

Volterra operator norms : a brief survey

THOMAS RANSFORD¹

À la mémoire de Mohamed Zarrabi

ABSTRACT. In this expository article, we discuss the evaluation and estimation of the operator norms of various functions of the Volterra operator.

Mathematics Subject Classification (2020). 47G10, 47A12.

Key words and phrases. Volterra operator, Operator norm, Numerical range.

1. Introduction

Let $V : L^2[0, 1] \rightarrow L^2[0, 1]$ be the Volterra operator, defined by

$$Vf(x) := \int_0^x f(t) dt. \quad (1)$$

It is well known that V is a compact, quasi-nilpotent operator, and that its adjoint is given by

$$V^*f(x) = \int_x^1 f(t) dt. \quad (2)$$

In this expository article, we discuss the problem of evaluating or estimating the operator norm of $f(V)$ for various functions f . This subject arises naturally in connection with the (apparently still open) problem of determining whether the Volterra operator satisfies the Crouzeix conjecture: namely whether $\|f(V)\| \leq 2 \max_{W(V)} |f(z)|$ for every polynomial $f(z)$, where $W(V)$

Received July 29, 2022 - Accepted: November 26, 2022.

© The Author(s) 2023. This article is published with open access by Sidi Mohamed Ben Abdallah University.

¹Département de mathématiques et de statistique, Université Laval, Québec (QC) G1V 0A6, Canada.

e-mail : thomas.ransford@mat.ulaval.ca (Corresponding Author).

Supported by grants from NSERC and the Canada Research Chairs program .

denotes the numerical range of V . We shall return to discuss Crouzeix's conjecture at the end of the article.

Our goal is to highlight the principal results and some of their more unexpected consequences. There are few proofs, and the focus is narrow. We concentrate exclusively on the classical Volterra operator, as opposed to more general integral operators. Also, we consider the action of V only on $L^2[0, 1]$, as opposed to $L^p[0, 1]$ for other values of p , or indeed more general function spaces. Despite these restrictions, we hope that the material is sufficiently rich to pique the reader's interest.

2. The norm of V

We begin with the problem of computing the operator norm of V itself. It is obvious, just by looking at the definition of V , that it is a bounded operator and that $\|V\| \leq 1$. The true value of $\|V\|$ was found by Halmos [8, Problem 188].

Theorem 2.1. $\|V\| = 2/\pi \approx 0.6366$.

Exceptionally, we shall give the proof. Not only is it short and beautiful, but it provides the basis for much of what is to follow. We shall refer to this general technique as Halmos's method.

Proof of the theorem. In fact, we shall compute all the singular values σ of V . This amounts to solving the eigenvalue problem $V^*Vf = \sigma^2 f$. Writing this out in full, we seek solutions $f \in L^2[0, 1]$ to the integral equation

$$\int_x^1 \int_0^x f(t) dt = \sigma^2 f(x) \quad \text{a.e.} \quad (3)$$

If $\sigma = 0$, then it is easily deduced that $f = 0$ a.e., which is forbidden for an eigenfunction. So we may suppose that $\sigma > 0$. In that case (3) expresses f as an indefinite integral, so, after changing f on a set of measure zero, we may suppose that f is continuous. Feeding this information back into (3) and using a bootstrap argument, we deduce that in fact f is C^∞ .

The integral equation (3) is thus equivalent to the following ODE with two boundary conditions:

$$\begin{cases} \sigma^2 f'' + f = 0, \\ f(1) = 0, \\ f'(0) = 0. \end{cases}$$

The general solution to the ODE is $f(x) = A \cos(x/\sigma) + B \sin(x/\sigma)$. The boundary condition $f'(0) = 0$ implies that $B = 0$. Finally, the boundary condition $f(1) = 0$ implies that $\cos(1/\sigma) = 0$, in other words, that $\sigma = \sigma_n := 2/((2n-1)\pi)$ for some integer $n \geq 1$. These σ_n then are the singular values of V , and, in particular, the largest, namely $\sigma_1 = 2/\pi$, is the norm of V . $\square$

As is well known, this theorem has an interpretation as a Wirtinger-type inequality.

Corollary 2.2. *If $g \in C^1[0, 1]$ and $g(0) = 0$, then*

$$\int_0^1 |g(t)|^2 dt \leq \frac{4}{\pi^2} \int_0^1 |g'(t)|^2 dt. \quad (4)$$

The constant $4/\pi^2$ is best possible.

Proof. If we set $f := g'$, then $g = Vf$, and the desired inequality becomes $\|Vf\|_2 \leq (2/\pi)\|f\|_2$. The theorem implies that this inequality is indeed true. Equality holds in (4) if $g(t) = \sin(\pi t/2)$. $\square$

3. Higher powers

Once $\|V\|$ has been calculated, a natural next step is to consider $\|V^2\|$. Following the Halmos method, one is led to consider a fourth-order ODE with four boundary conditions. The computations are thus more complicated, but still feasible. According to Thorpe [19], the result was known to Halmos, who learned of it from A. Brown. The earliest reference that we have been able to track down is in a paper of Horgan [9].

