sciendo
International Conference KNOWLEDGE-BASED ORGANIZATION
Vol. XXX No 3 2024

A NEW METHOD FOR SOLVING THE INTEGRALS OF THE MOHR-MAXWELL METHOD FOR DISPLACEMENTS CALCULUS OF BENT STRAIGHT BARS

Vasile NĂSTĂSESCU*, Ghiță BÂRSAN**

***"Ferdinand I" Military Technical Academy, Bucharest, Romania**
****"Nicolae Bălcescu" Land Forces Academy, Sibiu, Romania**
nastasescuv@gmail.com

Abstract: *The paper presents a new way of solving the integrals that appear in the Mohr-Maxwell energy method for calculating the displacements or rotations of straight bars subjected to bending. The method proposed by the authors, studied and tested for many years in the Military Technical Academy in the Strength of Materials group led by Col. Prof. Vasile Palacianu, eliminates the need to build effort diagrams. To solve the integral on a certain domain, a formula is applied that takes into consideration only the value of the moments at the ends of the integration interval. The well-known restriction for Veresceaghin grapho-analytical integration is maintained: on the integration domain, the variation of the bending moment produced by the generalized load equal to unity must vary linearly. Therefore, the method proposed by the authors cannot be applied to curved bars. Our new method can be used for a beam and also for a beams system. After the presentation of the theoretical foundations of the method and the establishment of the calculation relationship in three variants: without the load distributed over the integration interval, with the load uniformly distributed and with the load distributed according to a linear law, some edifying examples are presented, which highlight the efficiency of the method and the modality for work.*

Keywords: bending moment, distribuited load, load of unity

1. Introduction

In mechanical engineering, the problem of calculating the deformations of bars (displacements, rotations) subjected to different demands is a frequently encountered one in practice and is often not a simple problem.

Of course, the numerical calculation using the finite element method, the Galerkin free element method and other methods (meshfree/meshless type) is a correct, efficient and relatively easy alternative to apply by those specialized in using the respective methods. In addition to these, a dedicated software is also required.

Analytical or grapho-analytical calculation is an often accessible path. Over time, specialists have developed numerous methods, such as direct integration, the conjugate beam method (reciprocal), the method of parameters at the origin, Mohr's grapho-analytical method, the Klebsch s.a. method. Energy methods (perhaps the most effective) such as Castigliano or Mohr-Maxwell also belong to the category of analytical methods. For the latter, there is a grapho-analytic method of integration, known with the name of Veresceaghin method. Each of the methods mentioned above has advantages and disadvantages, which are more accentuated or not, in relation to other methods, but especially to some personal professional skills.

DOI: 10.2478/kbo-2024-0092

The method of solving the respective integral consists in applying a calculation relation that takes into account only the moment values at the ends of the integration interval and the extreme values of the distributed load. The relationship proposed by the authors takes into consideration three situations regarding the distributed load: zero value, constant value and linear variation. The method proposed by the authors can be applied to straight bars individually or within structures, such as frames.

2. Fundamentals of the New Method

In the Mohr-Maxwell method, an integral of the form appears,

$$I(x) = \int_l M(x) \cdot m(x) \cdot \mathrm{dx} \qquad (1)$$

which must be resolved. Let it be a portion of a straight bar, loaded on the integration interval with a distributed load, with linear variation, as in Figure 1.

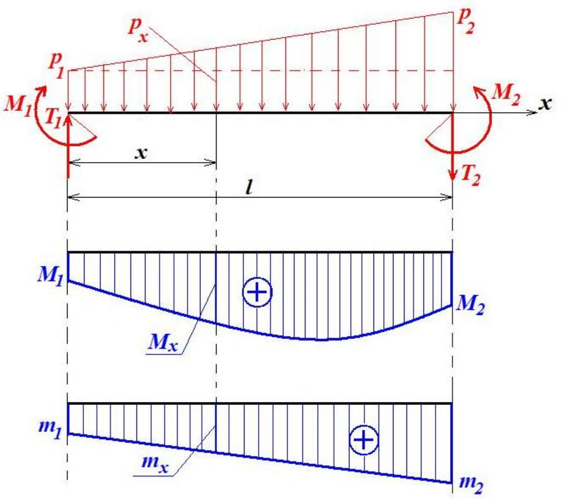


Figure 1: Calculus model

In a section, having x coordinate (Figure 1), on the integration interval, we can write:

