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Q-BERNSTEIN-SCHURER OPERATORS ON A TRIANGLE WITH ONE CURVED SIDE

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Abstract: *We construct q-Bernstein-Schurer type operators defined on a triangle with one curved side. They are extensions of the Bernstein-Schurer type operators, given by Schurer F., to the case of a curved domain. There are constructed the univariate q-Bernstein-Schurer type operators and their product operator and are studied some approximation properties of these operators.*

Keywords: **q-integers, Bernstein operator, Bernstein-Schurer operators**

1. Introduction

"Let $T_h = \{(x,y) \in R \mid x \geq 0, y \geq 0, x+y \leq h\}$, for $h \in R_+$, be the standard triangle" [3].

In paper [3] authors introduce some interpolation operators on triangle with one curved side.

"They consider a standard triangle, $\widetilde{T}_h$, having the vertices $V_1 = (h,0), V_2 = (0,h)$ and $V_3 = (0,0)$, two straight sides Γ_1, Γ_2, along the coordinate axes, and the third side Γ_3 (opposite to the vertex V_3), which is defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h.$" [3]

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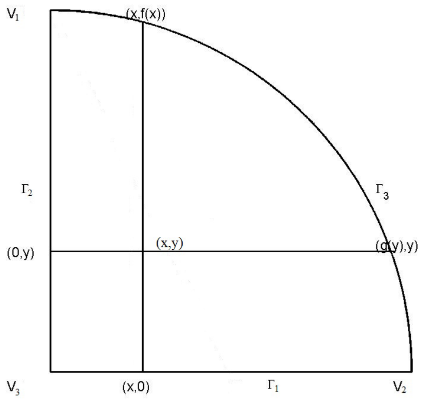

Figure 1: Triangle $\widetilde{T}_h$

In [1] the authors construct Bernstein-type operators B_m^x and B_n^y on the triangle $\widetilde{T}_h$. The most important approximation properties of the operators B_m^x and B_n^y are contained in the following theorem

Theorem 1 "*[1] If F is a real-valued function then*

1. $B_m^x F = F$ on $\Gamma_2 \cup \Gamma_3$,
2. $B_n^y F = F$ on $\Gamma_1 \cup \Gamma_3$,
3. $(B_m^x e_{ij})(x,y) = x^i y^j, i = 0,1, j \in N$,
   $$(B_m^x e_{2j})(x,y) = \left[x^2 + \frac{x(g(y)-x)}{m} \right] y^j,$$
4. $(B_n^y e_{ij})(x,y) = x^i y^j, i \in N, j = 0,1$,
   $$(B_n^y e_{i2})(x,y) = x^i \left[y^2 + \frac{y(f(x)-y)}{n} \right].$$

"

In 1962, Schurer F., (see [6]) introduced and studied the linear positive operators "$\tilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, defined for any function $f \in C([0,1+p])$ by:

$$(\tilde{B}_{m,p} f)(x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f\left(\frac{k}{m}\right) \qquad (1)$$

where

$$\tilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} \qquad (2)$$

are the so called "fundamental polynomials" of Bernstein-Schurer.
Note that in (1) and (2), p denotes a non-negative fixed integer.
Clearly, for p=0 the operators $\tilde{B}_{m,0}$ are the classical operators of Bernstein." [6]

We recall elements of q-Calculus, see, e.g., [4].

“For any fixed real number $q>0$, the q-integer $\left[k\right]_q$, for $k\in N$ is defined as:

$$\left[k\right]_q=\begin{cases}(1-q^k)/(1-q), & q\neq 1\\ \text{k} & , q=1\end{cases}\qquad (3)$$

Set $\left[0\right]_q=0$. The q-factorial $\left[k\right]_q!$ and q-binomial coefficient $\begin{bmatrix}n\\k\end{bmatrix}_q$ are defined as follows

$$\left[k\right]_q!=\begin{cases}[k]_q[k-1]_q...[1]_q, & \text{k=1,2,...}\\ 1 & \text{, k=0}\end{cases}\qquad (4)$$

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}(0\leq k\leq n)\qquad (5)$$

The q-analogue of $(x-a)^n$ is the polynomial

$$(x-a)^n=\begin{cases}1 & \text{, if } n=0\\ (x-a)(x-qa)...(x-q^{n-1}a), & \text{if } n\geq 1\end{cases}$$

”

In [5] the authors introduce a generalized q-Bernstein-Schurer operators as follows:

“

$$\tilde{B}_{m,p}(f,q,x)=\sum_{k=0}^{m+p}\begin{bmatrix}m+p\\k\end{bmatrix}_q x^k\cdot \prod_{s=0}^{m+p-k-1}(1-q^s x)f\left(\frac{[k]_q}{[m]_q}\right)$$

for $q\in(0,1)$, $m\in N$ and $f\in C\left([0,1+p]\right)$, p fixed.”