Theorem 3.1. $\|V^2\| = \rho^{-2}$, where ρ is the smallest positive solution to

$$(\cos \rho)(\cosh \rho) + 1 = 0.$$

Numerically, $\|V^2\| \approx 0.2844$.

For $\|V^3\|$, there is a similar, but even more complicated result, due to Böttcher and Widom [4].

Theorem 3.2. $\|V^3\| = \rho^{-3}$, where ρ is the smallest positive solution to

$$\cos \rho \cosh(\rho\sqrt{3}) + 8 \cos(\rho/2) \cosh(\rho\sqrt{3}/2) + (2 + \cos \rho)^2 = 0.$$

Numerically, $\|V^3\| \approx 0.09081$.

Numerical values of $\|V^n\|$ were tabulated in Lao–Whitley [13] (for $1 \leq n \leq 10$), in Thorpe [19] (for $n = 1, 2, 3, 5, 10, 15, 20$), and in Böttcher–Dörfler [2] (for $1 \leq n \leq 10$ and $n = 20, 40, 60$). In [13], they were obtained by a discrete approximation process, and in [19] they were deduced from earlier results concerning sharp constants in Kolmogorov-type inequalities [7]. In [2] they arose from some very accurate general estimates for $\|V^n\|$. We shall now describe how these estimates were derived.

One way to obtain an upper bound for the operator norm of V^n is to compute its Hilbert–Schmidt norm $\|V^n\|_{HS}$. It turns out that this is quite easy to do.

Theorem 3.3. For all $n \geq 1$,

$$\|V^n\|_{HS} = \frac{1}{2(n!)} \frac{1}{\sqrt{1 - 1/(2n)}}.$$

Proof. The Hilbert–Schmidt norm of an integral operator on $L^2[0, 1]$ with kernel K is just the norm of K in $L^2([0, 1] \times [0, 1])$. An easy induction shows that

$$V^n f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt.$$

Hence

$$\|V^n\|_{HS}^2 = \int_0^1 \int_0^x \frac{(x-t)^{2n-2}}{(n-1)!^2} dt dx = \frac{1}{(2n)(2n-1)(n-1)!^2}.$$

$\square$

What about lower bounds for $\|V^n\|$? The computations of Lao and Whitley led them to conjecture that $\|V^n\| \sim 1/(2n!)$ as $n \rightarrow \infty$ (in other words, that $\|V^n\| \sim \|V^n\|_{HS}$). This conjecture was proved, more-or-less simultaneously, by Thorpe [19], Little and Reade [14], and Kershaw [10], all by different methods. Böttcher and Dörfler [2] later gave a particularly slick proof of this result based on the following elementary inequality, valid for all Hilbert–Schmidt operators T :

$$\frac{\|T^*T\|_{HS}}{\|T\|_{HS}} \leq \|T\| \leq \|T\|_{HS}. \quad (5)$$

The point is that, taking $T := V^n$, both V^n and $(V^n)^*V^n$ are integral operators, so it is feasible to compute their Hilbert–Schmidt norms. This leads to the following theorem [2, Theorem 1.2].

Theorem 3.4. *For all $n \geq 1$,*

$$\frac{1}{2n!} \frac{1}{1 - 1/(2n)} \leq \|V^n\| \leq \frac{1}{2n!} \frac{1}{\sqrt{1 - 1/(2n)}}. \quad (6)$$

Of course, this result implies that $\|V^n\| \sim 1/(2n!)$. However, as pointed out in [2], this is not quite the end of the story. It turns out that neither of the bounds in (5) is optimal. By expressing the operator norms and Hilbert–Schmidt norms in terms of singular values, it is easy to show that

$$\|T\|^2 \leq \|T^*T\|_{HS} \quad \text{and} \quad \|T^*T\|_{HS}^2 \leq \left(\|T\|_{HS}^2 - \|T\|^2\right)^2 + \|T\|^4.$$

Applying these inequalities with $T := V^n$ eventually leads to an improvement of (6) that is tight enough to allow the tabulation of values of $\|V^n\|$ to an accuracy exceeding those in [13] and [19]. The exact statement of this improved inequality is quite complicated, so we do not give it here, but refer the interested reader to [2, Theorem 1.3]. We also mention the paper [3], where these ideas are developed further.

4. Linear polynomials

After powers of V , the natural next step is to consider linear polynomials of V , namely $V + \mu I$ where $\mu \in \mathbb{C}$. Using the Halmos approach, Lyubich and Tsedenbayar [16] determined the singular values of $I + \nu V$ for all $\nu \in \mathbb{C}$, and thence computed the operator norm of $\|I + \nu V\|$ [16, Theorems 2.2 and 2.3]. Evidently $\|V + \mu I\| = |\mu| \|I + (1/\mu)V\|$, so we can deduce the value of $\|V + \mu I\|$.