$$p_x = p_1 + (p_2 - p_1) \cdot \frac{x}{l}$$

(2)

$$M_x = M_1 + T_1 \cdot x - p_1 \frac{x^2}{2} - (p_x - p_1) \cdot \frac{x^2}{6} \qquad (3)$$

$$m_x = m_1 + (m_2 - m_1) \cdot \frac{x}{a} \qquad (4)$$

Taking into account relations (2)...(4), relation (1) is written:

$$I = \int_0^l M \cdot m \cdot \mathrm{dx} = \left[M_1 + T_1 x - p_1 \frac{x^2}{2} - (p_x - p_1) \frac{x^2}{6} \right] \cdot \left[m_1 + (m_2 - m_1) \frac{x}{a} \right] \cdot \mathrm{dx} \qquad (5)$$

In the relation (5), the shear force T_1 will be expressed as a function of the applied loads and the efforts M_1 and M_2, writing a balance equation with respect to the right end of the bar:

$$M_1 + T_1 \cdot l - M_2 - p_1 \cdot \frac{l^2}{2} - (p_2 - p_1) \cdot \frac{l^2}{6} = 0 \qquad (6)$$

From equation (6), shear force T_1 is:

$$T_1 = \frac{M_2 - M_1}{l} + p_1 \cdot \frac{a}{2} + (p_2 - p_1) \cdot \frac{a}{6}$$

(7)

By relations (2) and (7) into equation (1):

$$I = \int_l M m \mathrm{dx} = \int_0^l \left[M_1 + (M_2 - M_1) \frac{x}{l} - \right.$$
$$+ p_1 \frac{l}{2} x + (p_2 - p_1) \frac{l}{6} x - p_1 \frac{x^2}{2} - p_1 \frac{x^2}{6} -$$
$$\left. - (p_2 - p_1) \frac{x^3}{6l} + p_1 \frac{x^2}{6} \right] \times \qquad (8)$$
$$\times \left[m_1 + (m_2 - m_1) \frac{x}{l} \right] \cdot \mathrm{dx}$$

$$I=\int_0^l\Big[M_1m_1+M_1m_2\frac{x}{l}-M_1m_1\frac{x}{l}+$$
$$+M_2m_1\frac{x}{l}-M_1m_1\frac{x}{l}+\left(M_2-M_1\right)\cdot$$
$$\cdot\left(m_2-m_1\right)\frac{x^2}{l^2}+\frac{p_1}{2}\left(lx-\frac{l}{3}x-x^2-\right. \qquad (9)$$
$$\left.-\frac{x^2}{3}+\frac{x^3}{3l}+\frac{x^2}{3}\right)\cdot\left(m_1+m_2\frac{x}{l}-m_1\frac{x}{l}\right)$$
$$+\frac{p_2}{2}\left(\frac{lx}{3}-\frac{x^3}{3l}\right)\cdot\left(m_1+m_2\frac{x}{l}-m_1\frac{x}{l}\right)\Big]\mathrm{dx}$$

By reorganizing the terms in relation (8), we can write relation (9).

