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# The average and closure problem of turbulence theory resolved in random space

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**Abstract:** The problem above marked as resolved is more than a hundred years known as the closure problem of turbulence. Extending its name follows from below presented knowledge that to be its solution successful it is necessary to find an effective averaging tool enabling one to describe and smooth down any random turbulent field without any phenomenological limitations. To convince of necessity of such tool author in the article previously proved the non-differentiability of random fields of measurable turbulence characteristics. But the decisive momentum of his solution strategy arose from the idea that randomness is an autonomous factor of physical processes and, therefore, this property can be utilized as a property of independent variables of the governing PDEs. To realize this idea author picked random frequences of turbulent fluctuations. Author then postulated the dual property as well as bifunctionality hypothesis and found suitable constitutive equations enabling him: (i) to express the instantaneous behaving of any random vector and scalar turbulent fields; (ii) to average the non-linear N–S system for the thermally known turbulent flow over the characteristic domains in the 5–D random space; (iii) to close the averaged equations systems with the set of four relationships named the Energy Distribution Equations (EDE) as the key result of the closure process. The energy invariance principle was used to find a closing equation for the energy distribution factor**.** The resultant EDEs were successfully verified meanwhile by comparing them with data from four independent sources of experiments made in boundary layers of wind tunnel flows of high anisotropy. This closure problem solution was obtained without the use of any auxiliary parameters or assumptions of phenomenological or experimental origin. From the nature of EDEs it follows that all turbulent mean flows are always 3–Dimensional. The use of randomness autonomy as the property of independent variables at describing turbulent flows is not limited upon Newtonian fluids.

**Keywords:** Randomness as autonomous factor; 5–D random space; Energy distribution equations; Mathematical expectations; Energy distribution factor; Tensor of anisotropy.

## 1 INTRODUCTION

The problem of closure in turbulence occurs when trying to describe random processes of turbulent fluid flow by means of deterministic tools. These consist of the systems of partial differential equations (PDE), all of which are based on their respective conservation laws. The cause and circumstances of the problem are usually described as follows: if liquids or gases move slowly enough, the flow remains smooth and predictable. Increasing the kinetic energy of the movement beyond a certain limit will lead to a chaotic, turbulent flow and problems with describing it by means of the existing PDEs. When non-linear PDEs are adapted to turbulent flow by averaging, then new unknown parameters arise of the fluctuating non-linearities. The unknown means of the non-linearities are problematic because they create an unclosed system of the closed one and, thus, the closure problem. Its solution requires finding the missing physically justified relations between the new and the original flow parameters. Unsuccessful attempts to close the system by obtaining new relations through derivation and averaging operations directly from N-S equations have triggered the era of turbulence modelling. Turbulence models retrieve missing relations by evaluating their phenomenological manifestations. The domain of application of the numerical simulation of turbulence effects obtained by these models is demarcated therefore by the validity limits of the phenomenological assumptions on which they are based. The “direct numerical solution” of the Navier–Stokes (N–S) equations is based on the numerical treatment of PDEs as an alternative to turbulence models. Therefore, it should be assessed in the context of the issue of the existence of solutions to deterministic PDE systems when applying them to random turbulent phenomena.

The closure problem of turbulence arose after Reynolds (1895) published his crucial work involving his (apparent) turbulent stress tensor as an analogy to the viscous stress tensor. But his stress tensor had occurred to be a new unknown variable being generated by averaging as it is said above. The long absence of a universally accepted solution to the problem has inspired the development of turbulence modelling to extent, the sufficient info on which cannot be pressed even in a review article. The qualified review of the state of the discipline 30 years ago can be found in Lumley (1989) as well as in Ecke, R. (2005). Statistical fluid mechanics as the main basis for the study of turbulence are described in detail in the monograph by Monin and Yaglom (1975).

In Volume I of the book series on Advances in Fluid Mechanics edited by L. Debnath and D.N. Riahy (1998) the state of knowledge of turbulence was evaluated by examining two crucial aspects: the physical background of the phenomenon and the mathematical techniques used to describe it. There are considered the problems involved with the use of the N–S equations as the closure problem as well as the need for an explanation of phenomena, such as intermittency and coherent structures.

Here is suitable to write several quotations of notions which mostly influenced creation of strategy, objectives and title of this article. The first notions are picked out from the Barenblatt, G.I. (1996): "turbulence is considered with good reason to be the number one problem of contemporary classical physics..., it remains an open problem: none of the results available has been obtained from the first principles. Obtained results are based essentially on strong additional assumptions, which may or not be correct ".

A view of turbulence as a phenomenon that needs to be described using chaos theory can be found in the study by Li (2013), which also touches the main tool of statistical mechanics: namely, averaging (i.e., taking the mean flow as the mathematical expectation). Indeed, a quotation from his paper is particularly appropriate here: "Chaos is understood, but untamed as far as turbulence is considered, it is not known what kind of (method) of averaging should be used. The search for an effective description of turbulence started from Reynolds average of a stochastic signal. But Reynolds average is far from an effective description of turbulence. Its applying to chaos and turbulence leads to an unsolvable closure problem ".

Developments and advances in the knowledge of turbulence have been considerably influenced by Kolmogorov's cascade theory of isotropic turbulence (Kolmogorov, 1941a, b). He defined the length, time and velocity scales of turbulent vortices, together with a mean rate of energy dissipation and the relations between them, and these have remained the focus of turbulence research to the present day (see Hunt and Vassilicos, 1991). It needs to say: Inter-scale relations resulting from Kolmogorov's theory are used in this study to define the characteristic domains of averaging, but after author preceding extension of their validity upon the directional components of kinetic energy of non-isotropic turbulence.

## 2 ON THE GENESIS OF THE MEANS FOR DESCRIPTING RANDOM DYNAMICS OF REAL FLUIDS

The above-mentioned PDR systems were created by applying the physical laws of conservation to the process of their creation as tools for the deterministic description of the flow of real fluids. Given that the defining characteristic of this flow is the chaotic dynamics of a discontinuous microworld of molecules, it is useful to know the steps that make it possible to describe random phenomena by means of the deterministic tools of the field theory.

### 2.1 The non-differentiability of random fields of measurable turbulence characteristics

The randomness of turbulent flow characteristics is clearly demonstrated by experiments. It is manifested by the uncertainty of the values of the relevant parameters, resulting from the unpredictability of their occurrence. However, the continuity properties of functions as the deterministic tools of the field theory require the certainty of the values presented by these functions to be differentiable. This is clearly stated in *Vygodsky´s* continuity and differentiability conditions written in Sec. 1.6 below. But random fields of turbulence characteristics are non-differentiable because they do not meet the conditions § 231 and § 424 of the Sec. 2.6. There is proof of the non-differentiability of random turbulent fields and origin of the average and closure problem.

A direct consequence of the non-differentiability of the turbulent fields is the uncertainty of the partial derivatives, which, by the mere **i**ntention of using the corresponding PDEs to simulate a turbulent flow in any way, turns any closed PDE systems defined above in $G(\boldsymbol{x},t)$ into indeterminate, unclosed ones.

It coincides with the Reynolds decomposition which splits any random function $f$ in $G(\boldsymbol{x},t)$ into two dependent ones and doubles the number of unknowns in the respective PDE. This means that the property of the non-differentiability of turbulent fields justifie**s** the use of decomposition in "apparently" closed systems, because such use does not change the uncertain status of the system.

The main consequence of the proven non-differentiability of turbulent fields is that it does not allow turbulent flow to be described by existing "apparently" closed PDE systems of the Newtonian fluid dynamics without an aid of statistical mechanics.

### 2.2 The bifunctionality of position coordinates of the coordinate system

For the Euler expressing conservation laws the mathematical continuum concept is necessary. It is procured by the known relations defining the total (material) derivatives of dependent variables

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}, \quad \frac{dx_j}{dt} = u_j(\boldsymbol{x},t), \; i,j = 1,2,3 \qquad (2.1)$$

which at the same time are sources of problematic non-linearities. Nevertheless, we put (2.1) there as an old sample of the bifunctionality, the functional property we want effectively to apply.

The mutual independence of position coordinates $x_i$ among themself and with respect to time $t$ in the first equation of (2.1) is unconstrained. However, the same coordinates $x_i$ define in the second eq. of (2.1) the paths of mass, momentum and energy transfer in the role of dependent variables**,** evidencing by such a way property the of the bifunctionality of space coordinates.

Over and above after applying Reynolds' velocity decomposition in (2.1) one can adjust the second (reference) equations in (2.1) to an integrable form

$$u_i = \frac{dx_i}{dt} = \frac{d\overline{x_i}}{dt} + \frac{dx'_i}{dt} \quad \rightarrow \quad dx_i = d\overline{x_i} + dx'_i \qquad (2.2)$$

But this obtained can be integrated even back resulting in the decomposition of dependently variable

$$x_i = \overline{x_i} + x'_i \; at \; \frac{d\overline{x_i}}{dt} = \bar{u}_i \; and \; \frac{dx'_i}{dt} = u'_i \qquad (2.3)$$

It is important to emphasize that only thanks to this bifunctionality the $x_i$ given in (2.3) does not represent coordinates $x_i$ in the role of independent variables in partial derivations $\partial f/\partial x_i$. In opposite case it would imply the oscillation of the system of coordinates and the invalidity of the operations performed. In the case of turbulent flow, of course.

The possibility to apply bifunctionality as property of the coordinates $x_i$ has influenced the strategy at solving the average and closure problem in this study.

### 2.3 Sources and types of problematic non-linearities of turbulences in respective PDEs system

The respective PDEs system is a closed system involving seven dependent variables $f\,(x,t|)$. Five of them, i.e., three components $u_i$ of the velocity vector $u\,(u_i)$, density $\rho$, and temperature $T$ shall be considered as primary dependent variables, all governed by five equations of three conservation laws. Momentum, kinetic energy and internal energy belong to derived dependent variables, included pressure $p(\rho,T)$, given by the thermodynamic equation of state. All respective PDEs contain non-linearities as well as a momentum flux tensor $\rho u_i u_j$. Besides these, viscosity $\mu(T)$ and thermal conductivity $\varkappa(T)$ as properties of moving fluid are of experimental origin. Based on the laws of conservation PDEs can be presented in the common form of its left sides,

$$\rho \frac{df}{dt} = P_f \tag{2.4}$$

by means of notations in the conservation form suitable for dynamical systems by using the tools of differential calculus (2) and (2.1) and the properties of the mass conservation equation itself

$$\frac{d\rho}{dt} + \rho \mathrm{div}\boldsymbol{u} = 0 \rightarrow \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) = 0; \tag{2.5}$$

in which $\rho$ indicates the specific mass and $\boldsymbol{u}$ velocity vector. The second of the equations in (2.5) has already obtained the desired conservation form after eliminating the full derivation $d\rho/dt$ from the first one, by using the pair (2.0) and (2.1). Eq. (2.4) expresses the momentum conservation law if $f = \boldsymbol{u}$, or the energy conservation law if $f = \epsilon + E$. We obtain these conservation laws in the desired form by making the following adjustments based on the continuity equation and replacing $d(\rho f)/dt$:

$$\rho \frac{df}{dt} + f\left(\frac{d\rho}{dt} + \rho \mathrm{div}\boldsymbol{u}\right) = \frac{d(\rho f)}{dt} + \rho f \mathrm{div}\boldsymbol{u} = P_f \rightarrow \frac{\partial(\rho f)}{\partial t} + \mathrm{div}(\rho f \boldsymbol{u}) = P_f; \tag{2.6}$$

If (2.6|) reflects the conservation of momentum $\rho\boldsymbol{u}$, then $P_i$ are components of the resulting vector of body forces due to gravity and surface forces of pressure and friction. Its scalar notation is then a system of three N-S equations

$$\frac{\partial(\rho u_i)}{\partial t} + \mathrm{div}(\rho u_i u_j) = P_i, \quad i,j = 1,2,3 \tag{2.7}$$

In the case of isothermal (or thermally known) flow, their right sides $P_i$ do not contain problematic nonlinearities as it follows from the next section. The total (material) derivative on the left side of (2.4) are the source of the first, basic type of problematic turbulence non-linearities shown by eq. (2.6) and (2.7).

### 2.4 Influence of physical properties of fluids upon generating problematic non-linearities

The dependences of pressure $p$, internal energy $\epsilon$, viscosity coefficient $\mu$ and thermal conductivity $\varkappa$ on the mass $\rho$ and absolute temperature $T$ necessary for the closure of systems (2.3) to (2.7) are of thermodynamic origin, including the parameters $c_v$, $c_{p\,v}$ indicating specific heat at the same volume or pressure.

The dependence of coefficients $\mu$ and $\varkappa$ on temperature also means dependence on flow mode and dynamics. Therefore, we will comment on the need to assess these relations in terms of the possible formation of problematic non-linearities in the event of PDR adaptation to flow with turbulent temperature or density fluctuations.

Given that the relations for coefficients $\mu$ and $\varkappa$ are always of an experimental (or empirical) nature, this threat can be ruled out if the results of experimental measurements (or observations) have been processed only into regular dependences, devoid of random fluctuations in the measured values. In the case of using the averaged PDR system to simulate a turbulent flow, this allows us to postulate the assumption that if the coefficients $\mu$ and $\varkappa$ are determined as functions of the average temperature values $\overline{T}$, then $\mu(\overline{T})$ and $\varkappa(\overline{T})$ shall not be a source of problematic non-linearities.

Nevertheless, any use of the terms isothermal flows or incompressible fluids concerns only the case of justified neglect of the temperature/compressibility effects due to practical reasons.

### 2.5 The deformation energy as the effect of turbulent density fluctuations.

The formation of the constitutive functions of the desired properties requires the use of the vortical properties of the kinetic energy of the turbulence. One of these may be surprising, so we shall bring it up first. The current kinetic energy $E$ of the instantaneous velocity field per volume unit is defined (see L.M. Milne-Thomson, L.M. 1960, paragraph 3.50) as the product of the components $u_i$ of the velocity vector $\boldsymbol{u}$ and the specific mass $\rho$ in the form of.

$$E = \frac{1}{2}\sum_{i=1}^{3} \rho u_i^2, \qquad i = 1,2,3 \tag{2.8}$$

After applying Reynolds's decomposition $u_i = \bar{u}_i + u'_i$, $\rho = \bar{\rho} + \rho'$ and implicitly averaging in (2.8), we obtain the averaged kinetic energy $\overline{E}$ as a combination of averaged non-linearities

$$\overline{E} = \frac{1}{2}\sum_{i=1}^{3} \overline{\rho u_i^2} = \frac{1}{2}\sum_{i=1}^{3}\left[\bar{\rho}\left(\bar{u}_i^2 + \overline{u'^2_i}\right) + 2\bar{u}_i\overline{\rho' u'_i} + \overline{\rho' u'^2_i}\right] \tag{2.9}$$

in which the last two terms of the sum satisfy the inequalities.

$$2\bar{u}_i\overline{\rho' u'_i} \geq 0 \quad or \quad 2\bar{u}_i\overline{\rho' u'_i} \leq 0 \text{ and } \overline{\rho' u'^2_i} \geq 0 \quad or \quad \overline{\rho' u'^2_i} \leq 0 \tag{2.10}$$

Inequalities in (2.10) are valid due to fact that both fluctuating non-linearities $\rho' u'_i$ and $\rho' u'^2_i$ alternate positive and negative values before averaging as the result of the randomness and mutual independence of the fluctuations $\rho'$ and $u'_i$ and the nature of their products. After averaging, they behave in this way because the characteristic averaging domains $\Delta$ are final and their boundaries are unknown and unconditional; see the conclusion of Chapter 3.

For the above reasons, the area $\Delta$ shows a statistically equal probability of occurrence for both positive and negative values of the averaged fluctuating non-linearities in (2.10). However, the same probability of occurrence for both positive and negative values should also apply to their sum

$$E_d = \frac{1}{2}\sum_{i=1}^{3}\left[2\bar{u}_i\overline{\rho' u'_i} + \overline{\rho' u'^2_i}\right], \quad E_d \geq 0 \quad or \quad E_d \leq 0 \tag{2.11}$$

From the inequalities in (2.11), it follows that the process of averaging the kinetic energy E in (2.9) resulted into the sum of two qualitatively different kinds, elastic $E_e$ and deformation $E_d$ giving

$$\overline{\mathrm{E}} = \mathrm{E_e} + \mathrm{E_d} \tag{2.12}$$

Of these, the only the elastic part $E_e$ given by

$E_e = \bar{\rho}(K + k);\ K = \frac{1}{2}\sum_{i=1}^{3}\bar{u}_i^2;\ k = \frac{1}{2}\sum_{i=1}^{3}\overline{u'^2_i}\ ;\ E_e > 0$ (2.13)

remains after averaging always positively**,** while its part $E_d$ according to (2.11) can take also negative values.

However, this means that the implicit averaging (2.8) → (2.9) caused the separating transformation of kinetic energy $E$ into three qualitatively different types of energy. Two of these, *K and k*, denote the two known types of kinetic energy entraining per mass unit. $K$ is produced by the velocity field $\bar{u}_i$ and $k$ by its fluctuations $u'_i$. The third of them, $E_d$ is the result of the qualitative transformation of part of kinetic energy ***E*** and becomes part of the internal energy of the flow as an averaging effect when the following inequalities are valid,

$E > 0 \rightarrow \overline{E} = E_e + E_d > 0 \rightarrow E_e > abs(E_d)\ if\ E_d < 0$ (2.14)

The sign of the energy $E_d$ is determined by the result of the sum of the averaged fluctuation products of $\rho'$ and $u'_i$ creating $E_d$ in (2.11). It expresses a mass compressing/diluting effect on the energy exchange through pressure, density and temperature fluctuations. Since in atmospheric physics the effects of compressing/diluting are recollected in connection with the potential temperature of air, see Bednar, J – Zikmunda, O., (1985), the deforming energy $E_d$ can be named also by the term potential energy expressed through the potential temperature fluctuations.