2. q-Bernstein-Schurer Operators

For $m,n\in N, p,r\in N_0$ we consider the following extensions of the Bernstein-Schurer operators given in [1]:

$$(\tilde{B}^x_{m,p}F)(x,y)=\sum_{i=0}^{m+p}\tilde{p}_{m,i}(x,y)F\left(\frac{i}{m}g(y),y\right)$$

with

$$\tilde{p}_{m,i}(x,y)=\binom{m+p}{i}\left(\frac{x}{g(y)}\right)^i\left(1-\frac{x}{g(y)}\right)^{m+p-i}$$

respectively,

$$(\tilde{B}^y_{n,r}F)(x,y)=\sum_{j=0}^{n+r}\tilde{r}_{n,j}(x,y)F\left(x,\frac{j}{n}f(x)\right)$$

with

$$\tilde{r}_{n,j}(x,y)=\binom{n+r}{j}\left(\frac{y}{f(x)}\right)^j\left(1-\frac{y}{f(x)}\right)^{n+r-j}$$

where

$$\Delta^x_m=\left\{\frac{i}{m}g(y)\,|\,i=\overline{0,m}\right\}$$ and

$$\Delta^y_n=\left\{\frac{j}{n}f(x)\,|\,j=\overline{0,n}\right\}$$

are uniform partitions of the intervals $[0,f(x)]$ and $[0,g(y)]$.

For $q\in(0,1)$ we construct the q-Bernstein-Schurer operators as follows:

$$(\tilde{B}^x_{m,p}F)(x,y,q)=\sum_{i=0}^{m+p}\tilde{p}_{m,i}(x,y,q)\cdot F\left(\frac{[i]_q}{[m]_q}g(y),y\right)$$

with

$$\tilde{p}_{m,i}(x,y,q)=\begin{bmatrix}m+p\\ i\end{bmatrix}_q\left(\frac{x}{g(y)}\right)^i\cdot \prod_{s=0}^{m+p-i-1}\left(1-q^s\frac{x}{g(y)}\right)$$

and

$$(\tilde{B}^y_{n,r}F)(x,y,q)=\sum_{j=0}^{n+r}\tilde{p}_{n,r}(x,y,q)\cdot F\left(x,\frac{[j]_q}{[n]_q}f(x)\right)$$

with

$$\tilde{r}_{n,j}(x,y,q)=\begin{bmatrix}n+r\\ j\end{bmatrix}_q\left(\frac{y}{f(x)}\right)^j\cdot \prod_{t=0}^{n+r-j-1}\left(1-q^t\frac{y}{f(x)}\right)$$

We obtain the following results regarding this operators:
If $e_{ij}(x,y)=x^iy^j\ (0\le i+j\le 2, i,j\in N)$ are the test functions, we have the following equalities:

1. $(\tilde{B}^x_{m,p}e_{00})(x,q)=1$;
2. $(\tilde{B}^x_{m,p}e_{10})(x,q)=\dfrac{x[m+p]_q}{[m]_q}$;
3. $(\tilde{B}^x_{m,p}e_{20})(x,q)=\dfrac{[m+p]_q}{[m]^2_q}\cdot\left[[m+p]_qx^2+x[g(y)-x]\right]$
4. $(\tilde{B}^x_{m,p}e_{11})(x,q)=\dfrac{xy[m+p]_q}{[m]_q}$.

The approximation properties of the operator $\tilde{B}^y_{n,r}$ are constructed in a similar way.
By direct computation we can obtain the product of the operators $\tilde{B}^x_{m,p}$ and $\tilde{B}^y_{n,r}$, given by the operator:

$$(\tilde{B}_{m,n,p,r}F)(x,y,q)=\sum_{i=0}^{m+p}\sum_{j=0}^{n+r}\tilde{p}_{m,i}(x,y,q)\cdot\tilde{r}_{n,j}(x,y,q)\cdot F\left(\frac{[i]_q}{[m]_q}g(y),\frac{[j]_q}{[n]_q}f(x)\right)$$

Also by direct computation we obtain the approximation properties of the this operator .
If $e_{ij}(x,y)=x^iy^j\ (0\le i+j\le 2, i,j\in N)$ are the test functions, we have the following equalities :

1. $(\tilde{B}_{m,n,p,r}e_{00})(x,y,q)=1$
2. $(\tilde{B}_{m,n,p,r}e_{10})(x,y,q)=\dfrac{x[m+p]_q}{[m]_q}$;
3. $(\tilde{B}_{m,n,p,r}e_{01})(x,y,q)=\dfrac{y[n+r]_q}{[n]_q}$;
4. $(\tilde{B}_{m,n,p,r}e_{11})(x,y,q)=xy\dfrac{[m+p]_q}{[m]_q}\dfrac{[n+r]_q}{[n]_q}$
5. $(\tilde{B}_{m,n,p,r}e_{20})(x,y,q)=\dfrac{[m+p]_q}{[m]^2_q}\cdot\left[[m+p]_qx^2+x[g(y)-x]\right]$
6. $(\tilde{B}_{m,n,p,r}e_{02})(x,y,q)=\dfrac{[n+r]_q}{[n]^2_q}\cdot\left[[n+r]_qy^2+y[f(x)-y]\right]$

3. Conclusion

Because the Bernstein-Schurer operators are a particular case of Bernstein type operators we can use these type of operators both on triangles with straight and curved sides (being an extension).

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