$$I=\int_0^l\Big\{M_1m_1+M_1m_2\frac{x}{l}-M_1m_1\frac{x}{l}+$$
$$+M_2m_1\frac{x}{l}-M_1m_1\frac{x}{l}+\left(M_2-M_1\right)\cdot$$
$$\cdot\left(m_2-m_1\right)\frac{x^2}{l^2}+\frac{p_1}{2}\left(\frac{2l}{3}x-x^2+\frac{x^3}{3l}\right)\cdot \qquad (10)$$
$$\cdot\left[m_1\left(1-\frac{x}{l}\right)+m_2\frac{x}{l}\right]+\frac{p_2}{2}\left(\frac{l}{3}x-\frac{x^3}{3l}\right)\cdot$$
$$\cdot\left[m_1\left(1-\frac{x}{l}\right)+m_2\frac{x}{l}\right]\Big\}\cdot\mathrm{dx}$$

$$I=\int_0^l\Big\{M_1m_1+M_1m_2\frac{x}{l}+M_2m_1\frac{x}{l}$$
$$-2M_1m_1\frac{x}{l}+\left(M_2-M_1\right)\left(m_2-m_1\right)\frac{x^2}{l^2}+$$
$$+\frac{p_1m_1}{2}\left(\frac{2l}{3}x-x^2+\frac{x^3}{3l}-\frac{2}{3}x^2+\frac{x^3}{l}-\right.$$
$$\left.-\frac{x^4}{3l^2}\right)+\frac{p_1m_2}{2}\left(\frac{2}{3}x^2-\frac{x^3}{l}+\frac{x^4}{3l^2}\right)+ \qquad (11)$$
$$+\frac{p_2m_1}{2}\left(\frac{l}{3}x-\frac{x^3}{3l}-\frac{x^2}{3}+\frac{x^4}{3l^2}\right)+$$
$$+\frac{p_2m_2}{2}\left(\frac{x^2}{3}-\frac{x^4}{3l^2}\right)\Big]\cdot\mathrm{dx}$$

After integration,

$$I=M_1m_1l+\frac{M_1m_2}{2}l+\frac{M_2m_1}{2}l-$$
$$-M_1m_1l+\left(M_2-M_1\right)\left(m_2-m_1\right)\frac{l}{3}+$$
$$+\frac{p_1m_1}{2}\left(+\frac{l^3}{12}-2\frac{l^3}{9}+\frac{l^3}{4}-\frac{l^3}{15}\right)+ \qquad (12)$$
$$+\frac{p_1m_2}{2}\left(2\frac{l^3}{9}-\frac{l^3}{4}+\frac{l^3}{15}\right)+\frac{p_2m_1}{2}\left(\frac{l^3}{6}-\right.$$
$$\left.-\frac{l^3}{12}-\frac{l^3}{9}+\frac{l^3}{15}\right)+\frac{p_2m_2}{2}\left(\frac{l^3}{9}-\frac{l^3}{15}\right)$$

Finally, after some simple mathematical operations, relation (12) becomes:

$$I=\frac{2M_1m_1+M_1m_2+M_2m_1+2M_2m_2}{6}l+$$
$$+\frac{\left(8m_1+7m_2\right)p_1+\left(7m_1+8m_2\right)p_2}{360}l^3 \qquad (13)$$

If the distributed load is a constant one, $p_1=p_2=p$, the calculus relation (13) of the integral I is:

$$I=\frac{2M_1m_1+M_1m_2+M_2m_1+2M_2m_2}{6}l+$$
$$+\frac{\left(m_1+m_2\right)}{24}pl^3 \qquad 14)$$

If there is no distributed load on the integration domain (the beam or a portion of beam), $p_1=p_2=0$, the calculus relation of the integral I becomes:

$$I=\frac{2M_1m_1+M_1m_2+M_2m_1+2M_2m_2}{6}l \qquad (15)$$

In practice, relations (13), (14) and (15) must be retained and applied; they practically replace the graphic construction of the effort diagrams and the respective integration rule.
Integral I is calculated over the entire length of the structure. It can have several integration domains, for which the conditions in Figure 1 are met. The parameter l from relations (13)...(15) represents the length of the integration domain.

3. Case Studies

In this chapter, some simple problems, but significant in structure and loading, are presented, to illustrate the use of relations (13), (14) or (15) as the case may be. The validation of the results of the application of these relationships can be done by anyone, using any of the known methods.

3.1. Problem 1

A simply supported beam is subjected to a concentrated force P just in the middle of the beam. Calculate de vertical displacement of the section 3, where the force P is placed.

