}_{v-\vartheta}(\xi) \leq 1 - \lambda\}. \end{aligned}$$

Proof. Via Lemma 4.1 and Lemma 4.2, the mapping T fulfill all conditions of Theorem 3.1 where $(\mathcal{H}, \{p_\lambda\}_\lambda)$ is the locally convex space correspond to the probabilistic normed space $(\mathcal{H}, \mathfrak{F}, \min)$. Therefore, Theorem 3.1 implies the conclusion.

Definition 4.4. *We call a probabilistic normed space $(\mathcal{H}, \mathfrak{F}, \min)$ probabilistic strictly convex if, for any $v, \vartheta \in \mathcal{H}, v \neq \vartheta$, there exists $k > 1$ so that*

$$\mathcal{F}_{\frac{v+\vartheta}{2}}(\xi) \geq \min\{\mathcal{F}_v(k\xi); \mathcal{F}_\vartheta(k\xi)\}, \text{ for all } \xi \geq 0.$$

Definition 4.5. *A nonempty pair $(\mathcal{U}, \mathcal{V})$ of a probabilistic normed space $(\mathcal{H}, \mathfrak{F}, \min)$ exhibits the d -property, if and only if, for any $(v_1, \vartheta_1) \in \mathcal{U} \times \mathcal{V}$ and $(v_2, \vartheta_2) \in \mathcal{U} \times \mathcal{V}$ such that if*

$$\mathcal{F}_{v_1-\vartheta_1}(\xi) = \mathcal{F}_{v_2-\vartheta_2}(\xi) = \sup_{(v,\vartheta) \in \mathcal{U} \times \mathcal{V}} \mathcal{F}_{v-\vartheta}(\xi) \text{ implies } \mathcal{F}_{v_1-v_2}(\xi) = \mathcal{F}_{\vartheta_1-\vartheta_2}(\xi)$$

Lemma 4.3. *Every probabilistic strictly convex normed space is strictly convex in the corresponding locally convex space.*

Theorem 4.2. *A probabilistic normed space $(\mathcal{H}, \mathfrak{F}, \min)$ is considered probabilistic strictly convex, if and only if it possesses the d -property.*

Theorem 4.3. *Given a nonempty weakly compact and convex pair $(\mathcal{U}, \mathcal{V})$ within a probabilistic normed space $(\mathcal{H}, \mathfrak{F}, \min)$, which possess the probabilistic uniformly normal structure, and for any fixed $\xi \in \mathbb{R}$, $\mathcal{F}(\xi) : \mathcal{H} \rightarrow [0, 1]$ is weakly lower semi-continuous in $u \in \mathcal{H}$. Consider a probabilistic noncyclic nonexpansive mapping denoted $T : \mathcal{U} \cup \mathcal{V} \rightarrow \mathcal{U} \cup \mathcal{V}$. Then, there exists $(\theta, \eta) \in \mathcal{U} \times \mathcal{V}$ such that*

$$T\theta = \theta, T\eta = \eta \text{ and } \mathcal{F}_{\theta-\eta}(\xi) = \sup_{(v,\vartheta) \in \mathcal{U} \times \mathcal{V}} \mathcal{F}_{v-\vartheta}(\xi) = \inf_{(v,\vartheta) \in \mathcal{U} \times \mathcal{V}} \sup\{\xi \in \mathbb{R} : \mathcal{F}_{v-\vartheta}(\xi) \leq 1 - \lambda\}.$$

References

- [1] K. Chaira & S. Lazaiz, "Best proximity point theorems for cyclic relatively ρ -nonexpansive mappings in modular spaces", *Abstract and Applied Analysis*, vol.(2018), 8 pages.
- [2] SS. Chang, BS. Lee, YJ. Cho, YQ. Chen, SM. Kang & JS. Jung, "Generalized contraction mapping principles and differential equations in probabilistic metric spaces". *Proc Am Math Soc*, Vol.(1996); pages: 2367-2376.
- [3] J. Chi, C. Yan, M. Zhang & B. Ju, "On locally convex probabilistic normed spaces". *J Inequal Appl*, Vol.(2016), pages: 1-13.
- [4] A. Eldred, W. Kirk & P. Veeramani, "Proximal normal structure and relatively nonexpansive mappings". *Studia Mathematica*, Vol.(2005), pages: 283-293.
- [5] N. E. El Harmouchi, K. Chaira, & E. M. Marhrani, "Fixed Point Theorems for Noncyclic Monotone Relatively-Nonexpansive Mappings in Modular Spaces". *International Journal of Mathematics and Mathematical Sciences*, Vol.(2020), pages: 1-8.
- [6] R. Espínola, "A new approach to relatively nonexpansive mappings". *Proceedings of the American Mathematical Society*, Vol.(2008), pages: 1987-1995.
- [7] K. Fan, "Extensions of two fixed point theorems of F. E. Browder". *Math Z.* Vol.(1969), pages: 234-240.
- [8] M. Gabeleh, "Cyclic relatively nonexpansive mappings in convex metric spaces". *Miskolc Mathematical Notes*, Vol.(2015), pages: 133-144.
- [9] J. B. Prolla, "Fixed point theorems for set valued mappings and existence of best approximations". *Numer Funct Anal Optim.*, Vol.(1982), pages: 449-455.
- [10] V. S. Raj & P. Veeramani, "Best proximity pair theorems for relatively nonexpansive mappings", In *Applied General Topology*, vol.(2009), pages 21-28.
- [11] S. Reich, "Approximate selections, best approximations, fixed points and invariant sets". *J Math Anal Appl.*, vol.(1978), pages: 104-113.
- [12] B. Saadaoui, S. Lazaiz & M. Aamri, "On best proximity point theorems in locally convex spaces endowed with a graph", *International Journal of Mathematics and Mathematical Sciences*, vol.(2020), pages: 17.
- [13] V. M. Sehgal & S. P. Singh, "A generalization to multifunctions of Fan's best approximation theorem". *Proc Am Math Soc.*, vol.(1988), pages: 534-537.
- [14] V. M. Sehgal & S. P. Singh, "A theorem on best approximations". *Numer Funct Anal Optim.*, Vol.(1989), pages: 181-184.
- [15] M. Shams & S. M. Vaezpour, "Best approximation on probabilistic normed spaces". *Chaos, Solitons & Fractals*, vol.(2009), pages: 1661-1667.
- [16] A. N. Sherstnev, "On the notion of a random normed space". *Dokl Akad Nauk SSSR* 1963;149; pages: 280-283.
- [17] H. D. Vuong, "A fixed point theorem for nonexpansive mappings in locally convex spaces". *Vietnam J. Math.*, Vol.(2006), pages: 149-155.