The resulting system of averaged equations is also expected to determine the current value $E_d$ to be used in energy analysis or in the direct solution of the energy balance equation**.** However, the following two unifying effects of kinetic energy averaging, arising from the previous analysis, are also important for the current solution of the closure problem itself:

i. The elastic kinetic energy of flow $E_e$ defined in (2.13) is not affected by the compressibility effect. It is applied in adiabatic processes through the strain energy $E_d$ generated by fluctuations $\rho', u'_i$. This means that the kinetic energy of both compressible gases and "incompressible liquids" are defined in the same way by the same relationship (2.13).

ii. A.N. Kolmogorov assumed his cascade theory of the kinetic energy of turbulence dissipation to be valid for incompressible fluid flows. Since the object of his theory is the dissipation rate of fluctuating kinetic energy $k$, which is part of $E_e$, it follows from (i) that this property of $E_e$ can also be expected at compressible fluids. This justified its use in Section 3 below at defining the characteristic domains of needed properties for averaging relevant PDEs.

### 2.6 Information Support (1) and (2)

***(1) Vygodsky´s (1971) Conditions of Continuity and Differentiability of Functions of Several Arguments***

*231. Differentiable Functions*

*A continuous function which (at a given point) has a differential is called differentiable at that point. A discontinuous function cannot have either a derivative or a differential at a point of discontinuity.*

*424 Continuity of a Function of Several Arguments*

*Definition: A Function f(x,y) is called continuous at a point $M_0(x_0,y_0)$ if the following two conditions are fulfilled:*

1. *the function has definite value **l** at $M_0$.*
2. *the function has a limit, also equal to **l**, at $M_0$*

*If even one of these conditions is violated, the function is called discontinuous at the point $M_0$.*

*The same holds for the case of three and more arguments.*

***(2) Reynolds' decomposition of linear forms of random dependently variable functions.***

Applying the decomposition $f = \overline{f} + f'$ of random function $f(x_i, t)$ on its averaged part $\overline{f}$ and random fluctuating deviation $f'$ and establishing the averaging rules for operations with linear decomposition forms, Reynolds put into action an important tool of statistical mechanics. Written for velocity $u_i, u_j\ \ i,j = 1,2,3,$ Reynolds rules valid for mean values $\overline{f}$ consist of the relations

$u_i = \bar{u}_i + u'_i,\ \overline{u'_i} = 0,\ \overline{\bar{u}_i} = \bar{u}_i,\ \overline{u_i + u_j} = \bar{u}_i + \bar{u}_j,\ \overline{u_i u_j} = \bar{u}_i\bar{u}_j + \overline{u'_i u'_j}$ (2.15)

Using them on decomposition $f = \overline{f} + f'$ one obtains also rules for averaging the partial derivatives

$\overline{\frac{\partial f}{\partial t}} = \frac{\partial \overline{f}}{\partial t};\quad \overline{\frac{\partial f}{\partial x_i}} = \frac{\partial \overline{f}}{\partial x_i}\ ;$ (2.16)

## 3 RANDOMNESS OF TURBULENCE PHENOMENA IN THE ROLE OF A PHYSICALLY AUTONOMOUS FACTOR OF CHAOS AS THE BASIC STRATEGY OF THE AVERAGE AND CLOSURE PROBLEM SOLUTION

As the one of possible steps in the effort to get rid of any non-differentiability of the random turbulent fields proven above in Section 1, it is solution strategy arisen from the idea that randomness is an autonomous factor of physical processes and, therefore, this property can be utilized as a property of independent variables of the governing PDEs.

Applying Reynolds' decomposition in the "originally" closed PDEs one transforms this system into an unclosed one creating the closure problem. But use of above stated strategy to solution of the problem enables one to analytically express any random turbulent flow in the domain *G'(x,t)* and its mathematical expectation in the domain $\overline{G}$*(x,t)*, performing in such manner the transformation

*random fields* $f = \overline{f} + f'$ in $G'(\boldsymbol{x},t)$ → *smooth fields* $\overline{f}(\boldsymbol{x},t)$ in $\overline{G}(\boldsymbol{x},t)$

without any phenomenological presumptions on turbulent flow properties or behaviour.

To perform this the above postulated randomness and unpredictability of turbulent phenomena will be defined and used as the properties of independently variable parameters of turbulent flow.

### 3.1 Definition of randomness and physical independence of random turbulent parameters. Dual property and bifunctionality hypothesis.

The mean flow characteristics $\overline{f}$ are regular if they remain the same, after the mean flow, with the same initial and boundary conditions being repeated. Otherwise, they are random. Two starting steps were made on the way to the desired solution of the problem: (i) we considered randomness as an autonomous factor of random processes and (ii) we have applied it as a useful property enabling the presentation of the random behaving of turbulent fields by standard tools of applied mathematics. In this case it shall be constitutive equations consisting of regular functions of random variables. In the presented study, the idea of randomness is realized through the so-called dual property and to other random parameters as well as with respect to time *t* and position vector ***x***, and (ii) they are simultaneously formal functions of time and position in the sense that such random dependence can be determined and described only through measurement. It is impossible to know these parameters before measurements have been made, because they behave in time and space without causal coherence, i.e. randomly. This is a mark of physical independence. The frequency $\omega$ of turbulent fluctuations of vector and scalar fields is chosen to define the class of independent variables in functions creating constitutive equations.

### 3.2 On some tools of statistical mechanics and their properties applied in the article

This concerns, namely, the averaging of a function defining a mean value, also called the mathematical expectation, and the Reynolds' rules for operations with decomposed random functions. Two forms of averaging will be used. The first, known as the Reynolds' average, means an unknown function denoted by an overbar, which we call the implicit average. The second one, called the explicit average, defines the mean value of a given function through a definite integral over its characteristic domain. Integrands of these definite integrals will be created explicitly from the constitutive functions. In such specified integrands the products of any dependent variables will be expressed through constitutive functions as well. Both forms, implicit and explicit, assume the validity of the Reynolds' rules for the implicit form of averaging. The explicit averaging of constitutive functions will be carried out in the space of random independent variables $G(\omega, t)$ but with its mean values located in the space $\bar{G}(x, t)$ satisfying the conditions (3.1) below. Therefore, when specifying the domains of averaging, there shall be neither the possibility nor the need to consider any use of correlating or autocorrelating functions, or other aspects of probability theory. Particular attention is required when averaging a random but statistically steady field. The integral time scale $T \rightarrow \infty$ in this case, and the result of averaging for one position point in $\bar{G}(x, t)$ becomes constant regardless of the method of averaging. Therefore, the Reynolds average in the domain $-T/2 < 0 < T/2$ currently applied in experimental research on turbulence is consistent. This includes the experimental resources used in this study to confront theory with experimental data. Although the mean value of random functions can be defined differently, it does not exclude the possibility of comparing computation results with the results of experimental measurements under these conditions.

The domain of definition for variables to appear in constitutive equations belongs to $G(\omega, t)$. Random velocity components $u_i$ of the vector $u(x,t)$ and the scalar dependent variables (e.g. of density $\rho$) will be defined in the regular space $\bar{G}(x, t)$ and, through constitutive functions, also in the space $G(\omega, t)$.. The space $\overline{G}(x, t)$ is the standard four-dimensional space-time. The space $G(\omega, t)$. is $(N + 1) = 5$-D, (5-dimensional), where $N = 4$ is the number of active random frequency components. $N = 4$ because of three vector frequencies $\omega_i$ and plus one due to scalars $\omega_\varrho$. The domains of definition of the constitutive functions in 5-D space $G(\omega, t)$. satisfy the inequalities

$$\omega_{iD} \leq \omega_i \leq \omega_{iH}, \quad 0 \leq t \leq T, \quad i = 1,2,3, ... \tag{3.1}$$

Where $\omega_{iD}$, $\omega_{iH}$, $T$ are regular, non-random boundary values of the random frequency $\omega_i$ and time *t*. The following scheme depicts the role of both dependent as well as independent variable $\omega$ in

$$u_i = u_i\,(\omega_i, t) \quad \text{in } 5\text{-}D\ G(\omega, t); \rightarrow \bar{u}_i = \bar{u}_i(\overline{\omega}, (x, t), t) \quad \text{in } 4\text{-}D\ \overline{G}(\overline{\omega}, t) \tag{3.2}$$

before and after averaged turbulent velocity field in accordance with the dual property hypothesis.

### 3.3 The characteristic speed of turbulent fluctuations and the turbulent pressure of fluctuating energy

The directional components of the fluctuating kinetic energy $\bar{\rho}\overline{u'^2_i}$ are also known as normal turbulent stresses. This bifunctionality is also manifested in Kolmogorov's cascade theory of energy dissipation when defining velocity scale V by means of the characteristic velocity of turbulent fluctuations according to relations

$$V \approx \sqrt{\overline{u'^2_i}}; \quad \overline{u'^2_i} = \overline{u'^2_j}, \quad i = 1, 2, 3 \tag{3.3}$$

valid for isotropic turbulence. Cascade theory determines the velocity scale *V* together with the scales of length $L$ and time $T$ to be functions of the kinematic viscosity $\nu$ and the average dissipation rate of fluctuating energy $\bar{\varepsilon}$. By assuming the possibility of using some of the insights of Kolmogorov's cascade theory within TED also for non-isotropic turbulences, we shall find an equivalent of relation (3.3) for velocity scale V which satisfies this requirement.

An important feature of scale *V* in (3.3) is that, together with scale $L$, its values determine Re, which characterizes the local state of turbulence throughout the entire range of turbulent vortices from maximum vortices to micro-vortices, whereas Re => 1 values. Such a property in the non-isotropic region of turbulence has a velocity scale *V* only if it is determined by the average of three normal stresses $\bar{\rho}\overline{u'^2_i}$, i.e., by

$$V^2 = \frac{1}{3}\sum_{i=1}^{3} \overline{u'^2_i} \quad i = 1, 2, 3 \tag{3.4}$$

i.e. like how the kinetic energy of molecules defines pressure in thermodynamics, see Soo, S. L. (1962).

The average of the normal stresses (3.4) defines in such way the turbulent pressure $P_t$ as well as the velocity scale *V* by the relations

$$P_t = \frac{1}{3}\bar{\rho}\sum_{i=1}^{3} \overline{u'^2_i}; \quad V = (P_t/\bar{\rho})^{\frac{1}{2}} \tag{3.5}$$

and enables us to apply some tools of the cascade theory of turbulent energy dissipation also to the description of non-isotropic turbulence processes applying (3.4) in an analogy from thermodynamics. defining the validity domain and averaging the constitutive functions. This theory of course considers energy as a scalar whole. Since anisotropy of turbulent flows to be realized in the study and suitable method for averaging independent of any fluid flow state and property has to be found, it requires to extend the validity of the cascade scale relations to the directional components of energy. With this aim in mind, it is useful to begin with the survey of the steps leading to

Kolmogorov scales and inter-scales relations, see Kolmogorov, A.N., (1941b), or Dúbrava, L and Vajcik, S., (1988).

## 4 EXTENSION OF THE VALIDITY OF THE KOLMOGOROV'S CASCADE INTER-SCALE RELATIONS

### 4.1 Information support (3): Inter-scales relations of the Kolmogorov's energy cascade theory

Kolmogorov's energy cascade theory of turbulence provides a quantitative description of internal structure of turbulent fluid flow. This structure is understood as a system of vortices that are constantly forming and disintegrating. Smaller vortices are always more stable than larger ones. This process is random. But the Reynold´s criterion $Re = VL/\nu$ that characterizes a local state of flow, has been determined by the mean characteristic parameters

$$V = ( P_t /\bar{\rho})^{\frac{1}{2}} = (2k/3)^{\frac{1}{2}} ; \; k = \frac{1}{2}\sum_{i=1}^{3} \overline{u_i'^2} ; \; i = 1,\ 2,\ 3 \qquad (4.1)$$

These are the fluctuation kinetic energy. $k$, the kinematic viscosity $\nu$, the characteristic diameter $L$ and the characteristic velocity scale $V$. The last is taken from the average of three components of $2k$ in (4.1) to obtain $V$ approximately valid as well as for the case of not isotropic turbulence. It stems from the turbulent pressure formulations by (3.4) and (3.5) in the Sec. 3. According to Kolmogorov's first similarity hypothesis (Kolmogorov, 1941a), the dimensions of sufficiently small isotropic vortices are functionally dependent on just two factors: $\nu$ [ m$^2$/s] and the mean kinetic energy dissipation rate $\bar{\varepsilon}$ [m$^2$/s$^3$]. By assuming a threshold value of the Reynolds number $Re_\eta$, Kolmogorov was able to determine the threshold magnitudes of length scale $L = \eta$ and velocity scale $V = u_\eta$ of the smallest vortices below which viscous frictional forces prevail over inertial forces. Since $Re$ represents the ratio between those forces, the value $Re = Re_\eta = 1$ is the boundary at which the ratio changes in favor of the frictional forces. Using the threshold values in (4.1) leads to a relation that we shall call a locating equation for the sought-after microscales $\eta$ and $u_\eta$:

$$\eta u_\eta = \nu \qquad (4.2)$$

This equation follows from Kolmogorov's hypothesis on existence of the relationship $\eta(\nu, \bar{\varepsilon})$.

From dimensional analysis, such a relationship takes the form $\eta = c\bar{\varepsilon}^\alpha \nu^\beta$, with $\alpha = -1/4$ and $\beta = 3/4$. The dimensionless constant $c$ is determined to be 1, so we achieve the sought-after smallest size scale of turbulence:

$$\eta = \left(\frac{\nu^3}{\bar{\varepsilon}}\right)^{\frac{1}{4}} \qquad (4.3)$$

Using this result in (4.2) gives the microscale for velocity:

$$u_\eta = (\nu\bar{\varepsilon})^{\frac{1}{4}} \qquad (4.4)$$

According to the first similarity hypothesis, the validity of which the author himself called approximate, for characteristic scales of turbulent vortices on each $n$th level of size, the following dimensional relationships are applicable:

$$L = VT \rightarrow L_n = V_n T_n \rightarrow \eta = u_\eta \tau \qquad (4.5)$$

The final equality in (4.5) in combination with (4.3) and (4.4) leads directly to a microscale for time:

$$\tau = \left(\frac{\nu}{\bar{\varepsilon}}\right)^{\frac{1}{2}} \qquad (4.6)$$

Finally, the ratio of the square of the velocity from (4.4) to the time microscale $\tau$ from (4.6) provides a dimensionally consistent relationship when applied to the whole energy cascade under the assumption of a one-way flow of energy from larger to smaller vortices, as a result of which an average dissipation rate $\bar{\varepsilon}$ can be written as

$$\bar{\varepsilon} = \frac{u_\eta^2}{\tau} = \frac{V_n^2}{T_n} = \frac{V^2}{T} \ . \qquad (4.7)$$

### 4.2 Extension of the validity of the Kolmogorov's cascade inter-scale relations up the directional components of the fluctuating energy of a non-isotropic turbulence

Kolmogorov's cascade theory of energy dissipation refers to the fluctuating kinetic energy $k$ as a scalar whole. It assumes "incompressible" liquids and isotropic turbulence. The intention to extend its validity to compressible gases is justified above in the Section (2.7). The extension of the approximate validity of the cascade theory onto domains of large vortices (where $\overline{u_i'^2} \neq \overline{u_j'^2}$) ensures a "common" rate of energy dissipation

$$\bar{\varepsilon} = \frac{V^2}{T}, \qquad i = 1, 2, 3 \qquad (4.10)$$

by the fact that the square of the velocity scale used in (4.10) is determined in (3.4) by the arithmetic mean of the three components of fluctuating energy $k$, as defined in (4.1), and by the turbulent pressure $P_t$. While the average dissipation rate $\bar{\varepsilon}$ in (4.10) is studied in Kolmogorov's cascade theory as a parameter of one scalar whole of isotropic turbulence, solving the closure problem requires finding properties and interrelations for the three dissipation rates of fluctuating kinetic energy

$$\bar{\varepsilon}_i = \frac{V_i^2}{T_i}, \;\; V_i^2 = \overline{u_i'^2}, \;\; \overline{u_i'^2} \neq \overline{u_j'^2}, \;\; i = 1, 2, 3 \qquad (4.11)$$

and their three directional components $V_i^2 = \overline{u_i'^2}$. The third object of our interest is the interrelation between the integral time scale $T$ and its three components $T_i$. The basis of this relationship is obtained from (3.4) after eliminating the squares of velocity scales $V^2$ and $V_i^2$ obtained from (4.10) and (4.11). It will be simple,

$$\bar{\varepsilon}T = \frac{1}{3}\sum_{i=1}^{3} \bar{\varepsilon}_i T_i \qquad (4.12)$$

By this way, we expand the concept of an energy cascade to a vector variant that allows inter-scale relations to be defined for each of the three components $V_i^2$ of the fluctuation energy field that make up the sum (4.12). This brings us to a vector space whose determining parameters are the characteristic velocities of turbulent fluctuations, further named as the characteristic velocities of turbulence:

$$V_i = \left(\overline{u_i'^2}\right)^{\frac{1}{2}}, \quad i = 1,\ 2,\ 3 \qquad (4.13)$$

The number of characteristic parameters is tripled by the transition to a vector space, as are the number of inter-scale relations. This also applies to the dissipation rates $\bar{\varepsilon}_i$, in which the individual components will be different:

$$\bar{\varepsilon}_1 \neq \bar{\varepsilon}_2 \neq \bar{\varepsilon}_3. \qquad (4.14)$$

The transition to a vector field requires that we apply the relations (4.1) – (4.7) for each of the three components (4.13) of the characteristic velocity $V$ and for each of the three components of the velocity microscale $u_\eta$. Equation (4.1) will give us three relations

$$V_i L_i = \nu Re_i, \quad i = 1,\ 2,\ 3, \qquad (4.15)$$

to define three systems of energy cascades by the $Re$ values for all the directional components of energy. Since (4.2) is a location equation, the same goes for its three directional variants for the microscales $\eta_i$ and $u_{\eta i}$ , which are connected by a common locating $Re = Re_c$ or locating viscosity $\nu_c$ according to the equation

$$u_{\eta i}\, \eta_i = \nu\, Re_c = \nu_c \qquad (4.16)$$

In the case of a cascade located in the scalar space, the control $Re$ is determined by the value $Re = Re_\eta = 1$. Now, the value $Re_c$ in (4.16) is not known, but it will be the same for all $i$ =1,2,3 in (4.16) and will provide a domain in which averaging is assumed, the results of which will satisfy the inequalities

$$V_i^2 = \overline{u_i^2} > 0, \quad i = 1,\ 2,\ 3, \qquad (4.17)$$

Equations (3.7) also triple in number, but they perform the same function as before in the form

$$\bar{\varepsilon}_i = \frac{u_{\eta i}^2}{\tau_i} = \frac{V_i^2}{T_i}; \qquad i = 1,2,3, \qquad (4.18)$$

Now we just have to write three - dimensional relations equivalent to both of (4.5):

$$L_i = V_i T_i; \quad \eta_i = u_{\eta i}\tau_i, \quad i = 1,\ 2,\ 3 \qquad (4.19)$$

These complete a system of equations for defining nine microscales and nine inter scale relations. We can do this for each $i$ separately, just repeating the procedure from the previous scalar case. By solving the system of equations (4.16), (4.20), and (4.21) with respect to the microscales $\eta_i$, $\tau_i$, and $u_{\eta i}$ , the functions for the dissipation rate $\bar{\varepsilon}_i$ and locating viscosity $\nu_c$ are obtained:

$$\eta_i = \left(\frac{\nu_c^3}{\bar{\varepsilon}_i}\right)^{\frac{1}{4}}; \quad \tau_i = \left(\frac{\nu_c}{\bar{\varepsilon}_i}\right)^{\frac{1}{2}}; \quad u_{\eta i} = (\nu_c\, \bar{\varepsilon}_i\,)^{\frac{1}{4}}, \ i = 1,2,3 \qquad (4.20)$$

in which $\nu_c$ is the common parameter for all three directions $i$, formed by the control $Re_c$ in (4.16).