Theorem 4.1. *Let $\mu = \alpha + i\beta \in \mathbb{C}$. Then*

$$\|V + \mu I\| = \sqrt{\alpha^2 + 1/\rho^2},$$

where ρ is the unique positive number such that

$$\rho/(1 - \beta^2\rho^2) \in (0, \pi) \quad \text{and} \quad \cot\left(\frac{\rho}{1 - \beta^2\rho^2}\right) = \alpha\rho.$$

In the rest of this section, we explore some consequences of Theorem 4.1. Figure 1 is a contour plot of the function $\mu \mapsto \|V + \mu I\|$, computed using the theorem.

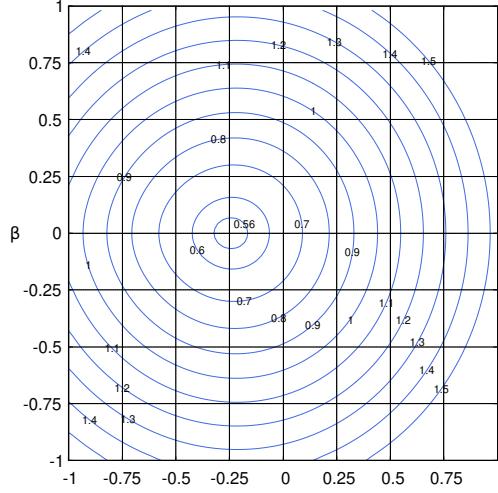


FIGURE 1. Contour plot of the function $\mu \mapsto \|V + \mu I\|$

Several features are immediately apparent. First, the function appears to be symmetric about the horizontal axis. This is obvious from Theorem 4.1, since $\|V + \mu I\|$ depends on β only through β^2 . It could also have been proved directly by exploiting the fact that V is unitarily equivalent to V^* via $V^* = U^* V U$, where $U f(x) := f(1 - x)$.

Another obvious feature in Figure 1 is the apparent existence of a minimum, attained near the point -0.25 , with value about 0.55 . The next theorem confirms this observation.

Theorem 4.2. *The minimum of $\|V + \mu I\|$ is attained at a unique point $\mu_0 \approx -0.2429626850$ and has value $\sqrt{\mu_0^2 - \mu_0} \approx 0.5495393994$. The exact value of μ_0 is $-1/\rho_0^2$, where ρ_0 is the unique solution in $(\pi/2, \pi)$ of the equation $\rho_0 + \tan \rho_0 = 0$.*

Proof. The existence of a minimum follows from a simple compactness argument. Its location may be determined using Lagrange multipliers. The details can be found in [17]. $\square$

It is perhaps tempting to believe that the function $\alpha \mapsto \|V + \alpha I\|$ is symmetric about $\alpha = \mu_0$, but in fact this is not true. For example, a calculation using Theorem 4.1 gives

$$\|V + (\mu_0 + 5)I\| = 5.265379338 \dots \quad \text{and} \quad \|V + (\mu_0 - 5)I\| = 5.253007667 \dots$$

The behaviour of $\|V + \mu I\|$ on the imaginary axis is more straightforward. There is even an explicit expression for $\|V + i\beta I\|$, generalizing Halmos's result Theorem 2.1. The following result is essentially [16, Corollary 2.4].

Theorem 4.3. *For all $\beta \in \mathbb{R}$,*

$$\|V + i\beta I\| = \frac{1}{\pi} + \sqrt{\beta^2 + \frac{1}{\pi^2}}.$$

Proof. Applying Theorem 4.1 with $\alpha = 0$, we have $\|V + i\beta I\| = 1/\rho$, where ρ is the unique positive solution of

$$\frac{\rho}{1 - \beta^2 \rho^2} \in (0, \pi) \quad \text{and} \quad \cot\left(\frac{\rho}{1 - \beta^2 \rho^2}\right) = 0.$$

These conditions imply that $\rho/(1 - \beta^2\rho^2) = \pi/2$, or, equivalently, $(1/\rho)^2 - (2/\pi)(1/\rho) - \beta^2 = 0$. Solving this equation for $1/\rho$ and substituting into the formula for $\|V + i\beta I\|$ gives the result. $\square$

It is also instructive to consider the function $\|I + \nu V\|$ for $\nu \in \mathbb{C}$. Figure 2 is a contour plot of the function $\nu \mapsto \|I + \nu V\|$.

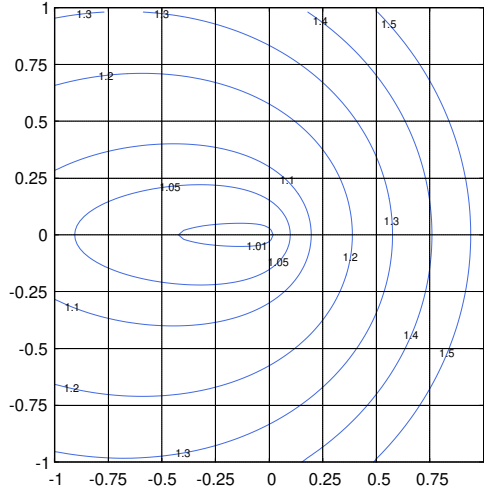


FIGURE 2. Contour plot of the function $\nu \mapsto \|I + \nu V\|$

In this case, the minimum is attained at $\nu = 0$. Indeed, since the operator norm is always at least as large as the spectral radius, we clearly have $\|I + \nu V\| \geq 1$ for all $\nu \in \mathbb{C}$. The following result, due to Lyubich and Tsedenbayar [16, Corollary 2.5], shows that, in fact, the minimum is attained only at 0.