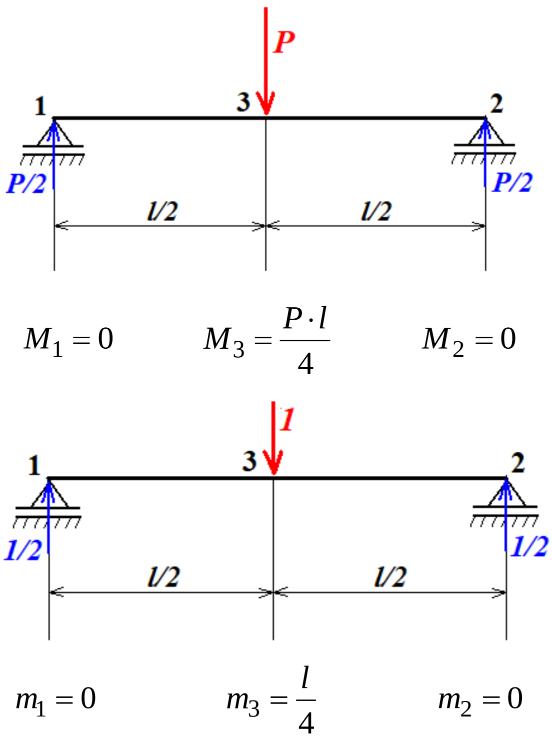


Figure 2: The calculus model by energetic method Mohr-Maxwell

$$EI \cdot \delta_3 = \int_0^l M \cdot m \cdot \mathrm{d}x = I \tag{16}$$

$$I = I^{(1-3)} + I^{(3-2)} \tag{17}$$

$$I^{(1-3)} = \frac{0+0+0+2 \cdot \frac{Pl}{4} \cdot \frac{l}{4}}{6} \cdot \frac{l}{2} = \frac{Pl^3}{96} \tag{18}$$

$$I^{(3-2)} = \frac{2 \cdot \frac{Pl}{4} \cdot \frac{l}{4} + 0+0+0}{6} \cdot \frac{l}{2} = \frac{Pl^3}{96}$$

(19)

Applying relation (17), it results:

$$I = \frac{Pl^3}{48}$$

(20)

The result of the problem is:

$$\delta_3 = \frac{Pl^3}{48EI} \tag{21}$$

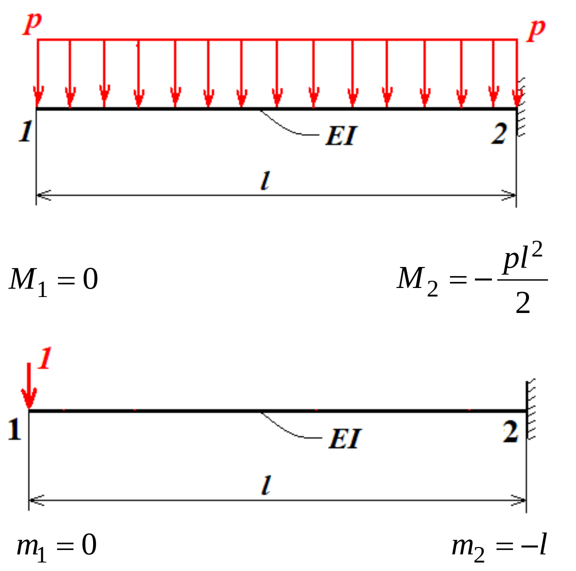
If the concentrated force P is applied at distances a, respectively b from the support *1*, respectively *2*, following an analogous procedure is obtained:

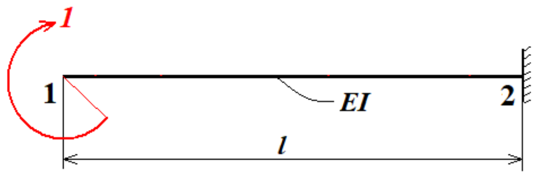
$$\delta_3 = \frac{Pa^2b^2}{3lEI} \tag{22}$$

The results obtained are identical to those obtained by any of the well-known methods.