When compiling the equations for the three inter-scale ratios, the first equation comes from the ratios between the two equations in (4.19). The second comes from the ratio between the equations (4.15) and (4.16). Finally, the second equation in (4.18) will close the nonlinear system for the three inter-scale ratios. Its solution in the form of power functions of Reynolds numbers gives the sought for inter-scale relations

$$\frac{T_i}{\tau_i} = \left(\frac{Re_i}{Re_c}\right)^{\frac{1}{2}} = \left(\frac{V_i L_i}{\nu Re_c}\right)^{\frac{1}{2}} = \left(\frac{V_i^2 T_i}{\nu_c}\right)^{\frac{1}{2}}, \ i = 1,\ 2,\ 3, \qquad (4.21)$$

for time scales and

$$\frac{V_i}{u_{\eta i}} = \left(\frac{V_i^2 T_i}{\nu_c}\right)^{\frac{1}{4}}; \ \frac{L_i}{\eta_i} = \left(\frac{V_i^2 T_i}{\nu_c}\right)^{\frac{3}{4}}, \quad i = 1,\ 2,\ 3, \qquad (4.22)$$

for scales of velocity and length. To find a way to define the space of averaging of the N–S equations, the ratios from time scales in (4.21) will be sufficient. The actual value of the "control" viscosity $\nu_c$ is unknown, but it may be specified within the inequality (4.17) on a physical basis. We do this by using characteristic turbulent frequencies defined by time scales through the relations

$$\Omega_i = \frac{1}{T_i}, \quad i = 1,\ 2,\ 3 \qquad (4.23)$$

The common locating value for these frequencies is unknown, but, owing to the relation

$$\Omega_c = \frac{1}{T_c} = \frac{1}{\tau_c}, \qquad (4.24)$$

it will combine three inter-scale relations (4.21) through a common value of the time microscales $\tau_i$:

$$\tau_1 = \tau_2 = \tau_3 = \tau_c = T_c. \qquad (4.25)$$

By rewriting (4.21) with a common $\tau_i$ according to (4.25), we get the desired inter scale relations connecting any vector values of the dissipation rates $\bar{\varepsilon}_i$ with their common scalar $\bar{\varepsilon}_c$ which follows from a common fluctuation frequency (4.24) for all three $i$ = 1,2,3.

$$\frac{T_i}{T_c} = V_i \left(\frac{T_i}{\nu_c}\right)^{\frac{1}{2}} \ \rightarrow\ \frac{V_i^{\,2}}{T_i} = \frac{\nu_c}{T_c^{\,2}} = \bar{\varepsilon}_c\,, \ i = 1,\ 2,\ 3. \qquad (4.26)$$

The elimination of the common right-hand side in the second triplet of equations (4.26) leads directly to farther usable step

$$\frac{T_i}{T_j} = \frac{V_i^2}{V_j^2} \ \rightarrow\ \frac{V_i^2}{T_i} = \frac{V_j^2}{T_j} = \bar{\varepsilon}_c \qquad \text{i, j = 1, 2, 3} \qquad (4.27)$$

The equations in (4.27) provide important relations to solve the closure problem accordingly with the strategy stated above. The first one determines the ratios between the time scales $T_i$ and $T_j$ directly by the ratios between the respective normal turbulent stresses**.** The second provides important physical information and namely that the dissipation rate of fluctuating energy $\bar{\varepsilon}$ is "approximately" constant not only for the "vertical" passage through a cascade of vortices of different sizes. It is valid also for the "contour-wise" motion in a single vortex (or multiple vortices) along a line of the same size $L$ and other parameters**.** This three-line information written with three characters indicates an approximate equality:

$$\bar{\varepsilon} \approx \bar{\varepsilon}_i \approx \bar{\varepsilon}_c \qquad (4.28)$$

The use of (4.28) in (4.12) leads to wanted relationship among the time scales

$$T = \frac{1}{3}\sum_{i=1}^{3} T_i \qquad (4.29)$$

enabling one to create the characteristic domain of averaging the respective PDEs systems.

It is important that (4.29) follows independently as vell as from this consideration: Numerator of the ratio in (4.10) defining the mean rate of energy dissipation $\bar{\varepsilon}$ is given by average (3.4) which stems from the thermodynamics analogy for turbulent pressure. But to keep this consideration consistent, denominator in (4.10) would be the same as follows from (4.29). It justifies validity of the Kolmogorov's cascade relations over the directional components of energy as well as.

Kolmogorov's energy cascade is fixed at its lower limit by the value of the Reynolds number $Re = 1$. The above-mentioned relations define three cascades $V_i^2/2$ in a positive vector space. A fourth, given by the sum for *k* in (4.12), remains in the original scalar space. These cascades are fixed by a common value of the Reynolds number $Re = Re_c$. We do not know the actual value of $Re_c$. However, it has a definite physical significance, which lies in the local (inner) isotropy of turbulence and in the common value of the averaged frequency of the velocity field fluctuations.

Above in Section 3 chosen fluctuation velocity frequencies $\omega$ for the bifunctional role of random independent variable parameters within the possibility to play dependent variables of turbulent flow has a demonstrable experimental justification. Its individual random numerical values $\omega_n = 1/T_n$ are clearly identifiable on the time axis of experimental records regarding measured velocities, see Dúbrava, L. and Vajcík, S. (1988). They are given by the inverted values of individual time periods $T_n$, n = 1, 2, between adjacent opposite (+/–) speed extremes. Values $T_n > 0$ are random in magnitude and occurrence and fill the continuous space on the time axis. They do the same with relationship $\omega_n = 1/T_n$ with frequencies $\omega_n$ ,satisfying the required properties of random independent variables.

## 5 CONSTITUTIVE FUNCTIONS AND CONSTITUTIVE EQUATIONS OF RANDOM TURBULENT FIELDS

The randomness of $f'$ and maximal probability of $\overline{f}$ in the Reynolds decomposition $f = \overline{f} + f'$ enabled him to open and define the closure problem. These properties though needed seem to be not sufficient for his decomposition to become a constitutive equation as a tool leading also to wanted solution in accordance with the dual property and bifunctionality hypothesis.

To avoid this limitation the general sinusoidal function was used as the constitutive function in suitable constitutive equations being able to describe any random oscillating flow due to its random frequency $\omega$ in the role of the independent variable.

### 5.1 Constitutive equations of random fields as the effective statistical tool of fluid dynamics

With the frequency of turbulent fluctuations playing the role of the random independent variable, the constitutive equations of the form

$$u_i = U_i \cos(\omega_i t + \varphi_i), \quad i = 1,\ 2,\ 3, \quad \text{in } G(\omega, t) \text{ as well as in } \bar{G}(x,t), \tag{5.1}$$

are chosen to define the vector of a random velocity field $u(u_1, u_2, u_3)$ via the regular cosine function in two spaces of independent variables. The first space, $G(\omega, t)$, is formed from time $t$ and the three components of a random frequency vector $\omega(\omega_1, \omega_2, \omega_3)$ of the velocity fluctuations. The random frequencies of velocity fluctuations $\omega_i$, although independent in $G(\omega, t)$, behave as

$$\omega_i = \omega_i(x, y, z, t) \quad \text{in } \bar{G}(x,t), \tag{5.2}$$

i.e., as random but formal functions of time and position. They are "formal" because, if we know them, we can record them in a deterministic space in the sense of the dual property hypothesis.

The random frequencies $\omega_i$ can also be understood as velocities of the vortex rotations caused by the fluctuations $u_i'$. The expression (5.1) contains the phase angles $\varphi_i$ and moduli $U_i \equiv |U_i| > 0$ of the velocity components $u_i = \bar{u}_i + u_i'$. The parameters $U_i$ and $\varphi_i$ are thought to be obtained as the functions of other mean flow parameters in order to connect $G(\omega, t)$ with the deterministic space $\bar{G}(x,t)$. Therefore $U_i$ and $\varphi_i$ can be named as the mean connecting parameters.

To obtain $U_i$ and $\varphi_i$ we will subject the constitutions (5.1) to averaging operations in the space $G(\omega, t)$. Before averaging (5.1) it is useful to use some substitutions and rewrite constitution (5.1) into its dimensionless variant in the form

$$u_{ri} = c_i \cos\varphi_i - s_i \sin\varphi_i, \tag{5.3}$$

where the dimensionless velocity $u_{ri}$ and the trigonometric functions of time *t* and frequencies $\omega_i$ are

$$u_{ri} = \frac{u_i}{U_i}; \quad c_i = \cos\omega_i t; \quad s_i = \sin\omega_i t \ ; \quad i = 1,\ 2,\ 3 \tag{5.4}$$

Since the averaging meanwhile can be only implicit, we average the constitutive equation (5.3) indicating resulted means meanwhile by overbars. The equation thus obtained can be used to determine the wanted connecting parameters $\sin\varphi_i$ and $\cos\varphi_i$. This is possible with help of the known relation $\sin^2\varphi_i + \cos^2\varphi_i = 1$. Utilizing this and the average of equation (5.3), we obtain the parameters to be eliminated in the constitution (5.3),

$$\sin\varphi_i = \frac{-\bar{u}_{ri}\,\bar{s}_i \pm \bar{c}_i\sqrt{D_i}}{M_i}, \quad \cos\varphi_i = \frac{\bar{u}_{ri}\,\bar{c}_i \pm \bar{s}_i\sqrt{D_i}}{M_i}, \tag{5.5}$$

as functions of the mean velocity field $\bar{u}_i$ and other mean quantities defined in $\bar{G}(x,t)$, i.e.:

$$\bar{s}_i = \overline{\sin\omega_i t}, \quad \bar{c}_i = \overline{\cos\omega_i t}, \tag{5.6}$$

$$M_i = \bar{s}^2{}_i + \bar{c}^2{}_i, \quad D_i = M_i - \bar{u}^2{}_{ri}, \quad \bar{u}_{ri} = \frac{\bar{u}_i}{U_i} \tag{5.7}$$

After elimination of the mean connecting parameters $\sin\varphi_i$ and $\cos\varphi_i$ as well as multiplication of the equation (5.3) by the modulus $U_i$, the constitutive equation (5.3) can be written in concise form

$$u_i = a_i\bar{u}_i \pm b_i U_i\sqrt{D_i}, \quad i = 1,\ 2,\ 3 \tag{5.8}$$

in which the key constitutive functions $a_i(\omega_i, t)$ and $b_i(\omega_i, t)$ are given by the relations

$$a_i = \frac{1}{M_i}\ (\bar{c}_i c_i + \bar{s}_i s_i), \quad b_i = \frac{1}{M_i}\ (\bar{s}_i c_i - \bar{c}_i s_i) \tag{5.9}$$

It can be seen immediately that these functions after their implicit averaging have constant values

$$\bar{a}_i = 1, \quad \overline{b_i} = 0, \quad i = 1,\ 2,\ 3,\ldots \tag{5.10}$$

which, thanks to (5.10) and $M_i$ in (5.7), changes the averaged constitutive equation (5.8) into an identity regardless of the current state of the random velocity field and the method by which it is averaged. This identity results from determining the parameters (5.5) from the implicitly averaged equation (5.3). The validity of (5.10) is important for the inner cohesion of solution of the problem**,** because if (5.10) does not hold, the averaging (5.8) introduces new relations among the mean flow parameters without any physical justification. The domain of definition of the expression (5.1) lies in $G(\omega,t)$. The process of obtaining and eliminating the fixing phase angles $\varphi_i$ as functions of mean flow parameters leads to a transformation of the expression into the form (5.8), which is also defined in the space $\bar{G}(x,t)$ in terms of averaged flow parameters. Therefore, the validity of (5.10) implies that the transformation is also correct.

### 5.2 Uniqueness of kinetic energy of a mean velocity field

According to (4.10) and (4.11), the kinetic energy of the random velocity field (5.8) is defined by the sum

$$K + k = \tfrac{1}{2}\textstyle\sum_{i=1}^{i=3} \overline{u_i u_i} \quad \text{for} \quad i = j, \quad i,j = 1,2,3,\ldots \tag{5.11}$$

in which the directional components

$$\overline{u_i u_i} = \overline{u_i}^2\overline{a_i a_i} \pm 2\,\overline{u_i}U_i\overline{a_i b_i}\sqrt{D_i} + D_i U_i^2\overline{b_i b_i}, \quad i = j, \quad i,j = 1,2,3 \tag{5.12}$$

have been obtained through implicit averaging of the squares of the velocities given by (5.8).

In the equation (5.12), two sources of possible non-uniqueness have appeared on the right-hand side, even in the case where the energy components themselves are $\overline{u_i u_i} = \overline{u_i^2} > 0$. The first of these is the existence of two signs + and −. The second lies in the value of the discriminant $D_i$. This is defined by the first equation in (5.4) and the second in (5.7). These equations do not guarantee a non-negative value of $D_i$. Besides this, the first equation in (5.4) determines its dependence on the unknown value of the velocity modulus. Since this energy is determined by averaging a system of positive elements (squares of velocity), it is natural to expect the positivity and uniqueness of the result of the operation prescribed by the equation (5.12). The stated properties of energy can be ensured by a suitable choice of an unknown velocity modulus $U_i$ in the second equation in (5.7). Uniqueness of all components of the turbulent stress tensor in (5.12) and a non-negative energy at $i = j$ will ensure a zero discriminant $D_i$, i.e.

$$D_i = M_i - \bar{u}^2{}_{ri} = 0 \tag{5.13}$$

By eliminating $\bar{u}_{ri}$ from (5.13) using the third equation in (4.7), we also determine the modulus of the velocity through the relation

$$U_i = \frac{|\bar{u}_i|}{\sqrt{M_i}}, \quad i = 1,\ 2,\ 3,\ldots \tag{5.14}$$

Since the condition (5.13) holds, the constitutive equation for the turbulent velocity field (5.8) can be written in the simple form

$$u_i = a_i\bar{u}_i\,, \quad i = 1,\ 2,\ 3,\ldots \tag{5.15}$$

which is a consequence of its being defined in the space $\bar{G}(x,t)$ and the assumption of the uniqueness and positivity of the kinetic energy applied in equation (5.12). The constitutive function $a_i = a_i(\omega_i,t)$ defined by the first equation in (5.9) plays a key role in EDT. The first equation in (5.9) with functions $c_i$ and $s_i$ from (5.4) puts the constitutive equation (5.15) into the open form

$$\frac{u_i}{\bar{u}_i} = \frac{1}{M_i}(\bar{s}_i \sin\omega_i t + \bar{c}_i \cos\omega_i t), \qquad i = 1,2,3,\ldots \tag{5.16}$$

Owing to the presence of the random independent variable $\omega_i$ and the mean velocity $\bar{u}_i$, equation (5.16) can be called the equation of random velocity oscillations around the equilibrium mean.

Integrating the products of the velocity vector components (5.15) over the characteristic domain of $G(\omega,t)$ one defines the explicitly averaged non-linearities

$$\overline{u_i u_j} = \bar{u}_i\bar{u}_j\overline{a_i a_j} \qquad i = 1,2,3 \tag{5.17}$$

which for $i = j$ imply three relationships between the directional components of the kinetic energies $\bar{u}^2{}_i$ and $\overline{u'^2_i}$**.** The following requirement was applied when deriving them: Any kinetic energy of turbulent flow if defined by the mean square of the random velocity field has to be unique and non-negative. For $i \neq j$ and $\rho$ = const. three tangential components of the turbulent. stress tensor $\overline{u_i' u_j'}$ are defined by (5.17) after its explicit averaging.

The constitutive equations of a random turbulent field (5.1) are defined in the space of random independent variables $G(\omega,t)$. Their normalized forms (5.15) and (5.16) resulting from the above operations are fixed by a mean velocity field and are thus also defined in the regular space $\bar{G}(x,t)$.

### 5.3 Constitutive equation of random scalar fields

Density $\rho$ in the case of compressible fluids as well as other scalar quantities also behave randomly in a turbulent flow. Since we see no reason to use another form of constitutive function for random scalar fields, we proceed with them in the same way as with the components of the velocity field. The resulting form of the constitutive equation, for example for $\rho$, will then be similar as (5.15), i.e.,

$$\rho = a_\rho\bar{\rho}. \tag{5.18}$$

The constitutive function for $\rho$ will be the same as for $u_i$ in the first equations in (5.9):

$$a_\rho = \frac{1}{M_\rho}\left(\bar{c}_\rho c_\rho + \bar{s}_\rho s_\rho\right), \quad M_\rho = \bar{s}^2{}_\rho + \bar{c}^2{}_\rho \tag{5.19}$$

Similarly, for explicit functions of time $t$ and random frequency $\omega_\rho(x,y,z,t)$,

$$\rho_{r\rho} = \frac{\rho_r}{\Omega_\rho}, \quad c_\rho = \cos\omega_\rho t, \quad s_\rho = sin\left(\omega_\rho t\right) \tag{5.20}$$

The constitutive function (5.18) will be applied in the averaging processes of nonlinear terms of the N–S equations for fluids with variable density $\rho$. The constitutive function of a random field (5.19) satisfies the condition

$$\bar{a}_\rho = 1 \tag{5.21}$$

as in the case of $a_i$ for the velocity field.