Theorem 4.4. $\|I + \nu V\| > 1$ for all $\nu \in \mathbb{C} \setminus \{0\}$.

Proof. Let $\mu = \alpha + i\beta$. By Theorem 4.1, $\|V + \mu I\| = \sqrt{\alpha^2 + 1/\rho^2}$, for some positive ρ with $\rho/(1 - \beta^2\rho^2) \in (0, \pi)$. Then $\beta^2\rho^2 < 1$, so $\|V + \mu I\| > \sqrt{\alpha^2 + \beta^2} = |\mu|$. The result follows. $\square$

Lyubich and Tsedenbayar conjectured in [16] that, if $p(z)$ is any non-constant polynomial with $p(0) = 1$, then $\|p(V)\| > 1$. Their conjecture was later refuted by ter Elst and Zemánek [18]. We shall return to this topic in §6.

Finally in this section, we mention that Khadkhoo and Tsedenbayar have computed $\|\alpha V + \beta V^*\|$ for $\alpha, \beta \in \mathbb{C}$. For the details, we refer to [11].

5. Resolvents

The singular values of an invertible operator permit us to determine not only its norm (the largest singular value) but also the norm of its inverse (the reciprocal of the smallest singular value). Thus the computation the singular values of $I + \nu V$ carried out by Lyubich and Tsedenbayar in [16] leads to a formula for the norms of $(I + \nu V)^{-1}$ and hence of the resolvent operators $(V - \mu I)^{-1}$. They did not state the formula explicitly, so we give it here.

Theorem 5.1. Let $\mu = \alpha + i\beta \in \mathbb{C} \setminus \{0\}$.

(i) If $\alpha > \beta^2$, then

$$\|(V - \mu I)^{-1}\| = \frac{1}{\sqrt{\alpha^2 - \gamma^2}},$$

where γ is the unique solution in $(0, \alpha)$ to the equation

$$\coth\left(\frac{\gamma}{\beta^2 + \gamma^2}\right) = \frac{\alpha}{\gamma}.$$

(ii) If $\alpha = \beta^2$, then

$$\|(V - \mu I)^{-1}\| = \frac{1}{\alpha}.$$

(iii) If $\alpha < \beta^2$ and $\beta \neq 0$, then

$$\|(V - \mu I)^{-1}\| = \frac{1}{\sqrt{\alpha^2 + \gamma^2}},$$

where γ is the unique positive number such that $\gamma/(\beta^2 - \gamma^2) \in (0, \pi)$ and

$$\cot\left(\frac{\gamma}{\beta^2 - \gamma^2}\right) = \frac{\alpha}{\gamma}.$$

(iv) If $\alpha < 0$ and $\beta = 0$, then

$$\|(V - \mu I)^{-1}\| = 1/|\alpha|.$$

Figure 3 illustrates the various curves and regions in (α, β) -space in which the parts (i)–(iv) of the theorem apply. The origin is excluded.

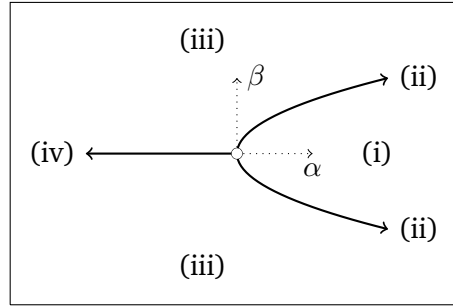
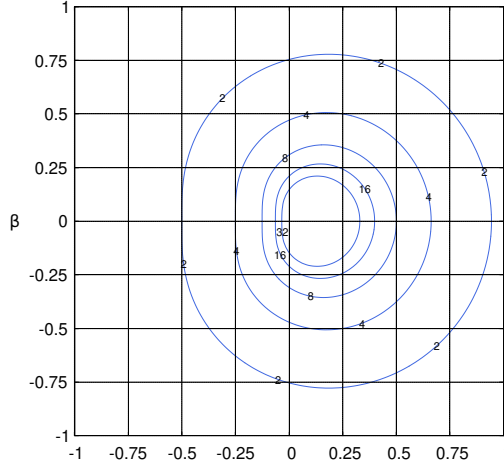


FIGURE 3. Curves and regions (i)–(iv) in Theorem 5.1

Why is Theorem 5.1 so much more complicated than Theorem 4.1? After all, they emerge from the same basic calculation, namely the Halmos method applied to the operator $I + \nu V$. It turns out this method leads to second-order linear ODE whose characteristic quadratic equation has negative discriminant in the case studied in Theorem 4.1, but may have positive, zero or negative discriminant in the case of Theorem 5.1, whence the extra complexity in the latter case.