3.2. Problem 2

The cantilever beam (bending rigidity EI) is loaded by a constant line load p. Find the vertical displacement and the rotation of section 1.





$m_1 = 1$ $m_2 = 1$

Figure 3: The calculus models for displacement and rotation calculus

Using relation (14), integral value is:

$$I = \frac{0+0+0+2\left(-\frac{pl^2}{2}\right)(-l)}{6} l + \quad (23)$$

$$+ \frac{(-l)}{24} pl^4$$

$$I = pl^4\left(\frac{1}{6} - \frac{1}{24}\right) = \frac{pl^4}{8} \quad (24)$$

$$\delta_1 = \frac{pl^4}{8EI} \quad (25)$$

$$I = \frac{0+0+\left(-\frac{pl^2}{2}\right)1+2\left(-\frac{pl^2}{2}\right)1}{6} l + \quad (26)$$

$$+ \frac{2}{24} pl^3 = pl^3\left(-\frac{3}{12} + \frac{1}{12}\right) = -\frac{pl^3}{6}$$

$$\varphi_1 = -\frac{pl^3}{6EI} \quad (27)$$

The minus sign shows us that the rotation of section 1 occurs in the opposite direction to the direction of the concentrated moment equal to unity, applied to section 1.

3.3. Problem 3

The structure presented in the Figure 4 is loaded by the force P. Calculate the vertical displacement of the section 1 of the structure.

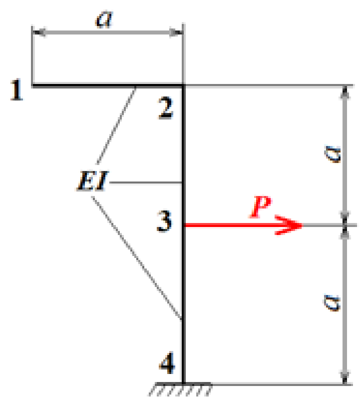


Figure 4: A 2D beam structure

The efforts at the ends of the integration intervals (Figure 4) are:

$M_1 = 0 \quad M_2 = 0 \quad M_3 = 0 \quad M_4 = P \cdot a$

The efforts at the ends of the integration intervals are (figure 5) are:

$m_1 = 0 \quad m_2 = -a \quad m_3 = -a \quad m_4 = -a$

Then,

$$I = I^{(1-2)} + I^{(2-3)} + I^{(3-4)} \quad (28)$$

$$I = 0 + 0 + \frac{Pa\cdot(-a) + 2Pa\cdot(-a)}{6} a \quad (29)$$

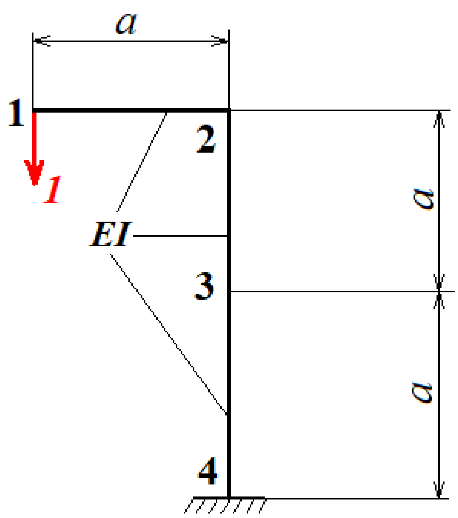


Figure 5: Calculus model

The efforts at the ends of the integration intervals are:

$$I = -\frac{Pa^3}{2} \quad (30)$$

$$\delta_1 = -\frac{Pa^3}{2EI} \quad (31)$$

3.4. Problem 4

Find the vertical displacement of section 2 of the beam, loaded like in the Figure 6.