## 6 DISTRIBUTION EQUATIONS OF THE KINETIC ENERGY OF TURBULENT FLOW

The desired relations in (5.17) are formulated through implicit averaging of given random functions. The final form of these relations will be found first for the normal stresses, i.e., for energy components. Let us create the square of the constitutive function given by the first equation in (5.9). The result will contain products and squares of the trigonometric functions defined in (5.4). After replacing them with their corresponding equivalents by means of double angles, the result of taking the square has the form

$$a_i^2 = \frac{1}{2M_i^2}[M_i + (\bar{c}^2{}_i - \bar{s}^2{}_i)c_{i2} + 2\bar{c}_i\bar{s}_i s_{i2}] \quad (6.1)$$

In this equation, two new functions of the random variables $\omega_i$ appear, namely

$$s_{i2} = sin(2\omega_i t), \quad c_{i2} = cos(2\omega_i t) \quad (6.2)$$

since $M_i$ is already defined in (5.7) as an averaged function.

Because the functions (6.2) appear linearly in (6.1), we obtain the average of the square $a_i^2$ still in its implicit form via

$$\overline{a_i a_i} = \frac{M_i + A_i}{2M_i^2}, \quad i = 1,\ 2,\ 3, \quad (6.3)$$

where

$$A_i = (\bar{c}^2{}_i - \bar{s}^2{}_i)\bar{c}_{i2} + 2\bar{c}_i\bar{s}_i\bar{s}_{i2} \quad (6.4)$$

The implicitly averaged functions of the random double angles in (6.4),

$$\bar{s}_{i2} = \overline{sin(2\omega_i t)}, \quad \bar{c}_{i2} = \overline{cos(2\omega_i t)} \quad (6.5)$$

will be replaced in the further development by their explicit equivalents. We shall call the average of the square of the constitutive function given by (6.3) the distribution function of the kinetic energy of turbulence since it will describe dividing total kinetic energy of flow on the energy of mean flow and the energy of its turbulent velocity fluctuations.

### 6.1 Characteristic domain of averaging and tensor of turbulence anisotropy

The averaging operations performed so far have been implicit in character. Forming the constitutive functions of vector and scalar fields allows us to utilize the strategic advantage of the theory, namely, the explicit form of averaging. The explicit average $\overline{f}$ of a function $f(x_1, x_2, \ldots x_N, t)$ , $N > 0$ , of $N$+1 independent variables will be given by the definite integral

$$\overline{f} = \frac{1}{\Delta}\int_{x_{1D}}^{x_{1H}}\int_{x_{2D}}^{x_{2H}}\int_0^T f \cdots\ dx_1\ dx_2 \cdots dx_N\ dt \ \text{ in } G(\omega, t) \quad (6.6)$$

through the characteristic domain

$$\Delta = T\Delta_1\Delta_2 \cdots \Delta_N, \quad \Delta_1 = x_{1H} - x_{1D}, \quad \Delta_2 = x_{2H} - x_{2D}, \quad \Delta_N = x_{NH} - x_{ND}\ , \quad (6.7)$$

where $x_1, x_2, \ldots, x_N$ are random and therefore physically independent parameters of the turbulent flow, and $T$ is an integral time scale. The domain of averaging $\Delta$ is called characteristic because its boundaries are characteristic. They are not random, but deterministic limits $T, x_{nD}, x_{nH}$. They are characteristic because they are not chosen arbitrarily, but as unknown dependent variables characterizing the state of the system of which they are a part. This recalls the role of the free liquid surface in problems with a moving unknown flow boundary (Kosorin, 1995, 2011).

When the theory is applied to the system of N–S equations (7.3) and (7,4) for isothermal flow several types of domains of averaging will come into consideration. The role of independent variables in (6.6) will be taken by frequencies of turbulent fluctuations of the scalars $\omega_\rho$ and velocities $\omega_i$ , $i = 1,\ 2,\ 3$. Mutual combination of indices $i$ and n as well as the resulting total number of random independent variables $N$ in each $f(x_n; t)$ is evident from the Table 1, which gives the dimensionalities of the characteristic domains $\Delta$ for the integral (6.6) in $G(\omega, t)$. The total dimensionality of any $\Delta$ is $N + 1$, where $N$ is the number of random variables $\omega_i$ and $\omega_\rho$ occurring in (6.6) and the 1 comes from the presence of time $t$.

The boundaries of the random frequencies in (6.6) are expected to characterize the state of turbulence between the upper $\omega_{iH}$ and lower $\omega_{iD}$ limits during the time period $0 \leq t \leq T$. The parameters derived from the characteristic velocities $V_i$ defined by (4.18) can reasonably be expected to meet the above requirement. They express the intensity of turbulence, which is directly proportional to the velocities $V_i$ , $i = 1,\ 2,\ 3$ . This reasoning leads to the definition of a lower limit $\omega_{iD}$ through the relation (4.23),

$$\omega_{iD} = \Omega_i = \frac{1}{T_i}, \quad i = 1,\ 2,\ 3, \quad (6.8)$$

and an upper limit $\omega_{iH}$ according to (4.24),

$$\omega_{iH} = \Omega_c = \frac{1}{T_c} \quad (6.9)$$

These characteristic times $T_i$ and $T_c$ together with the velocities $V_i$ and $V_c$ determine the length characteristics of vortices, $L_1 \equiv L_x$, $L_2 \equiv L_y$, $L_3 \equiv L_z$. If these vortices are larger, more unstable, or show signs of anisotropy, then $L_1 \neq L_2 \neq L_3$, and their various dimensions are approximated by the equations

$$L_i = V_i T_i, \quad i = 1,\ 2,\ 3 \quad (6.10)$$

Different values of $L_i$ simulate the axial dimensions of an unstable ellipsoid. With sufficiently small vortices, their dimensions $L_i$ become closer in value and more stable, and the vortices take on a spherical shape with $L_i = L_c$, $T_i = T_c$. Such a formulation of boundaries in (6.6) allows the integration in (6.6) to be interpreted as an integration of the function $f(\omega_i, t)$ in the domain of $G(\omega, t)$ with a random $\omega_i$ between the lower (6.8) and upper (6.9) boundaries in the given time period $T$. This is an integration of $f$ in (6.6) over domains ranging between large anisotropic vortices with dimensions $L_i$ and small isotropic vortices with dimensions $L_c$.

To derive the distribution equations in a definite form, it is necessary to find the explicit equivalents of the implicit averages (5.6) and (6.5) through the integral (6.6). The dimensionality of the integration space is in this case determined by the first and second columns of Table 1. The randomness for (6.6) has dimensionality $N = 1$ , and the integration space is two-dimensional. The result of the integration in (6.6) will be fixed in the deterministic four-dimensional space $\bar{G}(x,t)$ via the boundaries (6.8) and (6.9) and the integral time scale $T$.

The mean values (5.6) calculated according to (6.6) are given by the definite integrals

$$\bar{s}_i = \frac{1}{\Delta_i}\int_{\Omega_i}^{\Omega_c}\int_0^T sin\omega_i\ t\,dt\,d\omega_i = \frac{1}{\Delta_i}\left[C_s\left(\frac{T}{T_c}\right) - C_s\left(\frac{T}{T_i}\right)\right] \quad (6.11)$$

$$\bar{c}_i = \frac{1}{\Delta_i}\int_{\Omega_i}^{\Omega_c}\int_0^T cos\omega_i\ t\,dt\,d\omega_i = \frac{1}{\Delta_i}\left[S_i\left(\frac{T}{T_c}\right) - S_i\left(\frac{T}{T_i}\right)\right] \quad (6.12)$$

in which the indefinite integrals are expressed by the power series

$$S_i(x) = x - \frac{x^3}{3.3!} + \frac{x^5}{5.5!} + \cdots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)(2n-1)!}, \quad (6.13)$$

$$C_s(x) = \frac{x^2}{2.2!} - \frac{x^4}{4.4!} + \cdots + (-1)^{n+1}\frac{x^{2n}}{2n(2n)!}. \quad (6.14)$$

Here (6.13) is the integral sine, and (6.14) is related to the integral cosine $C_i(x)$ by $C_s(x) + C_i(x) + ln\,x = C$, where $C$ is the Euler–Mascheroni constant. The characteristic spaces $\Delta$ given in (6.7) – (6.9) become

$$\Delta_i = T(\Omega_c - \Omega_i) = \frac{T}{T_c} - \frac{T}{T_i} \quad (6.15)$$

The definite integrals of the functions of double angles, (6.5), differ from (6.11) and (6.12) by the double values of the arguments:

$$\bar{s}_{i2} = \frac{1}{2\Delta_i}\left[C_s\left(2\frac{T}{T_c}\right) - C_s\left(2\frac{T}{T_i}\right)\right] \quad (6.16)$$

$$\bar{c}_{i2} = \frac{1}{2\Delta_i}\left[S_i\left(2\frac{T}{T_c}\right) - S_i\left(2\frac{T}{T_i}\right)\right] \quad (6.17)$$

In the explicitly averaged functions, four unknown ratios between the time scales appear: three $T/T_i$ and a fourth $T/T_c$. We begin to solve the problem of determining the unknown ratios of time scales as functions of other flow parameters by asking about the relations between the scale $T$ and the scales $T_i$. The integral scale $T$ is a common time scale for all averaging operations, including those over the N-S equations. Since $T$ and $T_i$. cannot be independent of one another, we define the relation between them in the simplest possible way, namely, by the arithmetic average:

$$T = \frac{1}{3}(T_1 + T_2 + T_3) \quad (6.18)$$

The adoption of (6.18) recalls the expression for pressure $p$ as the arithmetic mean of three normal stresses $p_{11}, p_{22}, p_{33}$ of the stress tensor in the derivation of the basic equations of fluid mechanics (Lojcianskij, 1954 or Milne-Thomson, 1960, §19). Actually, the relation (6.18) also concerns the normal stresses, in this case the turbulent $\overline{u'^2_i}$.. The linearity of (6.18) allows us to use the inter-scale relations (4.27) to make an important shift in solving the problem. Let us divide the equation (6.18) by each $T_i$ in turn for $i$ =1,2,3. We get

$$\frac{T}{T_1} = \frac{1}{3}\left(1 + \frac{T_2}{T_1} + \frac{T_3}{T_1}\right) \quad (6.19)$$

$$\frac{T}{T_2} = \frac{1}{3}\left(\frac{T_1}{T_2} + 1 + \frac{T_3}{T_2}\right) \quad (6.20)$$

$$\frac{T}{T_3} = \frac{1}{3}\left(\frac{T_1}{T_3} + \frac{T_2}{T_3} + 1\right) \quad (6.21)$$

The equations (4.27) yield all the ratios between the time scales $T_i$ in (6.19) – (6.21) as functions of the ratios between the squares of the respective velocities $V_i^2 \equiv \overline{u'^2_i}$ . This allows us to eliminate them and thus obtain the desired relations for three of the four unknown ratios in (6.19) – (6.21). Let us call these ratios the anisotropy indices $i_i$ and write them

$$i_1 \equiv i_x = \frac{T}{T_1} = \frac{1}{3}\left(1 + \frac{\overline{u'^2_2}}{\overline{u'^2_1}} + \frac{\overline{u'^2_3}}{\overline{u'^2_1}}\right) \quad (6.22)$$

$$i_2 \equiv i_y = \frac{T}{T_2} = \frac{1}{3}\left(\frac{\overline{u'^2_1}}{\overline{u'^2_2}} + 1 + \frac{\overline{u'^2_3}}{\overline{u'^2_2}}\right) \quad (6.23)$$

$$i_3 \equiv i_z = \frac{T}{T_3} = \frac{1}{3}\left(\frac{\overline{u'^2_1}}{\overline{u'^2_3}} + \frac{\overline{u'^2_2}}{\overline{u'^2_3}} + 1\right) \quad (6.24)$$

The three anisotropy indices (6.22) – (6.24) form the components of the anisotropy vector $i_a(i_1, i_2, i_3)$ , which can also be considered as an anisotropy tensor

$$T_a = \frac{1}{3}\begin{Bmatrix} 1 & \frac{\overline{u'^2_2}}{\overline{u'^2_1}} & \frac{\overline{u'^2_3}}{\overline{u'^2_1}} \\ \frac{\overline{u'^2_1}}{\overline{u'^2_2}} & 1 & \frac{\overline{u'^2_3}}{\overline{u'^2_2}} \\ \frac{\overline{u'^2_1}}{\overline{u'^2_3}} & \frac{\overline{u'^2_2}}{\overline{u'^2_3}} & 1 \end{Bmatrix} = \begin{Bmatrix} i_1 \\ i_2 \\ i_3 \end{Bmatrix} \quad (6.25)$$

formed by the ratios of the normal turbulent stresses $\overline{u'^2_i}$. Meanwhile, it remains to determine the ratio of time scales $T/T_c$ in (6.11) – (6.17),

$$\xi = \frac{T}{T_c} \quad (6.26)$$

as a function of the other flow parameters. It can be named as the energy distribution factor.

After performing the operations prescribed by the series (6.13) and (6.14) and using (6.22) and (6.24), we find that the averaged functions (6.11) and (6.12) take the forms

$$\bar{s}_i = \frac{1}{\Delta_i}\left[\frac{\xi^2 - i_i^2}{2\cdot 2!} - \frac{\xi^4 - i_i^4}{4\cdot 4!} + \cdots + (-1)^{n+1}\frac{\xi^{2n} - i_i^{2n}}{2n\cdot(2n)!}\right] \quad (6.27)$$

$$\bar{c}_i = \frac{1}{\Delta_i}\left[\xi - i_i - \frac{\xi^3 - i_i^3}{3\cdot 3!} + \cdots + (-1)^{n+1}\frac{\xi^{2n-1} - i_i^{2n-1}}{(2n-1)\cdot(2n-1)!}\right] \quad (6.28)$$

After the same operations, the definite integrals (6.16) and (6.17) take the similar forms

$$\bar{s}_{i2} = \frac{1}{2\Delta_i}\left[\frac{2^2(\xi^2 - i_i^2)}{2\cdot 2!} - \frac{2^4(\xi^4 - i_i^4)}{4\cdot 4!} + \cdots + (-1)^{n+1}\frac{2^{2n}(\xi^{2n} - i_i^{2n})}{2n\cdot(2n)!}\right] \quad (6.29)$$

$$\bar{c}_{i2} = \frac{1}{2\Delta_i}\left[2(\xi - i_i) - \frac{2^3(\xi^3 - i_i^3)}{3\cdot 3!} + \cdots + (-1)^{n+1}\frac{2^{2n-1}(\xi^{2n-1} - i_i^{2n-1})}{(2n-1)\cdot(2n-1)!}\right] \quad (6.30)$$

**Table 1.** Dimensionalities of the characteristic domains $\Delta$ for the integral (6.6) in $G(\omega, t)$.

| Functions/products | Functions | Double products (dyads) | | | Triple products (triads) | |
|---|---|---|---|---|---|---|
| $f(x_n, t) \equiv f(\omega_i, t)$ | $\rho, u_i$ | $u_i\, u_j$ | $u_i\, u_j$ | $\rho u_i$ | $\rho u_i u_j$ | $\rho u_i u_j$ |
| | | $i = j$ | $i \neq j$ | | $i = j$ | $i \neq j$ |
| $n$=1, 2 ,.. $N$, $i$=1,2,3 | $N = 1$ | $N = 1$ | $N = 2$ | $N = 2$ | $N = 2$ | $N = 3$ |

As a consequence, each element of the series in (6.27) – (6.30) can be written as the product of the domain of averaging

$$\Delta_i = \xi - i_i \tag{6.31}$$

and polynomials $P_{im}$, which follows from the decomposition of the power functions $P_{in}$ for each element of the sums in the series (5.27) – (5.30). The result of this decomposition is given by the relations

$$P_{in} = (\xi - i_i)P_{im};\ m = m(n);\ \ n = 1,2,3, \dots N \tag{6.32}$$

$$P_{im} = \xi^m + i_i\xi^{m-1} + i_1^2\xi^{m-2} + \cdots + \xi^2 i_i^{m-2} + \xi i^{m-1} + i_i^m;\ \ m = n - 1 \tag{6.33}$$

Use of the decompositions (6.32) and (6.33) in (6.27) – (6.30) with $\Delta_i$ given by (6.31) leads to vanishing of $\Delta_i$ and thus to the independence of the averaged functions on the sizes $\Delta_i$ of the characteristic domains of averaging**.** The resulting explicit forms are then

$$\bar{s}_i = \frac{\xi + i_i}{2.2!} - \frac{\xi^3 + \xi^2 i_i + \xi i_i^3}{4.4!} + \cdots + \frac{(-1)^{n+1}\, P_{im}}{2n.(2n)!};\ m_s = 2n - 1 \tag{6.34}$$

$$\bar{c}_i = 1 - \frac{\xi^2 + \xi i_i + i_i^2}{3.3!} + \cdots + \frac{(-1)^{n+1}\, P_{im}}{(2n-1).(2n-1)!};\ m_c = 2(n - 1) \tag{6.35}$$

$$\bar{s}_{i2} = \frac{2(\xi + i_i)}{2.2!} - \frac{2^3(\xi^3 + \xi^2 i_i + \xi i_i^2 + i_i^3)}{4.4!} + \cdots + \frac{(-1)^{n+1}\, 2^{2n}\, P_{im}}{2n.(2n)!} \tag{6.36}$$

$$\bar{c}_{i2} = 1 - \frac{2^2(\xi^2 + \xi i_i + i_i^2)}{3.3!} + \cdots + \frac{(-1)^{n+1}\, 2^{2n-1}\, P_{im}}{(2n-1).(2n-1)!};\ \ i = 1,2,3 \tag{6.37}$$

The properties of the equations (6.31) – (6.33) ensure the validity of the mean values of the constitutive functions (6.34) – (6.37) for any values of the domain of averaging $-\infty < \Delta_i < \infty$, including $\Delta_i = 0$. The functions (6.34) – (6.37), being obtained by explicit averaging, complete the representation of the energy distribution functions (6.3) as functions of the anisotropy vector $i_a(i_1, i_2, i_3)$ and the energy distribution factor $\xi$. Becoming independent of the size of the averaging domains $\Delta_i$ eqs. (6.3) can be written in their final form as

$$\Phi_i(\xi, i_i) = \overline{a_i a_i} = \frac{M_i(\xi, i_i) + A_i(\xi, i_i)}{2M_i^2(\xi, i_i)},\ \ \ i = 1,2,3 \tag{6.38}$$

Their base functions (6.34) – (6.37) are depicted in Fig. 1 at the anisotropy index $i_i$ = 1.