Figure 4 is a contour plot of the function $\mu \mapsto \|(V - \mu I)^{-1}\|$, computed using the theorem.



Proof. By part (i) of Theorem 5.1, if $\alpha > 0$, then

$$\|(V - \alpha I)^{-1}\| = \frac{1}{\sqrt{\alpha^2 - \gamma^2}},$$

where γ is the unique solution in $(0, \alpha)$ to $\coth(1/\gamma) = \alpha/\gamma$. As $\alpha \rightarrow 0^+$, we also have $\gamma \rightarrow 0^+$, so

$$\gamma = \alpha \tanh(1/\gamma) = \alpha(1 + O(e^{-2/\gamma})) = \alpha(1 + O(e^{-2/\alpha})).$$

Hence, as $\alpha \rightarrow 0^+$,

$$\begin{aligned} \|(V - \alpha I)^{-1}\| &= \frac{1}{\sqrt{\alpha^2 - \gamma^2}} = \frac{1}{\sqrt{\alpha^2 - \alpha^2 \tanh^2(1/\gamma)}} = \frac{1}{\alpha} \cosh(1/\gamma) \\ &= \frac{1}{\alpha} \cosh\left(\frac{1}{\alpha}(1 + O(e^{-2/\alpha}))\right) = \frac{e^{1/\alpha}}{2\alpha} + O\left(\frac{e^{-1/\alpha}}{\alpha^2}\right). \end{aligned}$$

□

6. Polynomials of higher degree

The article [16] describes a general strategy, following the Halmos method, for computing $\|p(V)\|$ for general polynomials p . However, in practice, this rapidly becomes very complicated as the degree of p increases. As already remarked in §3, even when $p(z) = z^d$, it has only been carried out in detail for $1 \leq d \leq 3$. For more general polynomials, the situation is even more complex.

To our knowledge, the only case that has been investigated in detail, beyond the linear case, is when p is a quadratic polynomial with real coefficients. This was done recently in [17], which describes both the theoretical background and the numerical implementation. Figure 5 is a contour plot of the function $(\sigma, \tau) \mapsto \|V^2 + \sigma V + \tau I\|$, computed using the results in [17].

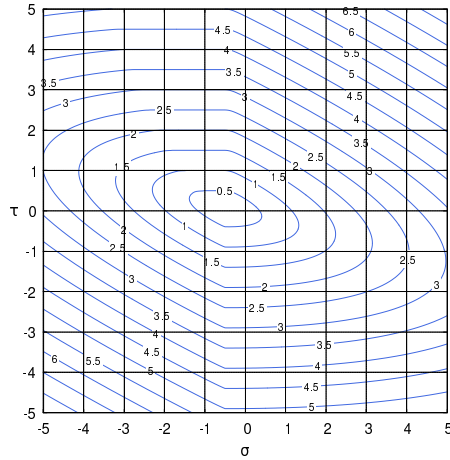
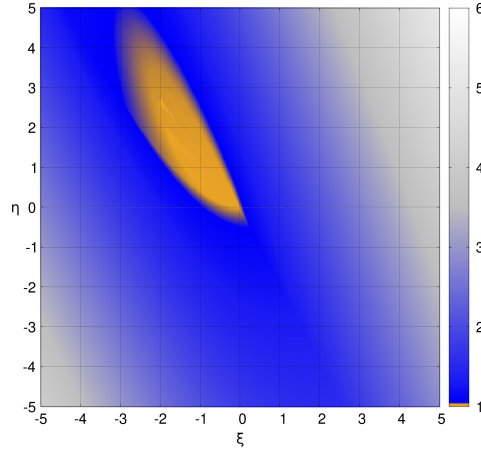


FIGURE 5. Contour plot of the function $(\sigma, \tau) \mapsto \|V^2 + \sigma V + \tau I\|$.

Just as in the case of linear polynomials, it is also instructive to consider $\|I + \xi V + \eta V^2\|$. Figure 6 is a colour plot of this norm as a function of (ξ, η) . We have chosen a colour plot since this helps bring to light the regions where the function attains a minimum.

FIGURE 6. Colour plot of the function $(\sigma, \tau) \mapsto \|V^2 + \sigma V + \tau I\|$