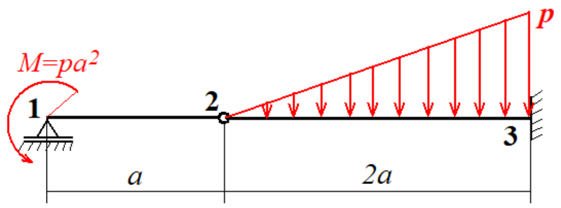
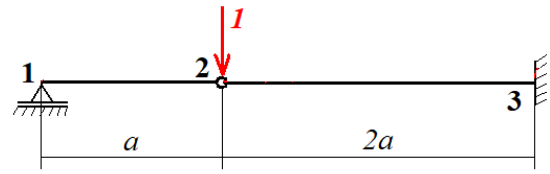


Figure 6: The loaded structure

$$M_1 = -pa^2 \qquad M_2 = 0 \qquad M_3 = \frac{4}{3}pa^2$$



$$m_1 = 0 \qquad m_2 = 0 \qquad m_3 = -2a$$

Figure 7: Calculus model

$$I = I^{(1-2)} + I^{(2-3)} = 0 + I^{(2-3)} \quad (32)$$

$$I = \frac{0+0+0+2\frac{4pa^2}{3}(-2a)}{6}2a + \frac{[0+8(-2a)]p}{360}(2a)^3 \quad (33)$$

$$I = \frac{2\cdot 4\cdot(-2)\cdot 2}{18}pa^4 - \frac{8\cdot 2\cdot 8}{360}pa^4 = -pa^4\left(\frac{16}{9}+\frac{128}{360}\right) \quad (34)$$

$$I = -\frac{32}{15}pa^4 = -2{,}13\cdot pa^4$$

(35)

$$\delta_2 = -\frac{32}{15}\cdot\frac{pa^4}{EI} = -2{,}13\frac{pa^4}{EI} \quad (36)$$

All results are confirmed by any other calculation methods. Besides this validation of the relationship found, the case studies also present the practical way of working, using accessible and representative examples.

4. Conclusions

The method proposed by us, which we would like to call the Palacianu Method, brings substantial simplifications to the use of the Mohr-Maxwell energy method, for calculating the displacements or rotations of straight bars subjected to bending.

The method is equally efficient in the case of application for straight bar systems. It can be seen that the method proposed by us is easier to use and more efficient than the grapho-analytical method of integration, known as the Veresceaghin Method.

Our method is likely to avoid mistakes, which can appear by drawing effort diagrams.

References List

[1] Bia, C., Ille, V., Soare, M., V., *Rezistenţa materialelor şi teoria elasticităţii,* Editura Didactică şi Pedagogică, Bucureşti, 1983.

[2] Buzdugan, Gh., *Proiectarea de rezistenţă în construcţia de maşini,* Editura Academiei Române, Bucureşti, 1998.

[3] Mocanu, D., R., *Rezistenţa materialelor,* Editura Tehnică, Bucureşti, 1980.

[4] Năstăsescu, V., Bârsan, Gh., *Rezistenţa materialelor - Probleme, vol. I,* Editura Academiei Tehnice Militare, Bucureşti, 1997.

[5] Năstăsescu, V., Bârsan, Gh., *Rezistenţa materialelor - Probleme, vol. II,* Editura Academiei Tehnice Militare, Bucureşti, 1997.

[6] Năstăsescu V., Ștefan, A., Mocian, O., *Rezistenţa materialelor, Vol. I,* Editura Academiei Tehnice Militare „Ferdinand I”, Bucureşti, 2020, ISBN: 978-973-640-311-8.
[7] Năstăsescu V., Ștefan, A., Mocian, O., *Rezistenţa materialelor, Vol. II,* Editura Academiei Tehnice Militare „Ferdinand I”, Bucureşti, 2021, ISBN.
[8] Posea, N., *Rezistenţa materialelor,* Editura Didactică şi Pedagogică, Bucureşti, 1979.
[9] Posea, N., ş. a., *Rezistenţa materialelor - Probleme,* Editura Ştiinţific şi Enciclopedică, Bucureşti, 1986.