Thus, the effort to complete the first three of the four desired relations for the closure problem solution has been successful. These follow from (6.38) written for $i$ = 1,2,3, as

$$\overline{u'^2_1} + \bar{u}_1^2\ = \bar{u}_1^2 \Phi_1(\xi, i_1) \tag{6.39}$$

$$\overline{u'^2_2} + \bar{u}_2^2 = \bar{u}_2^2 \Phi_2(\xi, i_2) \tag{6.40}$$

$$\overline{u'^2_3} + \bar{u}_3^2 = \bar{u}_3^2 \Phi_3(\xi, i_3) \tag{6.41}$$

where anisotropy indexes $i_i$ as arguments of the distribution functions $\Phi_i$ are defined by equations (6.22) – (6.24) as the functions of the normal turbulent stresses $\overline{u'^2_i}$.

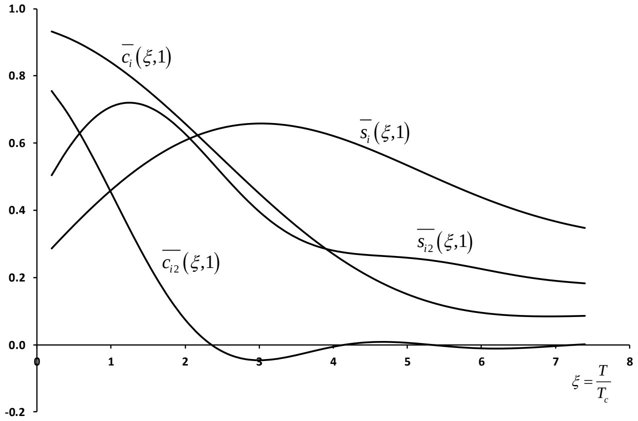


**Fig. 1** Means of the basic constitutive functions $\overline{s_i}(\xi, i_i) = \overline{sin\omega_i t}$, $\bar{c}_i = \overline{cos\omega_i t}$, $\overline{s_{i2}}(\xi, i_i) = \overline{sin\, 2\, \omega_i t}$, and $\overline{c_{i2}}(\xi, i_i) = \overline{cos\, 2\, \omega_i t}$ in the case of isotropy ($i_1 = i_2 = i_3 = 1$). Their anisotropic forms are given by (6.34) – (6.37). The energy distribution factor $\xi$ can be obtained from (6.52) as a function of *k*/*K*.

In the equations (6.39) – (6.41), in addition to the unknown ratio $\xi$, there occur only the directional components of energy $\bar{u}^2{}_i$ and $\overline{u'^2_i}$. These equations are therefore the equations of the kinetic energy distribution**,** further (EDE) of a turbulent velocity field in the four-dimensional space $\bar{G}(x, t)$. The unknown ratio of time scales $\xi = T/T_c$ has still to be found as a function of other flow parameters.

### 6.2 Characteristic velocity of turbulence and application of the energy invariance principle

Anisotropy of turbulent flow can be caused by two factors. The first physical, one follows from limits on the degree of freedom of movement imposed by solid boundaries of the flow. The second, formal one is connected with the mathematical tools applied to the description of vector fields in terms of a coordinate system. In EDT, the effect of anisotropy is expressed by the anisotropy index vector $i_a(i_1, i_2, i_3)$ given in (6.22) – (6.24). From (6.39) – (6.41), it can be seen that its values are positive but not limited from above. The energy invariance principle will be used when solving for the unknown parameter $\xi$ in the system (6.39) – (6.41).

The consideration starts with the system (4.10) – (4.13). The characteristic velocities of turbulence $V_i$, and the kinetic energies $K$ and $k$ are now related by

$V_i = \pm\sqrt{\overline{u_i'^2}}$ , $2K = \sum_{i=1}^{3} \bar{u}^2{}_i$, $2k = \sum_{i=1}^{3} V_i^2$, $K > 0$, $k > 0$, $i = 1, 2, 3$ in $\bar{G}$, (6.42)

which define a pair of vectors $\bar{\boldsymbol{u}}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ and $V(V_1, V_2, V_3)$ at each point of the turbulent flow. By projections of the absolute values of the vectors onto the coordinate axes, the components of the vectors

$$\bar{u}_i = |\bar{\boldsymbol{u}}| \cos\alpha_i,\ V_i = |V| \cos\beta_i,\ |\bar{\boldsymbol{u}}| = \sqrt{2K},\ |V| = \sqrt{2k} \quad (6.43)$$

are determined through their direction cosines $\cos\alpha_i$ and $\cos\beta_i$, $i = 1,2,3$. Let the vectors $\bar{\boldsymbol{u}}$ and $\boldsymbol{V}$ have a common initial point that coincides with the origin of the rectangular coordinate system. We now replace this coordinate system by a new one with the same origin, i.e., we rotate the axes. Let us do so in such a way that the axis of symmetry of the new coordinate trihedron coincides with the vector $\boldsymbol{V}$. Owing to symmetry, the direction cosines of the axis of the trihedron in the new coordinate system have the same value $1/\sqrt{3}$. Since the characteristic velocity vector $\boldsymbol{V}$ is collinear with the axis of the trihedron, all the $\cos\beta_i$, $i = 1,2,3$, in the new coordinates also have the same value $1/\sqrt{3}$.

Application of the energy invariance principle now means that the energies $K$ and $k$ are conserved after the above rotation of the coordinate system. Then, of course, one must accept changes in the direction energy components arising owing to changes in the velocity components through their projections on the new coordinates. The characteristic velocity components $V_i$ have acquired new values $V_{ir}$, but are equal to each other because the direction cosines have the same value. This reasoning applied to (6.43) leads to

$$V_{1r} = V_{2r} = V_{3r} = |\boldsymbol{V}|/\sqrt{3} = \sqrt{2k/3} \quad (6.44)$$

The equality of the velocity components in (6.44) implies a similar equality between the normal components of the stress tensor $\overline{u_i' u_i'} = V_i^2 = \tau_{ii}$. According to (6.39) – (6.41), this leads to the expected change in the anisotropy indices such that they take the common value

$$i_1 = i_2 = i_3 = 1 \quad (6.45)$$

The components of the velocity vector $\bar{u}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ change to $\bar{\boldsymbol{u}}_r(\bar{u}_{1r}, \bar{u}_{2r}, \bar{u}_{3r})$, and taking (6.45) in the distribution functions $\Phi_i(\xi, i_i)$, we find the distribution equations (6.39) – (6.41) taking the form

$$2k/3 + \bar{u}_{1r}^2 = \bar{u}_{1r}^2 \Phi_1(\xi, 1) \quad (6.46)$$

$$2k/3 + \bar{u}_{2r}^2 = \bar{u}_{2r}^2 \Phi_2(\xi, 1) \quad (6.47)$$

$$2k/3 + \bar{u}_{3r}^2 = \bar{u}_{3r}^2 \Phi_3(\xi, 1) \quad (6.48)$$

The distribution functions $\Phi_i(\xi, i_i)$ differ only in the anisotropy indices $i_i$. Therefore, when (6.45) is used in (6.46) – (6.48), they become equal:

$$\Phi_1(\xi, 1) = \Phi_2(\xi, 1) = \Phi_3(\xi, 1) = \Phi_e(\xi) \quad (6.49)$$

The sum of the three equations (6.46) – (6.48) now gives

$$2k + \left(1 - \Phi_e(\xi)\right)(\bar{u}_{1r}^2 + \bar{u}_{2r}^2 + \bar{u}_{3r}^2) = 0 \quad (6.50)$$

Energy invariance during rotation of the coordinate system requires from the sum of velocity squares in (6.50) to satisfy relation

$$\bar{u}_{1r}^2 + \bar{u}_{2r}^2 + \bar{u}_{3r}^2 = 2K \quad (6.51)$$

But this allows us to write (6.50) in the form

$$\Phi_e(\xi) = 1 + \frac{k}{K} \quad (6.52)$$

Since the system (6.46) – (6.48) contains three new dependent variables $\bar{u}_{ir}$ , none of its three equations provides useful information. This also applies in their possible combinations, except for the one that we used. This was simply their sum, which, owing to energy invariance, has led us to the new relation (6.52).

Obtaining (6.52) was the decisive step in solving the closure problem of turbulence. Fig.2 shows the key dependence $\Phi_e(\xi) = \Phi_i(\xi, 1)$, as well as $A_e(\xi) = A_i(\xi, 1)$ with $M_e(\xi) = M_i(\xi, 1)$.

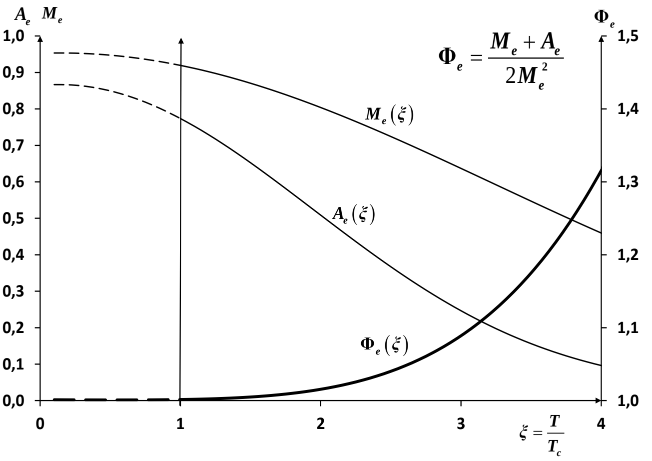


**Fig. 2.** Isotropic case $\Phi_e(\xi) = \Phi_i(\xi, 1)$ of the distribution function $\Phi_i(\xi, i_i)$and its components. The vertical line $\xi$ = 1 depicts the lower limit of the parameter $\xi$.

The free parameter $\xi = T/T_c$ in the distribution functions $\Phi_i(\xi, i_i)$ was the last of those that needed to be defined by an appropriate relation with other parameters of turbulent flow. $\Phi_e(\xi)$ itself is explicitly defined by (6.38) and (6.49) and allows the unknown parameter $\xi$ to be determined via (6.52) as a function of the energy ratio $k/K$. After that we can name $\xi$ the energy distribution factor.

The functions $\bar{s}_i = \overline{\sin\omega_i t}$, $\bar{c}_i = \overline{\cos\omega_i t}$, $\bar{s}_{i2} = \overline{\sin(2\omega_i t)}$, and $\bar{c}_{i2} = \overline{\cos(2\omega_i t)}$ are the underlying elements of EDT, through which all dependent variable parameters of turbulent flow are defined. They are explicitly given by (6.34) – (6.37) as functions of the distribution factor $\xi$ and the three components of the anisotropy vector $\boldsymbol{i}_a(i_1, i_2, i_3)$. Figure 1 depicts them for the one-dimensional isotropic case, when the anisotropy indices $i_1 = i_2 = i_3 = 1$. Their full four-dimensional form is numerically investigated through the distribution function (6.38) during the confrontation of EDT with experiment described in Section 8.

The energy distribution equations (6.39) – (6.41) together with (6.52) are the key output of the EDT, reflecting the current state of the kinetic energy distribution of the flow. The distribution function (6.38) is the determining element of the system of distribution equations. Its isotropic case from (6.52) for calculating the distribution factor $\xi$ is shown in Fig. 2. The behavior of the presented functions when the arguments grow without limit has not yet been rigorously investigated.

## 7 STRAIGHT AVERAGING OF THE ISOTHERMAL N–S AND CONTINUITY EQUATIONS

From the point of view of the turbulence closure problem, the equations of energy distribution (6.49) – (6.41) mean the crucial result of Part II. of this study. Using it as the main tool of the theory, the straight closure process can be completed successfully for governing equations of liquid as well as gas flows. For mass and momentum conservation laws it is shown below.

### 7.1 Implicit averaging of the N–S equations

The dynamics of real fluids are described by the following equations for the conservation of mass and momentum (Lojcianskij, 1954, II, p. 137) or (Milne-Thomson, L. M., (1960), § 19.03):

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \boldsymbol{u}) = 0 \tag{7.1}$$

$$\rho \frac{d\boldsymbol{u}}{dt} = \rho \boldsymbol{F} - \mathrm{grad}\, p + \mathrm{div}\, \boldsymbol{\tau} \tag{7.2}$$

where $\boldsymbol{F}$ denotes the vector of external forces, $p$ is a pressure and $\boldsymbol{\tau}$ means a viscous stress tensor, see subsection 7.5. For isothermal flow, viscosity $\mu$ will be constant and $\boldsymbol{\tau}$ a linear function of spatial derivatives of the velocity fields. The vector notation (7.2) of three momentum equations, after a small modification mentioned below, will be ready for the direct implicit averaging operation. The modified system can be implicitly averaged resulting in

$$\frac{\partial \overline{\rho \boldsymbol{u}}}{\partial t} + div(\overline{\rho \boldsymbol{u}\boldsymbol{u}}) = \bar{\rho}\boldsymbol{F} - \mathrm{grad}\,\bar{p} + \mathrm{div}\,\overline{\boldsymbol{\tau}} \tag{7.3}$$

$$\frac{\partial \bar{\rho}}{\partial t} + div\,\overline{\rho \boldsymbol{u}} = 0 \tag{7.4}$$

when eq. (7.2) before its averaging was rewritten in the conservation form (7.3) in the way as it is shown in equations line (2.6) of Sec. 2.5 in Part I. of the study. The product $\rho uu$ in. (7.2) creating tensor of nine nonlinearities would be explicitly averaged. We will call it the momentum flux tensor.

### 7.2 Straight explicit averaging of N–S and continuity equations in a random space $\boldsymbol{G}(\boldsymbol{\omega}, \boldsymbol{t})$

The energy distribution equations (6.39) – (6.41) together with (6.52) provide three implicit relations for the normal turbulent stresses defined in (5.17). Since all these relations were derived independently of the N-S equations, the definition of the tensor (5.17) provides explicit dependences also for the remaining three tangential components of the tensor. They will all depend on the energy and velocity fields of the mean flow. However, the constitutive functions and the existence of energy distribution equations allow us to close the averaged N-S system (7.3), (7.4) after averaging of all nonlinear terms of the system. We determine the mean values of nonlinearities in the system (7.3), (7.4) through the definite integrals of the type in (6.6). The integrand functions appearing in (7.3) and (7.4) in the form of twin and triple products of distribution functions $a_i$ for velocities $u_i$ and $a_\rho$ for density $\rho$ are given by (5.9) or (5.19).

For $a_\rho$, we must define the boundaries of the characteristic domain $\omega_{\rho D}$ and $\omega_{\rho H}$. Owing to isotropy, i.e., the same (higher) frequencies of turbulent disturbances, the upper limit will be the same as for velocities, i.e.

$$\omega_{\rho H} = \Omega_c = \frac{1}{T_c}. \tag{7.5}$$

We still have to choose the lower boundary $\omega_{\rho D}$. The mean specific density $\bar{\rho}$ is the result of turbulent mixing, and there is no reason to assume anisotropy. Therefore, the lower limit will be determined by the average time scale *T*, and so the lower limit of the frequency $\omega_{\rho D}$ will be

$$\omega_{\rho D} = \Omega_\rho = \frac{1}{T}. \tag{7.6}$$

To achieve the main goal of this study, it is still necessary to determine the explicit means of the dyad and triad non-linearities that appear in the system (7.3) – (7.4). The above steps allowed us to discover such integrand functions and its definite integrals which yield the means of the non-linearities having the needed property of the mathematical expectation. Such integrand functions representing in (7.3) – (7.4) the vector of momentum $\rho\boldsymbol{u}$ and tensor of the momentum stream $\rho\boldsymbol{u}\boldsymbol{u}$ for gases as well as $\boldsymbol{u}\boldsymbol{u}$ for liquids will be given by the products

$$\rho\boldsymbol{u} = [\bar{\rho}\,\bar{u}_i a_\rho\, a_i];\ \boldsymbol{u}\boldsymbol{u} = [\bar{u}_i \bar{u}_j a_i a_j];\ \rho\boldsymbol{u}\boldsymbol{u} = [\bar{\rho}\,\bar{u}_i\bar{u}_j a_\rho\, a_i a_j]\ ; \quad i,j = 1,2,3,.. \tag{7.7}$$

in which constitution functions $a_i$ and $a_j$ are given by (5.9) and $a_\rho$ by (4.19). To get $a_\rho$, it is still necessary to define averages $\overline{sin(\omega_\rho t)}$ and $\overline{cos(\omega_\rho t)}$ by integration between the limits given in (6.5) and (6.6), which yields

$$\bar{s}_\rho = \frac{1}{T(\Omega_c - \Omega_\rho)} \int_0^T \int_{\Omega_\rho}^{\Omega_c} sin(\omega_\rho t)\, d\omega_\rho dt = \bar{s}_\xi = \bar{s}_i(\xi, 1) \tag{7.8}$$

$$\bar{c}_\rho = \frac{1}{T(\Omega_c - \Omega_\rho)} \int_0^T \int_{\Omega_\rho}^{\Omega_c} cos(\omega_\rho t)\, d\omega_\rho dt = \bar{c}_\xi = \bar{c}_i(\xi, 1) \tag{7.9}$$

These results are identical with (6.11) and (6.12) because $i_i$ = 1 owing to the bound (7.6). One gets the explicit average of (7.7) by applying definite integral (6.6) to all integrands (7.7) whilst respecting the rules (2.15). Then the mean gas momentum field $\overline{\rho u_i}$ will be made by

$$\overline{\rho u_i} = \frac{\bar{\rho}\bar{u}_i}{M_\rho M_i \Delta_{i\rho}} \int_0^T \int_{\Omega_\rho}^{\Omega_c} \int_{\Omega_i}^{\Omega_c} (\bar{c}_\rho c_\rho + \bar{s}_\rho s_\rho)\,(\bar{c}_i\, c_i + \bar{s}_i s_i)\; d\omega_\rho\, d\omega_i dt, \quad i = 1,2,3 \tag{7.10}$$

The dyads $\overline{u_\iota u_J}$ yield by-products $\rho\overline{u_\iota u_J}$ the momentum flux tensor for liquids being given by

$$\overline{u_i u_j} = \frac{\bar{u}_i\, \bar{u}_j}{M_i M_j \Delta_{ij}} \int_0^T \int_{\Omega_i}^{\Omega_c} \int_{\Omega_j}^{\Omega_c} (\bar{c}_i c_i + \bar{s}_i s_i)\,(\bar{c}_j c_j + \bar{s}_j s_j)\; d\omega_j\, d\omega_i\, dt, \quad i \neq j, \quad i,j = 1,2,3. \tag{7.11}$$