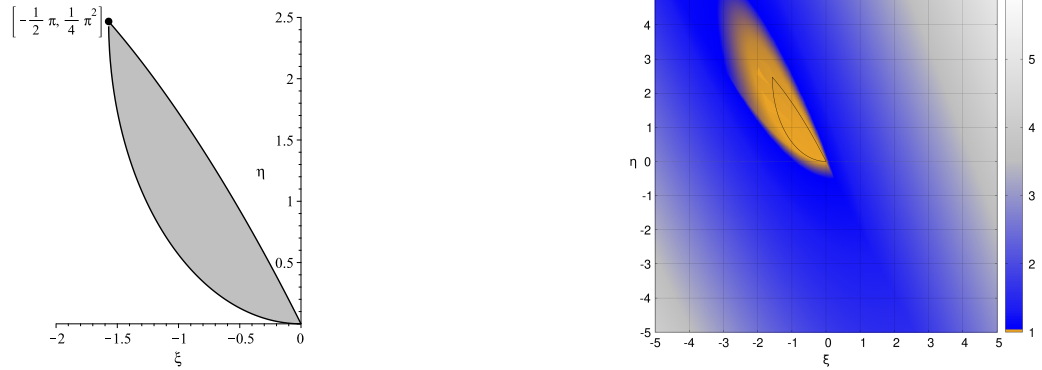
Of course, we always have $\|I + \xi V + \eta V^2\| \geq 1$, since the norm is at least as large as the spectral radius. Figure 6 suggests that there is a region of pairs (ξ, η) around $(-1, 1)$ where $\|I + \xi V + \eta V^2\| = 1$. The following result established in [17], proves this rigorously.

Theorem 6.1. *Suppose that $\xi, \eta \in \mathbb{R}$ satisfy*

$$\begin{cases} \xi \leq 0, \\ 4\eta^2/\pi^2 - 2\eta + \xi^2 \leq 0, \\ (4\xi/\pi^2 - 1)\eta^2 + (\xi^2 - 2\xi)\eta + \xi^3 \geq 0. \end{cases} \quad (7)$$

Then $\|I + \xi V + \eta V^2\| = 1$. In particular, we have $\|I - V + V^2\| = 1$.

The left-hand side of Figure 7 illustrates the range of pairs (ξ, η) satisfying the inequalities (7). The right-hand side of Figure 7 superimposes this set on the plot of the function $(\xi, \eta) \mapsto \|I + \xi V + \eta V^2\|$ previously obtained.

FIGURE 7. Range of pairs (ξ, η) satisfying (7)

Remarks. (1) As mentioned in §4, it was conjectured in [16] that $\|p(V)\| > 1$ whenever $p(z)$ is a non-constant polynomial with $p(0) = 1$. Theorem 6.1 disproves this conjecture. It is not the first such example: ter Elst and Zemánek [18], using a different method, showed that $\|1 - bV + bV^2\| = 1$ for all $b \in [0, 1/8]$. However, they were unable to show that this equality extends to $b = 1$ (see [18, Remark 3.3]). Not only does Theorem 6.1 establish this, but it provides a whole open region of coefficients (ξ, η) around $(-1, 1)$ for which $\|I + \xi V + \eta V^2\| = 1$.

(2) As remarked in [18], the set of polynomials $p(z)$ with the property that $\|p(V)\| = p(0) = 1$ is both a convex set and a multiplicative semigroup. Thus, the quadratic polynomials that arise in Theorem 6.1 can be used to generate a much larger set of polynomials p with $\|p(V)\| = p(0) = 1$.

7. Numerical range and Crouzeix's conjecture

We begin this section with a simple result.

Proposition 7.1. *The Volterra operator is accretive, i.e.,*

$$\operatorname{Re}\langle Vf, f \rangle \geq 0 \quad (f \in L^2[0, 1]).$$

Proof. Let $f \in L^2[0, 1]$. From the formulas (1) and (2), we have

$$(V + V^*)f = \int_0^x f + \int_x^1 f = \int_0^1 f.$$

Hence

$$2 \operatorname{Re}\langle Vf, f \rangle = \langle (V + V^*)f, f \rangle = \left| \int_0^1 f \right|^2 \geq 0. \quad \square$$

Despite its simplicity, this result has some interesting consequences. Both the following corollaries are well-known properties of accretive operators.

Corollary 7.2. *If $0 \leq s \leq t$, then*

$$\|I + tV\| \geq \|I - sV\| \quad \text{and} \quad \|(I - sV)(I + tV)^{-1}\| \leq 1.$$

Proof. Using Proposition 7.1, we see that $\|(I + tV)f\|_2^2 \geq \|(I - sV)f\|_2^2$ for all $f \in L^2[0, 1]$. The result follows easily from this. $\square$

Corollary 7.3. *If $t \geq 0$, then*

$$\|e^{-tV}\| \leq 1.$$

Proof. By Corollary 7.2 and induction, for all $t \geq 0$ and all $n \geq 1$ we have

$$\|(I + (t/n)V)^{-n}\| \leq \|(I + (t/n)V)^{-1}\|^n \leq 1.$$

Let $n \rightarrow \infty$ to get the result. $\square$

We recall that the *numerical range* of a bounded Hilbert-space operator T is defined by

$$W(T) := \left\{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \right\}.$$

It is a bounded convex set whose closure contains the spectrum of T . If further T is a compact operator and if $0 \in W(T)$, then $W(T)$ is compact (see [8, Problem 213]). Hence $W(V)$ is compact.

Proposition 7.1 may be interpreted as saying that the numerical range of V is contained in the half plane $\{z : \operatorname{Re} z \geq 0\}$. The following result identifies $W(V)$ precisely.

