Some results on deferred statistical convergence of double sequences in neutrosophic normed linear spaces

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Abstract

In this research, we extend the concept of statistical convergence by introducing the idea of deferred statistical convergence in the neutrosophic normed spaces. This work considers the deferred density proposed in 2016 by Küçükaslan and Yilmaztürk to define deferred statistical convergence of double sequences in neutrosophic normed spaces. We look into some of the core characteristics of the recently developed concept of deferred statistical convergence.

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Keywords: Deferred statistical convergence; $\mathfrak{Q}_{\beta,\gamma}$- summable; Double sequence; Bounded sequence.

1. INTRODUCTION

Fast [3] and Steinhaus [22] developed summability theory by independently establishing the notion of statistical convergence in 1951. Following their work, Hazarika and Esi [5], Mursaleen [15], Savas and Gurdal [19] and many others conducted several investigations and made numerous generalisations.

Statistical convergence has become one of the most active fields of research due to its many applications in other branches of mathematics, such as probability theory, mathematical analysis, number theory, etc. Agnew [1] generalised the Cesàro mean in 1932 to the deferred Cesàro mean and produced a method with stronger features that was more useful. Then, in 2016, Küçükaslan and Yilmaztürk [11] used the concept of the deferred Cesàro mean to present the idea of deferred statistical convergence, primarily as a natural generalisation of statistical convergence, lambda statistical convergence, and lacunary statistical convergence.

In 1965, Zadeh [23] was the first to present the idea of fuzzy sets, which was an expansion of classical set theoretical concept. These days, it can be used in a wide range of engineering and scientific fields. When membership degrees are unknown, the idea of fuzzy sets sometimes struggles to handle the situation. As an improvement of fuzzy sets, Atanassov[2] introduced intuitionistic fuzzy sets in 1986. Numerous

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decision-making issues have been solved using intuitionistic fuzzy sets.

Decision-makers frequently experienced some hesitations in addition to using yes-or-no approaches. Additionally, in some actual events like sports, voting, etc., we can get a tricomponent result. Smarandache [20], taking into account all the aspects, first proposed the idea of the neutrosophic set in 1998 as a generalisation of both the fuzzy set and the intuitionistic fuzzy set. A neutrosophic set's elements are made up of the triplets true- membership function(T), indeterminacy membership function (I) and falsity membership function (F). When all elements of the universe have certain degrees of T, F, and I, a set is said to be neutrosophic.

In 1992, Felbin [4] proposed the notion of fuzzy normed space. Later, in 2006, Saadati and Park [18] developed the idea of intuitionistic fuzzy normed spaces. In intuitionistic fuzzy normed spaces, the idea of statistical convergence was introduced and examined by Karakus et al. [9] in 2008. In addition, it was generalised to deferred statistical convergence by Melliani et al. [13] in 2018. Statistical Δ^m convergence in neutrosophic normed spaces was recently presented by Jeyaraman and Jenifer [6]. Neutrosophic normed linear spaces and the concept of statistical convergence were presented by Kirisci and Simsek [10]. Sequence convergence studies in neutrosophic normed linear space have not yet made much progress and are still in their early stages. In the framework of the neutrosophic normed space, we examine the deferred statistical convergence of double sequences in this research.

2. PRELIMINARIES

In 1932, Agnew [1] defined the deferred Cesaro mean $\mathfrak{Q}_{\mathfrak{P},\mathfrak{H}}$ of a sequence $\mathfrak{x} = (\mathfrak{x}_\mathfrak{n})$ by

$$(\mathfrak{Q}_{\mathfrak{P},\mathfrak{H}}\mathfrak{x})_\mathfrak{n} = \frac{1}{\mathfrak{H}(\mathfrak{n}) - \mathfrak{P}(\mathfrak{n})} \sum_{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1}^{\mathfrak{H}(\mathfrak{n})} (\mathfrak{x}_\mathfrak{p}),$$

where $\{\mathfrak{P}(\mathfrak{n})\}$ and $\{\mathfrak{H}(\mathfrak{n})\}$ are sequences of positive natural numbers satisfying

$$\mathfrak{P}(\mathfrak{n}) < \mathfrak{H}(\mathfrak{n}) \text{ and } \mathfrak{H}(\mathfrak{n}) \to \infty. \tag{1}$$