But (7.11) bears further information concerning liquids. Decomposition $\rho\overline{u_\iota u_J} = \rho\bar{u}_i\bar{u}_j + \rho\overline{u'_\iota u'_J}$ for $i \neq j$ yields the Reynolds' (apparent) stress tensor $\rho\overline{u'_\iota u'_J}$ and (6.11) for *i* = *j* is identical to the energy distribution equations we have obtained. The integration domains in (7.10) and (7.11) are

$$\Delta_{i\rho} = T(\Omega_c - \Omega_\rho)(\Omega_c - \Omega_i),\ \Delta_{ij} = T(\Omega_c - \Omega_i)(\Omega_c - \Omega_j),$$

$i \neq j.$ (7.12)

The triad in (7.7) describing the mean momentum flux tensor for gas is defined by the integral

$$\overline{\rho u_i u_j} = \frac{M_{\rho ij}}{\Delta_{\rho ij}} \int_0^T \int_{\Omega_\rho}^{\Omega_c} \int_{\Omega_i}^{\Omega_c} \int_{\Omega_j}^{\Omega_c} (\bar{c}_\rho c_\rho + \bar{s}_\rho s_\rho)(\bar{c}_i c_i + \bar{s}_i s_i)(\bar{c}_j c_j + \bar{s}_j s_j)\, d\omega_\rho\, d\omega_i\, d\omega_j\, dt, \quad (7.13)$$

with integration domains

$$\Delta_{\rho ij} = T\Delta_\rho \Delta_i \Delta_j, \quad \Delta_\rho = \Omega_c - \Omega_\rho, \quad \Delta_i = \Omega_c - \Omega_i, \quad \Delta_j = \Omega_c - \Omega_j, \quad (7.14)$$

and the factor

$$M_{\rho ij} = \frac{\bar{\rho}\, \bar{u}_i\, \bar{u}_j}{M_\rho\, M_i\, M_j}, \quad i = 1,2,3, \quad (7.15)$$

All the above multiple integrals can be evaluated directly by series expansion of the trigonometric functions. We will show that power series defined by the integrals (7.8) – (7.18) will have the same properties as (6.31) and (6.33), leading to independence of the averaged functions on the size of the domain of averaging. The definite integral (7.13) consists of eight separate products of three power series. Each term of each series resulting from the multiplication operation has the form of the *m*-th product

$$(\omega_i t)^p (\omega_j t)^q (\omega_\rho t)^r, \; p,q,r = 0,1,2,3, \ldots \; s = p + q + r \quad (7.16)$$

for *m* = 1, 2, 3, ..., *mc*, where *mc* follows from the amount of the power series terms included.

Realizing all averaging operations in random space $G(\omega, t)$ to be of probabilistic nature the validity of the Reynolds rule $\overline{\bar{u}_i} = \bar{u}_i$ was accepted during integration for all integrands, including those in (7.10) – (7.13). Therefore, the resultant definite integral in (7.13) divided by the integration domain defines the *m*-th averaged function $J_m$ as

$$J_m = \frac{1}{\Delta_{\rho ij}} \int_0^T \int_{\Omega_\rho}^{\Omega_c} \int_{\Omega_i}^{\Omega_c} \int_{\Omega_j}^{\Omega_c} (\omega_i t)^p (\omega_j t)^q (\omega_\rho t)^r\, d\omega_\rho\, d\omega_i\, d\omega_j\, dt = \frac{\left[\omega_i^{p+1} \omega_j^{q+1} \omega_\rho^{r+1} t^{s+1}\right]_{DH}^{HH}}{\Delta_{\rho ij}(p+1)(q+1)(r+1)(s+1)} \quad (7.17)$$

If we apply for the upper and lower limits *HH* and *DH* in (7.17) the limits of domains (7.14), then $J_m$ acquires the form

$$J_m = \frac{({\Omega_c}^{p+1} - {\Omega_i}^{p+1})({\Omega_c}^{q+1} - {\Omega_j}^{q+1})({\Omega_c}^{r+1} - {\Omega_\rho}^{r+1}) T^{s+1}}{\Delta_{\rho ij}(p+1)(q+1)(r+1)(s+1)} \quad (7.18)$$

If we decompose the first set of parentheses in the numerator of (7.18) into powers, multiply it by $T^p$, and take (7.14) into account, then we obtain

$$T^p \left({\Omega_c}^{p+1} - {\Omega_i}^{p+1}\right) = \Delta_i T^p \left({\Omega_c}^p + \Omega_i {\Omega_c}^{p-1} + {\Omega_i}^2 {\Omega_c}^{p-2} + \cdots + {\Omega_c}^2 {\Omega_i}^{p-2} + \Omega_c {\Omega_i}^{p-1} + {\Omega_i}^p\right). \quad (7.19)$$

Since each product of the integral time scale *T* with the bounding frequencies $\Omega$ satisfies the relations

$$T\Omega_c = \xi, \quad T\Omega_i = i_i, \quad T\Omega_j = i_j, \quad T\Omega_\rho = 1, \quad i = 1,2,3 \quad (7.20)$$

any *k*-th product ${\Omega_i}^k {\Omega_c}^{p-k}$ in (7.19) can be rewritten, after multiplication by $T^p$, in the form

$$T^p {\Omega_i}^k {\Omega_c}^{p-k} = (\Omega_i T)^k (\Omega_c T)^{p-k} = i_i^k\, \xi^{p-k}, k = 0,1,2,3, \ldots, p \quad (7.21)$$

By the operation (7.21) one can rewrite the right-hand side of (7.19) with polynomial $P_{ip}$ obtaining

$$T^p \left(\Omega_c^{p+1} - \Omega_i^{p+1}\right) = \Delta_i P_{ip} \quad (7.22)$$

and repeating the procedure used in (7.19) up (7.21) with the other two sets of parentheses in the numerator (7.18), but this time for powers *q* and *r* leads to

$$T^q \left(\Omega_c^{q+1} - \Omega_j^{q+1}\right) = \Delta_j P_{jq} \quad (7.23)$$

$$T^r \left(\Omega_c^{r+1} - \Omega_\rho^{r+1}\right) = \Delta_\rho P_{\rho r} \quad (7.24)$$

with the polynomials (7.33) given now for maximal powers $m = n - 1 = p, q, r;\;\; i, j = 1,2,3$

$$P_{ip} = \xi^p + i_i\, \xi^{p-1} + i_i^2\, \xi^{p-2} + \cdots + \xi^2\, i_i^{p-2} + \xi\, i_i^{p-1} + i_i^p \quad (7.25)$$

$$P_{jq} = \xi^q + i_j\, \xi^{q-1} + i_j^2\, \xi^{q-2} + \cdots + \xi^2\, i_j^{q-2} + \xi\, i_j^{q-1} + i_j^q \quad (7.26)$$

$$P_{\rho r} = \xi^r + \xi^{r-1} + \xi^{r-2} + \cdots + \xi^2 + \xi + 1 \quad (7.27)$$

Since the properties of the polynomials (7.25) – (7.27) are identical to those defined by (6.31) – (6.33), using (7.22) – (7.24) in (7.18) we get resulted form for the functions averaged by (7.17)

$$J_m = \frac{P_{ip} P_{jq} P_{\rho r}}{(p+1)(q+1)(r+1)(s+1)} \;;\; i,j = 1,2,3,\; p,q,r = 0,1,2,\ldots \text{ for triads } \overline{\rho u_i u_j} \quad (7.28)$$

for triads $\overline{\rho u_i u_j}$, while for dyads $\overline{\rho u_i}$ $q = 0$, $P_{jq} = 1$ and for $\overline{u_i u_j}$ $r = 0$, $P_{\rho r} = 1$. Since products $J_m$ do not depend on sizes $\boldsymbol{\Delta_i}$ of the domain of averaging (7.14), the averaged dyad and triad non-linearities gain the property of a mathematical expectation**.** They arose from the integrands defined by the constitutive functions in (7.7) and resulted in the wanted explicit means

$$\overline{u_i u_j} = \bar{u}_i \bar{u}_j \Phi_{ij}(\xi, i_i, i_j) \quad (7.29)$$

$$\overline{\rho u_i} = \bar{\rho} \bar{u}_i \Psi_i(\xi, i_i, 1) \quad (7.30)$$

$$\overline{\rho u_i u_j} = \bar{\rho} \bar{u}_i \bar{u}_j \Psi_{ij}(\xi, i_i, i_j, 1) \quad (7.31)$$

being expressed through distribution functions $\Psi_i$ and $\Psi_{ij}$ of the momentum vector $\overline{\rho u}$ and the momentum flux tensor $\overline{\rho u u}$ for gases as well as through distribution functions $\Phi_{ij}$ of the tensor of momentum flux $\overline{uu}$ for liquids. All these functions are defined by integrals (7.32) – (7.34) as

$$\Phi_{ij} = \frac{1}{\mathrm{M}_i \mathrm{M}_j} \left( \bar{s}_i \bar{s}_j \overline{s_i s_j} + \bar{s}_i \bar{c}_j \overline{s_i c_j} + \bar{c}_i \bar{s}_j \overline{c_i s_j} + \bar{c}_i \bar{c}_j \overline{c_i c_j} \right);\;\; i,j = 1,2,3 \quad (7.32)$$

$$\Psi_i = \frac{1}{\mathrm{M}_i \mathrm{M}_\rho} \left( \bar{s}_i \bar{s}_\rho \overline{s_i s_\rho} + \bar{s}_i \bar{c}_\rho \overline{s_i c_\rho} + \bar{c}_i \bar{s}_\rho \overline{c_i s_\rho} + \bar{c}_i \bar{c}_\rho \overline{c_i c_\rho} \right);\;\; i = 1,2,3 \quad (7.33)$$

$$\Psi_{ij} = \frac{1}{M_\rho M_i M_j}\left(\bar{c}_\rho \bar{c}_i \bar{c}_j \overline{c_\rho c_i c_j} + \bar{c}_\rho \bar{c}_i \bar{s}_j \overline{c_\rho c_i s_j} + \bar{c}_\rho \bar{s}_i \bar{c}_j \overline{c_\rho s_i c_j} + \bar{c}_\rho \bar{s}_i \bar{s}_j \overline{c_\rho s_i s_j} + \bar{s}_\rho \bar{c}_i \bar{c}_j \overline{s_\rho c_i c_j} + \bar{s}_\rho \bar{c}_i \bar{s}_j \overline{s_\rho c_i s_j} + \bar{s}_\rho \bar{s}_i \bar{c}_j \overline{s_\rho s_i c_j} + \bar{s}_\rho \bar{s}_i \bar{s}_j \overline{s_\rho s_i s_j}\right) \quad i,j = 1,2,3 \tag{7.34}$$

### 7.3 A few samples of averaging of the above dyad products of trigonometric functions

The means of deriving and the properties of these distribution functions have been consistently described above, beginning with equation (7.10). Their resultant explicit form was reached using routine tools of applied mathematics by means of the products of polynomials (7.28). The following samples illustrate the mentioned proceedings. The development of averaging of the dyad products of trigonometric functions in (7.32) and (7.33) can be shown by the following equations:

$$s_i s_\rho = t^2\omega_i\omega_\rho - \frac{t^4}{3!}\left(\omega_i^3\omega_\rho + ..\right) \to \overline{s_i s_\rho} = \frac{1}{3}\frac{P_{i1}P_{\rho 1}}{2\,.\,2} - \frac{1}{5\,.\,3!}\left(\frac{P_{i3}P_{\rho 1}}{4\,.\,2} + \ldots\right) \tag{7.35}$$

$$s_i c_j = t\omega_i - t^3\left(\frac{\omega_i^3}{3!} + \frac{\omega_i\omega_j^2}{2!}\right) + \ldots \to \overline{s_i c_j} = \frac{P_{i1}}{2\,.\,2} - \frac{1}{4}\left(\frac{P_{i3}}{4\,.\,3!} + \frac{P_{i1}P_{j2}}{2\,.\,3\,.\,2!}\right) + \ldots \tag{7.36}$$

The sample of averaging of any triad in (7.34) demands two lines,

$$s_i c_j s_\rho = t^2\omega_\rho\omega_i - t^4\left(\frac{\omega_\rho\omega_i^3 + \omega_\rho^3\omega_i}{3!} + \frac{\omega_\rho\omega_i\omega_j^2}{2!}\right) + t^6\left(\frac{\omega_\rho\omega_j^5 + ..}{5!} + \ldots\right) - \cdots \tag{7.37}$$

$$\overline{s_i c_j s_\rho} = \frac{P_{\rho 1}P_{i1}}{3\,.\,2\,.\,2} - \frac{1}{5}\left(\frac{P_{\rho 1}P_{i3} + P_{\rho 3}P_{i1}}{2\,.\,3!} + \frac{P_{\rho 1}P_{i1}P_{j2}}{2\,.\,2\,.\,3\,.\,2!}\right) + \frac{1}{7}\left(\frac{P_{\rho 1}P_{i5} + ..}{2\,.\,6\,.\,5} + \ldots\right) - \cdots \tag{7.38}$$

Finally, it is applicable to remember that polynomials $P_{ip}$, $P_{jq}$, $P_{\rho r}$ are explicitly defined in (7.25) – (7.27) as functions of the distribution factor $\xi$ and the anisotropy indexes $i_i$ , $i_j$, while $\xi$ given by (6.52) and indexes $i_i$ , $i_j$ in (6.22) – (6.24) are functions of mean turbulent energy and velocity fields. The number *mc* of products $J_m$ in (7.28) depends on the highest expansion level $p, q, r$considered.

Equations (7.10) – (7.29) define all the non-linear terms in the averaged N–S system (7.3), (7.4).

The most important averaged fluctuation non-linearities occurring in the closed system are the normal turbulent stresses $\overline{u'^2_i}$. They determine the anisotropy indices $\boldsymbol{i_i}$ by (6.22–6.24) and the *distribution factor* $\boldsymbol{\xi}$ by (6.52) creating in such way the arguments of all the distribution functions $\Phi_{ij}$ , $\Psi_i$ and $\Psi_{ij}$**,** written above as the products of integration in (7.10) – (7.13). The first of them, $\Phi_{ij}$ equals at $i = j$ the energy distribution function $\Phi_i$ given by (6.38), so

$$\Phi_{ii} = \Phi_i \text{ at } i = j; \qquad i,j = 1,2,3. \tag{7.39}$$

The above non-linearities of turbulent flow are defined and averaged in the multidimensional characteristic domain of averaging $G(\omega, t)$ of the finite but unknown size $\Delta_{\rho ij} = T\Delta_\rho\Delta_i\Delta_j$. Being averaged they become functions of the mean velocity field $\bar{u}$ and fluctuation energy *k*. These dependences are expressed through the distribution factor $\boldsymbol{\xi}$ and the anisotropy indices $\boldsymbol{i_i}$ bounded below by $\boldsymbol{\xi} > 1$ and $i_i > 1/3$.

### 7.4 The closed equation system as resultant of the closure problem solution for isothermal flows

In the vectorial notation of implicitly averaged N–S equations (7.3) and (7.4), vector $\overline{\rho\boldsymbol{u}} = [\overline{\rho u_i}]$ determines the momentum of the fluids per fluid volume unit. Tensor $\overline{\rho\boldsymbol{uu}} = [\,\overline{\rho u_i u_j}\,]$ denotes the implicitly averaged flux (propagation velocity) of momentum $\rho\boldsymbol{u}$. The direct explicit form of averaging dyadic and triadic non-linearities $\rho\boldsymbol{u}$, $\boldsymbol{uu}$ and $\rho\boldsymbol{uu}$ is provided by certain integrals of these non-linearities in (7.10) to (7.13) over the characteristic domains $G(\boldsymbol{\omega}, t)$ of time $t$ and of the random turbulent fluctuation frequencies of velocity $\omega_i$ and density $\omega_\rho$. Defining the integrands of averaging integrals by the relations (7.7) via the constitutive functions $\boldsymbol{a_i}$, $\boldsymbol{a_j}$ and $\boldsymbol{a_\rho}$ is the decisive theoretical step of this study. The wanted form of explicitly averaging the first kind of non-linearities of the system (7.3) – (7.4) has already been obtained by standard operations of applied mathematics. Accepting that the viscosity $\mu(\overline{T})$ is regular function of a mean absolute temperature $\overline{T}$ the viscous tensor $\tau$ components in this system are linear functions of products $\mu$ grad $u_i$; see Section 10 below. Averaging $\tau \to \overline{\tau}$ was therefore possible to carry out directly applying Reynolds' rule of averaging partial derivatives in the velocity gradients.