Theorem 7.4. $W(V)$ is the convex compact set bounded by the vertical segment $[-i/2\pi, i/2\pi]$ and the curves

$$t \mapsto \left(\frac{1 - \cos t}{t^2} \right) \pm i \left(\frac{t - \sin t}{t^2} \right) \quad (t \in [0, 2\pi]).$$

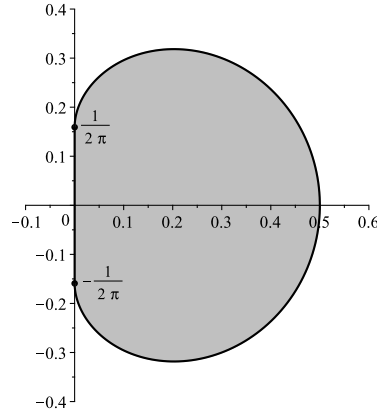


FIGURE 8. Numerical range of V

This result is folklore. It appears in the middle of a discussion in [8, p.113–114], where it is attributed to A. Brown. We sketch briefly how it may be obtained.

Proof. According to a well-known formula of Lumer [15, Lemma 12], if T is a bounded operator on a Hilbert space, then

$$\sup \left\{ \operatorname{Re} z : z \in W(T) \right\} = \lim_{r \rightarrow 0^+} \frac{\|I + rT\| - 1}{r}.$$

Taking $T = e^{i\theta}V$, and using the asymptotic formula

$$\|I + re^{i\theta}V\| = 1 + r \frac{\sin \theta}{2\theta} + O(r^2) \quad (r \rightarrow 0^+),$$

(which can be deduced, for example, from Theorem 4.1), we obtain

$$\sup \left\{ \operatorname{Re} z : z \in e^{i\theta}W(V) \right\} = \frac{\sin \theta}{2\theta} \quad (\theta \in [-\pi, \pi]).$$

This identifies the support function of $W(V)$. It turns out to be the same as the support function of the set described in the statement of Theorem 7.4. The details of this calculation can be found for example in [12, p.105]. As both sets are convex and compact, they must be equal. $\square$

Knowing the precise location of the numerical range of V yields further information about the norms of polynomials of V . Crouzeix and Palencia [6] showed that, for every Hilbert-space operator T , the numerical range of T is a $(1 + \sqrt{2})$ -spectral set. Explicitly, for every polynomial

p , we have

$$\|p(T)\| \leq (1 + \sqrt{2}) \sup_{z \in W(T)} |p(z)|. \quad (8)$$

In particular, this holds if $T = V$.

It is not known whether the constant $(1 + \sqrt{2})$ in (8) is sharp. There are simple examples to show that the best constant is at least 2. According to a celebrated conjecture of Crouzeix [5], the best constant is exactly 2, i.e., for every operator T and every polynomial p ,

$$\|p(T)\| \leq 2 \sup_{z \in W(T)} |p(z)|. \quad (9)$$

This conjecture is still unproved. Apparently, it is still an open question even in the special case when $T = V$, the Volterra operator.

Where might we find a counter-example when $T = V$? A heuristic argument suggests trying real-quadratic polynomials. Indeed, if there is a polynomial p such that $\|p(V)\| / \max_{W(V)} |p| > 2$, then, since $\|V^n\| \rightarrow 0$ very rapidly with n , one might expect lower powers of V to dominate in $p(V)$, and since $W(V)$ is symmetric about the x -axis, one might expect the same to be true of the roots of p . The authors of [17] computed the ratio $\|p(V)\| / \max_{W(V)} |p|$ for real-quadratic polynomials. Figure 9 illustrates the results of these computations.

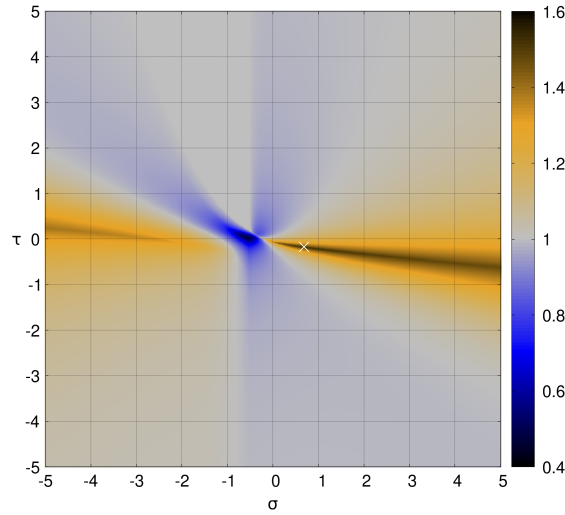


FIGURE 9. Colour plot of $(\sigma, \tau) \mapsto \|V^2 + \sigma V + \tau I\| / \max_{W(V)} |z^2 + \sigma z + \tau|$

The largest value found for $\|V^2 + \sigma V + \tau I\| / \max_{W(V)} |z^2 + \sigma z + \tau|$ was 1.5278, attained when $(\sigma, \tau) = (0.685, -0.167)$ (marked with a white cross on the figure). In this case, $\|V^2 + \sigma V + \tau I\| = 0.6501$ and $\max_{W(V)} |z^2 + \sigma z + \tau| = 0.4255$.