The momentum vector $\rho\boldsymbol{u}$ if explicitly averaged by integral (7.10) is given through its components

$$\overline{\rho u_i} = \bar{\rho}\,\bar{u}_i\Psi_i \quad i = 1,2,3 \tag{7.40}$$

The symmetric tensor of flux of momentum $h = \rho\boldsymbol{uu}$ after being averaged by means of integral (7.13) has been presented by the matrix form through its vector $\overline{\boldsymbol{h_i}}$ and tensor $\bar{u}_i\bar{u}_j\Psi_{ij}$ components

$$\overline{\rho\boldsymbol{uu}} = \overline{\rho u_i u_j} = \begin{Bmatrix} \overline{\boldsymbol{h_1}} \\ \overline{\boldsymbol{h_2}} \\ \overline{\boldsymbol{h_3}} \end{Bmatrix} = \bar{\rho}\begin{Bmatrix} \bar{u}_1\bar{u}_1\Psi_{11} & \bar{u}_1\bar{u}_2\Psi_{12} & \bar{u}_1\bar{u}_3\Psi_{13} \\ \bar{u}_1\bar{u}_2\Psi_{21} & \bar{u}_2\bar{u}_2\Psi_{22} & \bar{u}_2\bar{u}_3\Psi_{23} \\ \bar{u}_1\bar{u}_3\Psi_{31} & \bar{u}_2\bar{u}_3\Psi_{32} & \bar{u}_3\bar{u}_3\Psi_{33} \end{Bmatrix}, \quad i,j = 1,2,3 \tag{7.41}$$

In the case of liquids and assuming that $\rho = \bar{\rho}$ = const. the mean momentum flux tensor stems from the integral (7.11) written in the matrix form

$$\rho\overline{\boldsymbol{uu}} = \rho\,\overline{u_i u_j} = \begin{Bmatrix} \overline{\boldsymbol{h_1}} \\ \overline{\boldsymbol{h_2}} \\ \overline{\boldsymbol{h_3}} \end{Bmatrix} = \rho\begin{Bmatrix} \bar{u}_1\bar{u}_1\Phi_{11} & \bar{u}_1\bar{u}_2\Phi_{12} & \bar{u}_1\bar{u}_3\Phi_{13} \\ \bar{u}_1\bar{u}_2\Phi_{21} & \bar{u}_2\bar{u}_2\Phi_{22} & \bar{u}_2\bar{u}_3\Phi_{23} \\ \bar{u}_1\bar{u}_3\Phi_{31} & \bar{u}_2\bar{u}_3\Phi_{32} & \bar{u}_3\bar{u}_3\Phi_{33} \end{Bmatrix}, i,j = 1,2,3 \tag{7.42}$$

Equations (7.3) and (7.4) imply the vectorial but implicit notation of the wanted averaged N–S system governing turbulent flow of Newtonian fluids. Its resulting explicit notation

$$\frac{\partial(\bar{\rho}\,\bar{u}_1\Psi_1)}{\partial t} + \frac{\partial(\bar{\rho}\,\bar{u}_1\bar{u}_1\Psi_{11})}{\partial x_1} + \frac{\partial(\bar{\rho}\,\bar{u}_1\bar{u}_2\Psi_{12})}{\partial x_2} + \frac{\partial(\bar{\rho}\,\bar{u}_1\bar{u}_3\Psi_{13})}{\partial x_3} = \bar{\rho}F_x - \frac{\partial\bar{p}}{\partial x_1} + \text{div}\,\overline{\boldsymbol{\tau}}_{\boldsymbol{1}}$$

$$\frac{\partial(\bar\rho\,\bar u_2\Psi_2)}{\partial t}+\frac{\partial(\bar\rho\bar u_2\bar u_1\Psi_{21})}{\partial x_1}+\frac{\partial(\bar\rho\bar u_2\bar u_2\Psi_{22})}{\partial x_2}+\frac{\partial(\bar\rho\bar u_2\bar u_3\Psi_{23})}{\partial x_3}=\bar\rho F_y-\frac{\partial\bar p}{\partial x_2}+\operatorname{div}\bar{\boldsymbol\tau}_{\mathbf 2} \quad (7.43)$$

$$\frac{\partial(\bar\rho\,\bar u_3\Psi_3)}{\partial t}+\frac{\partial(\bar\rho\bar u_3\bar u_1\Psi_{31})}{\partial x_1}+\frac{\partial(\bar\rho\bar u_3\bar u_2\Psi_{32})}{\partial x_2}+\frac{\partial(\bar\rho\bar u_3\bar u_3\Psi_{33})}{\partial x_3}=\bar\rho F_z-\frac{\partial\bar p}{\partial x_3}+\operatorname{div}\bar{\boldsymbol\tau}_{\mathbf 3}$$

$$\frac{\partial\bar\rho}{\partial t}+\frac{\partial(\bar\rho\,\bar u_1\Psi_1)}{\partial x_1}+\frac{\partial(\bar\rho\,\bar u_2\Psi_2)}{\partial x_2}+\frac{\partial(\bar\rho\,\bar u_3\Psi_3)}{\partial x_3}=0 \quad (7.44)$$

has been completed into the final closed shape by the EDE (energy distribution equations)

$$\overline{u'^2_i}+\bar u_i^2[1-\Phi_i(\xi,i_i)]=0, i=1,2,3 \quad (7.45)$$

where the distribution factor $\xi = T/T_c$ depends on the energy ratio $k/K$ satisfying relations

$$\Phi_e(\xi)=1+k/K;\ \Phi_e(\xi)=\Phi_i(\xi,1) \text{ at } i_1=i_2=i_3=1; \quad (7.46)$$

and three anisotropy indices $i_i$ are determined as functions of normal turbulent stresses $\overline{u'^2_i}$ by

$$i_1=\frac13\left(1+\frac{\overline{u'^2_2}}{\overline{u'^2_1}}+\frac{\overline{u'^2_3}}{\overline{u'^2_1}}\right);\quad i_2=\frac13\left(\frac{\overline{u'^2_1}}{\overline{u'^2_2}}+1+\frac{\overline{u'^2_3}}{\overline{u'^2_2}}\right);\quad i_3=\frac13\left(\frac{\overline{u'^2_1}}{\overline{u'^2_3}}+\frac{\overline{u'^2_2}}{\overline{u'^2_3}}+1\right) \quad (7.47)$$

The resulted system of averaged PDEs for liquid flows with $\rho = const.$ differs from previous little:

$$\frac{\partial\bar u_1}{\partial t}+\frac{\partial(\bar u_1\bar u_1\Phi_{11})}{\partial x_1}+\frac{\partial(\bar u_1\bar u_2\Phi_{12})}{\partial x_2}+\frac{\partial(\bar u_1\bar u_3\Phi_{13})}{\partial x_3}=F_x-\frac{1}{\bar\rho}\left(\frac{\partial\bar p}{\partial x_1}+\operatorname{div}\bar{\boldsymbol\tau}_{\mathbf 1}\right)$$

$$\frac{\partial\bar u_2}{\partial t}+\frac{\partial(\bar u_2\bar u_1\Phi_{21})}{\partial x_1}+\frac{\partial(\bar u_2\bar u_2\Phi_{22})}{\partial x_2}+\frac{\partial(\bar u_2\bar u_3\Phi_{23})}{\partial x_3}=F_y-\frac{1}{\bar\rho}\left(\frac{\partial\bar p}{\partial x_2}+\operatorname{div}\bar{\boldsymbol\tau}_{\mathbf 2}\right)$$

$$\frac{\partial\bar u_3}{\partial t}+\frac{\partial(\bar u_3\bar u_1\Phi_{31})}{\partial x_1}+\frac{\partial(\bar u_3\bar u_2\Phi_{32})}{\partial x_2}+\frac{\partial(\bar u_3\bar u_3\Phi_{33})}{\partial x_3}=F_z-\frac{1}{\bar\rho}\left(\frac{\partial\bar p}{\partial x_3}+\operatorname{div}\bar{\boldsymbol\tau}_{\mathbf 3}\right) \quad (7.48)$$

$$\operatorname{div}(\bar{\boldsymbol u})=0 \quad (7.49)$$

The momentum flux tensors $\bar h$ given by (7.41) and (7.42) enable one to write down the resultant PDE systems (7.43) and (7.48) shortly. To do so we put these systems at the same time into purely evolution form for the mean velocity field $\overline{u_i}$

$$\frac{\partial\overline{u_i}}{\partial t}+\frac{1}{\bar\rho\,\Psi_i}\left(\overline{u_i}\frac{\partial(\bar\rho\,\Psi_i)}{\partial t}+\operatorname{div}\overline{\boldsymbol h_i}+\frac{\partial\bar p}{\partial x_i}\right)=\frac{1}{\bar\rho\,\Psi_i}\left(\bar\rho\boldsymbol F_i+\operatorname{div}\bar{\boldsymbol\tau}_{\boldsymbol i}\right) \quad (7.50)$$

which after including divergences of the vector components $\overline{\boldsymbol h_i}$

$$\operatorname{div}\overline{h_1}=\frac{\partial(\bar\rho\,\bar u_1\bar u_1\Psi_{11})}{\partial x_1}+\frac{\partial(\bar\rho\,\bar u_1\bar u_2\Psi_{12})}{\partial x_2}+\frac{\partial(\bar\rho\,\bar u_1\bar u_3\Psi_{13})}{\partial x_3}$$

$$\operatorname{div}\overline{h_2}=\frac{\partial(\bar\rho\bar u_2\bar u_1\Psi_{21})}{\partial x_1}+\frac{\partial(\bar\rho\bar u_2\bar u_2\Psi_{22})}{\partial x_2}+\frac{\partial(\bar\rho\bar u_2\bar u_3\Psi_{23})}{\partial x_3} \quad (7.51)$$

$$\operatorname{div}\overline{h_3}=\frac{\partial(\bar\rho\bar u_3\bar u_1\Psi_{31})}{\partial x_1}+\frac{\partial(\bar\rho\bar u_3\bar u_2\Psi_{32})}{\partial x_2}+\frac{\partial(\bar\rho\bar u_3\bar u_3\Psi_{33})}{\partial x_3}$$

complete the averaged and closed dynamic PDEs system for turbulent gas flows. Writing down

$$\frac{\partial\overline{u_i}}{\partial t}+\frac{1}{\bar\rho}\left(\operatorname{div}\overline{\boldsymbol h_i}+\operatorname{grad}\bar p\right)=\boldsymbol F_i+\frac{1}{\bar\rho}\operatorname{div}\bar\tau_i \quad (i=1,2,3=x,y,z) \quad (7.52)$$

together with divergence $\operatorname{div}(\bar{\boldsymbol u})=0$ and

$$\operatorname{div}\overline{h_1}=\frac{\partial(\bar\rho\,\bar u_1\bar u_1\Phi_{11})}{\partial x_1}+\frac{\partial(\bar\rho\,\bar u_1\bar u_2\Phi_{12})}{\partial x_2}+\frac{\partial(\bar\rho\,\bar u_1\bar u_3\Phi_{13})}{\partial x_3}$$

$$\operatorname{div}\overline{h_2}=\frac{\partial(\bar\rho\bar u_2\bar u_1\Phi_{21})}{\partial x_1}+\frac{\partial(\bar\rho\bar u_2\bar u_2\Phi_{22})}{\partial x_2}+\frac{\partial(\bar\rho\bar u_2\bar u_3\Phi_{23})}{\partial x_3} \quad (7.53)$$

$$\operatorname{div}\overline{h_3}=\frac{\partial(\bar\rho\bar u_3\bar u_1\Phi_{31})}{\partial x_1}+\frac{\partial(\bar\rho\bar u_3\bar u_2\Phi_{32})}{\partial x_2}+\frac{\partial(\bar\rho\bar u_3\bar u_3\Phi_{33})}{\partial x_3}$$

the same is done for turbulent liquid flows with $\bar\rho$ = const. leading the momentum distribution function $\Psi_{ij}$ to $\Psi_{ij}=1$. The formal definition of the vector components $h_i$ is analogical with $\boldsymbol\tau_{\boldsymbol i}$ and can be seen in the Section 11.

The momentum distribution functions $\Psi_i$ in (7.40) and the distribution functions $\Psi_{ij}$ and $\Phi_{ij}$ of the momentum flux in equations (7.41) and (7.42) are defined by relations (7.29) to (7.31) in combination with integrals (7.10), (7.11) and (7.13) as products of these averaging integrals.

The obtained averaged N–S system (7.43), (7.44) treats five unknown flow parameters, i.e., three velocity components, pressure $p$ and density $\rho$. To be closed, it requires the known equation of the state $p = p(\rho,T)$ to be added. The system (7.48), (7.49) for liquids becomes closed for given constant density $\rho$. Nevertheless, to do without the energy balance equation, the known relationship $\mu(\overline{T})$ as well a constant or known temperature are needed, but in case of turbulent flow both the equation of state as well as of energy balance need to be averaged.

## 7.5 Relation between constitution functions of turbulent flow and its random fluctuations. Computing the Reynolds (apparent) stress tensor and other averaged nonlinearities, if wanted

In the above averaged and closed system of the Navier-Stokes PDEs no averaged products of turbulent fluctuations appear to be needed if applying resultant equations (7.40) – (7.53) in any task. But relevant non-linearities, if wanted, can be averaged and used successively in current numerical models after satisfying some compatibility conditions.

Random velocity $u_i$ being expressed usually by the Reynolds' decomposition $u_i = \bar u_i + u'_i$ has been presented in this study by means of constitution function $a_i(\omega_i,t)$ through product $u_i = \bar u_i a_i$.

It yields the valid constitutional equation for velocity fluctuations $u'_i$

$$u_i=\bar u_i+u'_i=\bar u_i a_i \rightarrow u'_i=\bar u_i(a_i-1),\ \bar a_i=1,\ \overline{u'_i}=0,\ i=1,2,3 \quad (7.54)$$

After the same operation for random $\rho$ we obtain the following constitutional equation for density fluctuations

$$\rho'=\bar\rho\,(a_\varrho-1),\ \bar a_\varrho=1\ \ \overline{\rho'}=0 \quad (7.55)$$

and the need to compute the means of the products of fluctuations given in (7.54) and (7.55), integrating them through the characteristic domain in (7.10) – (7.13). But one can do so by a simpler way using the means $\overline{u_iu_j}$, $\overline{\rho u_i}$ and $\overline{\rho u_iu_j}$ already defined by the averaging integrals over the same characteristic domain and given in final form by equations (7.29) – (7.31). Replacing the random functions $\rho$ and $u_i$ on the left sides of (7.29) – (7.31) by Reynolds' decompositions and realising the required formal (implicit) average of these equations, one obtains the necessary linear algebraic system for computing wanted averaged non-linearities. Its solution is simple and yields

$$\overline{u'_i u'_j} = \bar{u}_i \bar{u}_j (\Phi_{ij} - 1) \quad (7.56)$$

$$\overline{\rho' u'_i} = \bar{\rho}\, \bar{u}_i (\Psi_i - 1) \quad (7.57)$$

$$\overline{\rho' u'_j} = \bar{\rho}\, \bar{u}_j (\Psi_j - 1) \quad (7.58)$$

$$\overline{\rho' u'_i u'_j} = \bar{\rho}\, \bar{u}_i \bar{u}_j (\Psi_{ij} - \Psi_i - \Psi_j - \Phi_{ij} + 2) \quad (7.59)$$

The first of them, in (7.56), is known to readers. Its products $\bar{\rho}\,\overline{u'_i u'_j}$ represent at $i \neq j$ the Reynolds' (apparent) stresses in turbulent liquid flows and became the most frequent object of approximate phenomenological modelling. Its applications through the Prandtl's mixing length theory, Von Kármán's similarity hypothesis, and others, occur still in the recent modelling of turbulence flow even in a compressible atmosphere, see Bednar, J. Zikmunda, O. (1985).

### 7.6 The relations defining deformation energy as the effect of turbulent density fluctuations

The instantaneous random kinetic energy field E of turbulent flow defined by

$$E = \frac{1}{2}\sum_{i=1}^{3} \rho u_i^2 \qquad i = 1, 2, \quad (7.60)$$

was in subsection 1.7 implicitly averaged after applying Reynolds' decomposition $u_i = \bar{u}_i + u'_i$, $\rho = \bar{\rho} + \rho'$ in (2.8). Analysis of its average $\overline{E}$ resulted in the following sum of two qualitatively different parts, elastic $E_e$ and deformation $E_d$

$$\overline{E} = E_e + E_d \quad (7.61)$$

Of these, the elastic part $E_e$ given by

$$E_e = \bar{\rho}(K + k);\ K = \frac{1}{2}\sum_{i=1}^{3} \bar{u}_i^2;\ k = \frac{1}{2}\sum_{i=1}^{3} \overline{u'^2_i};\ \ E_e > 0 \quad (7.62)$$

is known as the product of Reynolds' (implicit) averaging of the N-S system if written for liquid flows.

However, its deformation part

$$E_d = \frac{1}{2}\sum_{i=1}^{3}\left[2\bar{u}_i \overline{\rho' u'_i} + \overline{\rho' u'^2_i}\right], E_d \geq 0 \ \ or \ \ E_d \leq 0,\ \ i = 1, 2, 3 \quad (7.63)$$

acquiring due to pressure, velocity and density fluctuations both positive or negative values, belongs to internal (potential) energy. Therefore, it should be considered along with the $E_d$ energy in the treatement of properties and application of the energy balance equation. Such possibility follows from the above found averaged non-linearitíes (7.56) – (7.59). It enables one to obtain the means needed for (7.63)

$$\overline{\rho' u'_i} = \bar{\rho}\, \bar{u}_i (\Psi_i - 1),\ \overline{\rho' u'^2_i} = \bar{\rho} \bar{u}_i^2 (\Psi_{ii} - 1) \quad (7.64)$$

and compute wanted deformation energy by

$$E_d = \frac{\bar{\rho}}{2}\sum_{i=1}^{3}[2\Psi_i + \Psi_{ii} - 3]\,\bar{u}_i^2 \quad (7.65)$$

The above results allow us to describe the gas and energy flows without incompressibility approximation and to remove by such a way the frequent problem as treated by V. L. Yushkov (2015).

## 8 VERIFYING EDE BY EXPERIMENTAL DATA ON WALL BOUNDED TURBULENCE

The EDE (energy distribution equations) (6.39) – (6.41) play a key role at closing the averaged N–S system (7.3), (7.4). It is therefore important to answer the question of whether and how the distribution equations fit the data from relevant experiments. The part of the answer is written in Section 4. The sample and practical realizing this possibility has been shown in Section 9.

Therefore, in the selection of experimental sources, the focus here will be on normal turbulent stress and flows with sufficiently large anisotropy. The first reason for this stems from the fact that normal stresses are essential for the verification of distribution equations. The second is a surprising but important property of the obtained energy distribution equations that is relevant to the possibility of comparing measured \tangential turbulent stresses with theoretical ones. This property is the three-dimensionality of the mean velocity field of any turbulent flow. These reasons have led us to select four independent sources of experimental data: Reichardt (1938), Klebanoff (1954). All of these relate to experimental investigation of boundary layers in parallel turbulent flow in wind tunnels. Although there are other frequently cited studies that could also have been included, such as those by Mäsiar and Dúbrava (1975), the sources that we chose were favored owing to the negligible differences in error deviations $er_i$ between them. We can, however, assume that a similar closeness of the results would be the case for other sources.

An evaluation of relevant inputs and outputs of experimental sources is part of the supplementary materials available from the author upon request.