8. Conclusion

We have seen that Halmos's technique for determining the norm of the Volterra operator V can be extended to compute the norm of linear polynomials in V and their inverses. It also yields the exact value of the norms of V^2 and V^3 .

For higher powers of V , there are very precise upper and lower bounds for the norm, showing in particular that $\|V^n\| \sim 1/(2n!)$ as $n \rightarrow \infty$.

For more general polynomials p , we have the upper bound $\|p(V)\| \leq (1 + \sqrt{2}) \max_{W(V)} |p|$, where $W(V)$ denotes the numerical range of V . It is still an open problem whether $(1 + \sqrt{2})$ can be replaced by 2 in this inequality. The highest value of the ratio $\|p(V)\| / \max_{W(V)} |p|$ found to date is 1.5278, lending support to the belief that the Crouzeix conjecture holds for V .

Acknowledgement

The author expresses his sincere gratitude to Nathan Walsh for producing many of the figures in this article, namely those containing the contour plots and colour plots of various functions. The computations for these figures were performed using Haskell, and the graphics produced using Gnuplot.

References

- [1] Allan, G. R., *Power-bounded elements and radical Banach algebras*. Linear operators (Warsaw, 1994), 38, 9–16, Polish Acad. Sci. Inst. Math., Warsaw, 1997.
- [2] Böttcher, A. and Dörfler, P., *On the best constants in inequalities of the Markov and Wirtinger types for polynomials on the half-line*. Linear Algebra Appl. Vol 430, N 4, 1057–1069, 2009.
- [3] Böttcher, A. and Dörfler, P., *Weighted Markov-type inequalities, norms of Volterra operators, and zeros of Bessel functions*. Math. Nachr. Vol. 283, No. 1, 40–57, 2010.
- [4] Böttcher, A. and Widom, H., *On the eigenvalues of certain canonical higher-order ordinary differential operators*. J. Math. Anal. Appl. Vol. 322, No. 2, 990–1000, 2006.
- [5] Crouzeix, M., *Bounds for analytical functions of matrices*. Integral Equations Operator Theory. Vol. 48, No. 4, 461–477, 2004.
- [6] Crouzeix, M. and Palencia, C., *The numerical range is a $(1 + \sqrt{2})$ -spectral set*. SIAM J. Matrix Anal. Appl. Vol. 38, No. 2, 649–655, 2017.
- [7] Franco, Z. M. and Kaper, H. G. and Kwong, M. K. and Zettl, A., *Best constants in norm inequalities for derivatives on a half-line*. Proc. Roy. Soc. Edinburgh Sect. A. Vol. 100, No. 1-2, 67–84, 1985.
- [8] Halmos, P. R., *A Hilbert space problem book*. Encyclopedia of Mathematics and its Applications. Vol. 17, Edition Second, 1982.
- [9] Horgan, C. O., *A note on a class of integral inequalities*. Proc. Cambridge Philos. Soc. Vol. 74, 127–131, 1973.
- [10] Kershaw, D., *Operator norms of powers of the Volterra operator*. J. Integral Equations Appl. Vol. 11, No. 3, 351–362, 1999.
- [11] Khadkhuu, L. and Tsedenbayar, D., *A note about Volterra operator*. Math. Slovaca. Vol. 68, No. 5, 2018.
- [12] Khadkhuu, L. and Tsedenbayar, D., *On the numerical range and numerical radius of the Volterra operator*. Izv. Irkutsk. Gos. Univ. Ser. Mat. Vol. 24, pp. 102–108, 2018.
- [13] Lao, N. and Whitley, R., *Norms of powers of the Volterra operator*. Integral Equations Operator Theory. Vol. 27, No. 4, 419–425, 1997.
- [14] Little, G. and Reade, J. B., *Estimates for the norm of the n th indefinite integral*. Bull. London Math. Soc. Vol. 30, No. 5, 539–542, 1998.

- [15] Lumer, G., *Semi-inner-product spaces*. Trans. Amer. Math. Soc. Vol. 100, 29–43, 1961.
- [16] Lyubich, Y. and Tsedenbayar, D., *The norms and singular numbers of polynomials of the classical Volterra operator in $L_2(0, 1)$* . Studia Math. Vol. 199, No. 2, 171–184, 2010.
- [17] Ransford, T. and Walsh, N., *Norms of polynomials of the Volterra operator*, J. Math. Anal. Appl. 517 (2023), paper no. 126626.
- [18] Ter Elst, A. F. M. and Zemánek, J., *Contractive polynomials of the Volterra operator*. Studia Math. Vol. 240, No. 3, 201–211, 2018.
- [19] Thorpe, B., *The norm of powers of the indefinite integral operator on $(0, 1)$* . Bull. London Math. Soc. Vol. 30, No. 5, 543–548, 1998.