## 9 ON SOME PROPERTIES OF THE CONSTITUTIVE AND THE ENERGY DISTRIBUTION EQUATIONS

Explicit averaging of the products of the constitutive functions using the integrals in (7.10), (7.11), and (7.13) provided the required number of closing equations for the averaged N–S system (7.3), (7.4). The distribution equations of kinetic energy (6.39) – (6.41) and (6.52) played a decisive role in this completion. The basic form of these equations is surprisingly simple:

$$\overline{u_i u_i} = \Phi_i \bar{u}^2{}_i \rightarrow \Phi_i(\xi, i_a) = 1 + \frac{\overline{u'^2}}{\bar{u}^2}, \Phi_e(\xi) = 1 + \frac{k}{K},\ \ \overline{u'_i u'_i} > 0,$$
$$1 < \Phi_i < \infty,\ \ i = 1,2,3. \quad (9.1)$$

However, in (9.1), the three distribution functions themselves, $\Phi_i(\xi, i_i)$ are no longer simple. The evidence for this is the equation (6.38) and all the others through which it is defined, including, for example, (6.46) – (6.48). The structure of the distribution equations determines their validity and thus the validity of EDE for that part of the space $\bar{G}(x, t)$ in which the inequalities in (9.1) hold. But such conditions are satisfied only by a three-dimensional velocity field whose streamlines or vector lines are spatial curves at each point of the turbulent flow, excepting singularities.

### 9.1 Ability of constitution equation (5.16) to describe a random velocity field of turbulent flow

In all standard operations with a constitutive function, the declared property of mutual independence of random frequencies $\omega_i$ has been strictly respected, as has their independence of time and position in the space $G(\omega,t)$. In averaging operations via multiple integrals, the independence of $\omega_i$ will also be respected in operations with partial derivative in this section. Its formal dependence (5.2) comes into consideration when evaluating experimentally obtained velocity fields in $\bar{G}(x,t)$.

The ability mentioned in the title of this section concerns the equation of random oscillations (5.16) now written in the form

$$\frac{u_i}{\bar{u}_i} = \frac{1}{M_i}(\bar{s}_i \sin\omega_i t + \bar{c}_i \cos\omega_i t), \quad \bar{u}_i \neq 0, \quad i = 1,2,3. \qquad (9.2)$$

The constitutive function of the random frequency $\omega_i$ time $t$ and some mean parameters defines the random velocity field $u_i$ through the constitutive equation (9.2). We consider the constitutive equation (9.2) to be capable of describing an arbitrary random field $\boldsymbol{u_i}$ if the relation (9.2) yields the possibility of determining in an appropriate way the random frequency $\omega_i$ for each measured random $u_i$ in an arbitrary turbulent flow.

As the appropriate way in the above condition, we adopt the following approach. First, we need the inputs to be used in (9.2) as the time record of the measured velocity field $u_i$ at a point of the turbulent flow, together with the corresponding parameters of the averaged flows, $\bar{u}_i$, $\bar{s}_i$ , and $\bar{c}_i$, for (9.2). The time record will also provide the rate of change $\partial u/\partial t$ . The mean characteristics can be obtained either from the solution of the initial value problem for the closed system (7.3), (7.4) or from experimental data processed as in Section 8 in the case of statistically steady flow.

Let us seek the missing relation determining the random frequency $\omega_i$. Equation (9.2) applies to a domain bounded in $G(\omega,t)$ by the rectangle

$$\Delta_i = T\left(\frac{1}{T_c} - \frac{1}{T_i}\right), \quad 0 \leq t \leq T, \quad i = 1,2,3 \qquad (9.3)$$

This comes from averaging the original constitutive equation (5.1) on the rectangle $\Delta_i$ given in (9.3). Owing to the averaging of (5.1) and its connection to $\bar{G}(x,t)$ through the phase angle $\varphi_i$ (and not through a condition in time $t=0$), it is not possible to find a time coordinate $t$ in (9.2) corresponding to a time $t$ in the experimental record of $u_i$ ($\boldsymbol{x}_0$, $t$). Therefore, if we want to obtain a random frequency $\omega_i$ from (9.2) as a function of the given inputs at any point of the experimental record, we need to remove the explicit time $t$ from (9.2). This is possible by differentiating (9.2) with respect to time. Then, from (9.2), it leads to equation n

$$\frac{1}{\bar{u}_i}\frac{\partial u_i}{\partial t} = \frac{\omega_i}{M_i}(\bar{s}_i \cos\omega_i t - \bar{c}_i \sin\omega_i t) \text{ in } G(\omega,t),\ i = 1,2,3. \quad (9.4)$$

The functions $\sin\omega_i t$ and $\cos\omega_i t$ can be obtained from the two equations (9.2) and (9.4). Putting these into the still valid relations between their squares, after a small adjustment, one obtains the desired relation

$$\omega_i^2 = \frac{M_i}{\bar{u}^2{}_i - M_i u_i^2}\left(\frac{\partial u_i}{\partial t}\right)^2 \text{ in } G(\omega,t) \text{ and } \bar{G}(x,t),\ i = 1,2,3. \qquad (9.5)$$

If we have the given inputs at our disposal, then, from the random values of the velocity $u_i$ and its rate of change $\partial u_i/\partial t$, we can obtain the values of a random independent variable $\omega_i$ from the square root of (9.5). We will choose from the two signs of the root. If we take $\omega_i$ as a frequency of velocity fluctuations, then we accept that the sign of $\omega_i$ can only be positive. If we consider $\omega_i$ as the rotational velocity of a three-dimensional vortex, there is no restriction on the choice of sign. The relation (9.5) can also be obtained by inversion of (9.2) and taking the limit at the point $t=0$ using l'Hôpital's rule. The mean parameters in the differentiation of (9.2). are considered as constant due to Raynolds rules of averaging. Equation (9.5) provides real-valued $\omega_i$ only for

$$\bar{u}^2{}_i - M_i u_i^2 > 0. \qquad (9.6)$$

If it is not met, then $\omega_i$ will be complex - valued. The value $M_i$ in the condition (9.6) is given by

$$M_i(\xi, i_i) = \bar{s}^2{}_i + \bar{c}^2{}_i. \qquad (9.7)$$

For example, when processing the experimental data in Section 8, the values of the quantity $M_i$ ranged from $M_i(1.1)$ = 0.92 to $M_i(10,1)$ = 0.09. Validity of the relation (9.5) justifies the assumption that the constitutive equation (9.2) can describe any random turbulent scalar or vector field. It is important that constitutive equation (9.2) holds this ability as well as when the time derivatives of the mean flow velocities $\partial\bar{u}_i/\partial t$ in the deriving (9.5) are accepted. In such case the above treatment leads to other but again quadratic equation with respect to unknown random frequency $\omega_i$ yielding again possibility to determine its value and proving wanted ability of the constitutive equations.

### 9.2 On numerical solution of the Energy Distribution Equations

If there is a relationship between fluid pressure and density, or $\bar{\rho}$ is known, then the four PDEs in (7.3) and (7.4) together with the four energy distribution equations and integrals in (7.10), (7.11), and (7.13) form a closed system. After the integrations in (7.10) and (7.13) are performed, it becomes a system of nonlinear differential and algebraic equations. In the eventual application of this system to the solution of the initial (Cauchy) problem, the solution will start at each time step by solving the system of distribution equations (9.1), because, when the components of the velocity field $\bar{u}_i$ are entered as an initial condition, the system (9.1) becomes closed. Its solution will provide the normal turbulent stresses, and the equations (7.10) – (7.13) will provide the remaining dependent variables.

One of possible methods for solving a nonlinear system like (9.1) was successfully tested in section 8.

## 10 INFORMATION SUPPORT (4) AND (5)

### 10.1 Information Support (4): Viscous stress tensor and its vector and tensor components

In the vector notation of the momentum PDRs (7.3) all internal friction forces are presented by the viscous stress tensor

$$\tau = \begin{Bmatrix} \boldsymbol{\tau}_x \\ \boldsymbol{\tau}_y \\ \boldsymbol{\tau}_z \end{Bmatrix} = \begin{Bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{Bmatrix} \tag{10.1}$$

by means of three vector components $\boldsymbol{\tau}_i$, *( i = 1,2,3 = x, y, z)*

$$\boldsymbol{\tau}_x = \boldsymbol{\tau}_1 = \boldsymbol{i}\,\tau_{xx} + \boldsymbol{j}\,\tau_{xy} + \boldsymbol{k}\,\tau_{xz}$$

$$\boldsymbol{\tau}_y = \boldsymbol{\tau}_2 = \boldsymbol{i}\,\tau_{yx} + \boldsymbol{j}\,\tau_{yy} + \boldsymbol{k}\,\tau_{yz} \tag{10.2}$$

$$\boldsymbol{\tau}_z = \boldsymbol{\tau}_3 = \boldsymbol{i}\,\tau_{zx} + \boldsymbol{j}\,\tau_{zy} + \boldsymbol{k}\,\tau_{zz}$$

and its nine tensor components

$$\tau_{xx} = \mu\left(2\frac{\partial u_1}{\partial x} - \frac{2}{3} div\,\boldsymbol{u}\right); \quad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)$$

$$\tau_{yy} = \mu\left(2\frac{\partial u_2}{\partial y} - \frac{2}{3} div\,\boldsymbol{u}\right); \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right) \tag{10.3}$$

$$\tau_{zz} = \mu\left(2\frac{\partial u_3}{\partial z} - \frac{2}{3} div\,\boldsymbol{u}\right); \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}\right)$$

as linear functions of velocity gradients built on the Stokes hypothesis, see Lojcianskij, L. G. (1954), Milne-Thomson L.M. (1960) or Schlichting, H. (1960). The dynamics viscosity $\mu(\overline{T})$ assumes to be the known function of the mean temperature $\overline{T}$. The divergences of vectors defined in (10.2) follow from (10.3) and (10.4),

$$div\,\tau_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

$$div\,\tau_y = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \tag{10.4}$$

$$div\,\tau_z = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

### 10.2 Information Support (5): On momentum flux tensor *h* and its components including the Reynolds' stresses tensor

The symmetric tensor of the momentum flux $h = \rho\boldsymbol{uu}$ represents the effects of inertial forces in momentum equations (7.2) and (7.4). As the main source of non-linearities it occurs in the resultant equations systems (7.41) up to (7.53) in the explicitly averaged final shapes. Its vector components are defined equally as done for vectors $\tau_i$ in (10.2) as well as for divergences in (10.4).

The averaged products $\bar{\rho}\,\overline{u'_i u'_j}$ of its fluctuations represent at $i \neq j$ and constant $\bar{\rho}$ the components of the Reynolds' (apparent) stresses tensor in turbulent liquid flows which became the most frequent object of approximate phenomenological turbulent modelling. Section 7.5 contains more about this.

## 11. SURVEY OF RESULTS AND CONCLUSIONS

### 11.1 Summarizing remarks

The first three sections of the study were mainly on preparing a suitable strategy for the treatment of and solution to the closure problem. From the author's insight into the foundations of fluid mechanics it followed that, while thermodynamics has reflected no effects of turbulence phenomena into relevant PDEs, the statistical tools of Reynolds have directed attention upon the roll of its randomness. Applying the idea that randomness as an autonomous factor of physical processes could be utilized as a property of independent variables of PDEs, the frequency of turbulent fluctuations was chosen for this role. The property of bifunctionality of spatial coordinates treated in Sect. 2.4 helped this choice due to its obvious analogy.

The third Section has been devoted to creating the characteristic domain of averaging. Demonstrating the simultaneous existence of deformation (potential) energy together with elastic (kinetic) energy as the effects of turbulent density and velocity fluctuation, the author justified the extension of the validity of Kolmogorov's inter-scale relations upon anisotropic turbulence**.** It allowed the characteristic domains of needed properties for averaging relevant PDEs describing any random turbulent fields in the 5–D random space to be created.

The wanted equations closing the averaged N-S system (7.1) – (7.2)" in this study consist of the derived Energy Distribution Equations, EDEs (6.39) – (6.41) and equation (6.52) for computing the energy distribution factor$\xi$. This solution can be called "direct" since it is expressed through a mere three definite integrals yielding all wanted averaged non-linearities of the double and triple products as defined in (7.10) – (7.15). The resultant closed system of averaged N–S equations for isothermal fluid flow is presented in equations (7.40) – (7.49).

All results of the averaging process have been expressed through distribution functions $\Psi_i$ of the vector of momentum $\overline{\rho u_i}$ **,** $\Psi_{ij}$ of the momentum flax tensor $\overline{\rho u_i u_j}$ for gases**,** as well as through distribution functions $\Phi_{ij}$ of the momentum flux tensor $\rho\overline{u_i u_j}$ for liquids**,** all in (7.32) – (7.34).

The structure of the EDE requires anyone to realize that the turbulent mean flow is always three-dimensional, with a spatially curved stream or vector lines. This property led to expected contradictions when the theory was confronted with data from experiments made in wind tunnel boundary layers assuming straight parallel flow there. The conflict, caused also by different ways of averaging the random fields, was removed by redistribution of the measured energy into all its directional parts in accordance with the numerical solution of the relevant EDEs. The errors that remained were below 2.5% for all comparisons with experiments, see in supplementary material. It convinced the author that this verification of the resultant EDE was successful.

In Sect. 9.1 the important property of the constitution equation (4.16) is verified. It is its ability to describe any random velocity fields of turbulent flow. In Sect. 9.2 the method of unique numerical solution of the non-linear EDE algebraic system was described in connection with setting up the Cauchy initial value problem for (7.40) – (7.49).

### 11.2 On verifying resultant equations systems by experimental measurements

Mean turbulent steady as well as unsteady fields are defined as the mathematical expectations, which cannot be measured. This can lead to presumption, that results of the statistical methods in fluid dynamics cannot be tested by measurements. But below it is shown that the steady mean fields can be tested comparing with measurements by means of the experimental expectations. This analogy to mathematical expectations works as follows:

The term experimental expectation for $\overline{f}_e(\boldsymbol{x}) = f_e(\boldsymbol{x}, t) - f_e'(\boldsymbol{x}, t)$ stems from the time averaging the measured random $f_e(\boldsymbol{x},t)$ over sufficiently long time $t_e$ and signifies the comparing equivalent to any steady mean turbulent field $\overline{f}(\boldsymbol{x}) = f(\boldsymbol{x}, t) - f'(\boldsymbol{x}, t)$. If $\overline{f}(\boldsymbol{x})$ is solution of steady task to above resultant system and error deviations abs ($\overline{f}$ – s$\overline{f}_e$) is small enough during the whole $t_e$. in the tested points of $\overline{G}(\boldsymbol{x}, t)$, then such test by experiment of the found resultant system can be accepted as successful.

The validity of results of above treated testing of mean steady turbulent flow for the mean unsteady flow follows from assumption that used description tool is suitable to respect and record mean values of all flow characteristics including non-stationarity. Nevertheless, the particular limits for error deviations are needed to state for each mean flow testing by measurements as well as its extension of validity needs own specific justification.

### 11.3 On the ability to solve the average and closure problem at non-Newtonian fluids flows

The properties of the constitutive functions found in this article allow to effectively average also the non-linearities of other kinds including those in the energy balance equation. But this ability does not vouch for the possibility to apply these tools without any problems also at describing turbulent fields of non-Newtonian fluids flow.

**Availability of data**

The data supporting the findings of this study are available within the article and its supplementary material available from author, if asked.

## REFERENCES


Barenblatt, G.I., 1996. Scaling, Self-similarity, and Intermedate Asymptotics: Canbridge, 1996

Debnath, L., 1998. Wavelet transforms, fractals and turbulence, in Nonlinear Instability Chaos and Turbulence, Advances in Fluid Mechanics, ed. by L. Debnath and D.N. Riahi, WIT Press, 129–179.

Dúbrava, L., Vajcík, S., 1988. Energy balance of turbulent flow, VEDA, Slovak Academy of Sciences.

Ecke, R. (2005) The turbulence problem, Los Alamos Science, No. 29, 124–138.

Hunt, J.C.R., Vassilicos, J.C., 1991. Kolmogorov's contributions to the physical and geometrical understanding of small-scale turbulence and recent developments, Proc. R. Soc. Lond., Ser. A, 434 183–210.

Klebanoff, P.S., 1954. Characteristics of turbulence in a boundary layer with zero pressure gradient, Report 1247, NACA TN 3178, 1135–1138.

Kolmogorov, A.N., 1941a. The local structure of turbulence in an incompressible fluid with very large Reynolds number, Dokl. Akad. Nauk SSSR, 30, 301–305.

Kolmogorov, A.N., 1941b. Dissipation of energy under locally isotropic turbulence, Dokl. Akad. Nauk SSSR, 32, 16–18.

Kosorin, K., 1995. On solution of two problems in free surface seepage flow hydrodynamics, In Computer Modelling of Free and Moving Boundary Problems III, Comp. Mech, Publications, Southampton, 245–254.

Kosorin, K., 2011. Theory of orthogonal flow in elastic chambers and bio-bags, in Fluid Structure Interaction VI, WIT Press, Boston, 103–118.

Li, Y.C., 2013. Major open problems in chaos theory, turbulence and nonlinear dynamics, Dynam. PDE, 10, 4, 379–392.

Lojcianskij, L.G., 1954. Mechanics of Liquids and Gases, SNTL, Prague.

Lumley, J.L., 1989. The state of turbulence research, In Advances in Turbulence, eds. W. K. George, R. Arndt, Hemisphere, New York, 1–10.

Mäsiar, E., Dúbrava, L., 1975. Strain gauge scanner measurement of the water flow velocity fluctuations, In Proceedings, XVI Congress of AHR, Sao Paulo, 24–28.

Milne-Thomson, L. M., 1960. Theoretical Hydodynamics, Fourth ed., London -New York.

Monin, A.S., Yaglom, A.M., 1975. Statistical Fluid Mechanics: Vol. 2, MIT Press, Cambridge, MA.

Navier, M. Mémoire sur les Lois du Mouvements des Fluides. Mem. De l'Acad. D. Sci. 6, 389 (1827)

Reichardt, H., 1938. Messungen turbulenten Schwangkungen, Naturwissenschaften, 404.

Reynolds, O., 1895. On the dynamical theory of incompressible viscous fluids and the determination of the criterion, Phil. Trans. Roy. Soc., T. 186 A 123, or Sci. Papers 1. 355.

Schlichting, H., 1960. Boundary Layer Theory, McGraw-Hill, New York, 465–469.

Soo, S. L. (1962) Analytical Thermodynamics, Prentice-Hall.

Vygodsky, M. Mathematical Handbook, Higher Mathematics, MIR Publishers, Moscow 1971, p. 633.

Yushkov, V.P. (2015) The Hamiltonian Formalism and Quantum–Mechanical Analogy in the Probabilistic Description of Turbulence, Theoretical and Mathematical Physics, Moscow Univ. Phys. Bull.,2015, vol. 70


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