A sequence $\mathfrak{x} = (\mathfrak{x}_\mathfrak{n})$ is called

1. Deferred Cesaro convergent to $\mathfrak{L}$ if $\frac{1}{\mathfrak{H}(\mathfrak{n})-\mathfrak{P}(\mathfrak{n})} \sum_{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1}^{\mathfrak{H}(\mathfrak{n})} (\mathfrak{x}_\mathfrak{p} - \mathfrak{L}) \to 0$;
2. Strongly deferred Cesaro convergent to $\mathfrak{L}$ if $\frac{1}{\mathfrak{H}(\mathfrak{n})-\mathfrak{P}(\mathfrak{n})} \sum_{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1}^{\mathfrak{H}(\mathfrak{n})} |\mathfrak{x}_\mathfrak{p} - \mathfrak{L}| \to 0$.

We denote $\lim\limits_{\mathfrak{n}\to\infty} \mathfrak{x}_\mathfrak{n} = \mathfrak{L}(\mathfrak{Q}S[\mathfrak{P},\mathfrak{H}])$.

Let $\mathfrak{K}$ be an arbitrary subset of $\mathbb{N}$ and $\mathfrak{K}_{\mathfrak{P},\mathfrak{H}}(\mathfrak{n}) = \{\mathfrak{P}(\mathfrak{n}) < \mathfrak{p} \leqslant \mathfrak{H}(\mathfrak{n}), \mathfrak{p} \in \mathfrak{K}\}$ be an associated set of $\mathfrak{K}$ for the arbitrary sequences $\mathfrak{P}(\mathfrak{n})$ and $\mathfrak{H}(\mathfrak{n})$ satisfying (1).

Let $\mathfrak{K}$ be an arbitrary subset of $\mathbb{N}$.
If the following limit $\delta_{\mathfrak{P},\mathfrak{H}}(\mathfrak{K}) = \lim\limits_{\mathfrak{n}\to\infty} \dfrac{1}{\mathfrak{H}(\mathfrak{n}) - \mathfrak{P}(\mathfrak{n})} |\mathfrak{K}_{\mathfrak{P},\mathfrak{H}}(\mathfrak{n})|$ exists, then $\delta_{\mathfrak{P},\mathfrak{H}}(\mathfrak{K})$ is called a deferred density of the subset $\mathfrak{K}$.
A sequence $\mathfrak{x} = (\mathfrak{x}_\mathfrak{p})$ is called deferred statistically convergent to $\zeta \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$\lim_{\mathfrak{n}\to\infty} \frac{1}{\mathfrak{H}(\mathfrak{n}) - \mathfrak{P}(\mathfrak{n})} |\{\mathfrak{p}, \mathfrak{P}(\mathfrak{n}) < \mathfrak{p} \leqslant \mathfrak{H}(\mathfrak{n}) : |\mathfrak{x}_\mathfrak{p} - \mathfrak{L}| \geqslant \varepsilon\}| = 0.$$

It is denoted by $\lim\limits_{\mathfrak{n}\to\infty} \mathfrak{x}_\mathfrak{n} = \mathfrak{L}(\mathfrak{Q}[\mathfrak{P},\mathfrak{H}])$.

Definition 2.1 [7] The following axioms define a continuous t-norm as a binary operation $* : [0,1] \times [0,1] \to [0,1]$

(i) $*$ is continuous, associative, and commutative,

(ii) $\mathfrak{a} * 1 = \mathfrak{a}$ for all $\mathfrak{a} \in [0,1]$,

(iii) If $\mathfrak{a} \leqslant \mathfrak{c}$ and $\mathfrak{b} \leqslant \mathfrak{d}$ then $\mathfrak{a} * \mathfrak{b} \leqslant \mathfrak{c} * \mathfrak{d}$, for $0 \leqslant \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \leqslant 1$.

Definition 2.2 [7] The following axioms define a continuous t- conorm as a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$

(i) $\diamond$ is continuous, associative, and commutative,

(ii) $\mathfrak{a} \diamond 0 = \mathfrak{a}$ for all $\mathfrak{a} \in [0,1]$,

(iii) If $\mathfrak{a} \leqslant \mathfrak{c}$ and $\mathfrak{b} \leqslant \mathfrak{d}$ then $\mathfrak{a} \diamond \mathfrak{b} \leqslant \mathfrak{c} \diamond \mathfrak{s}$, for $0 \leqslant \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \leqslant 1$.

Definition 2.3 [7] The 7-tuple $(\mathfrak{Y}, \mu, \nu, \omega, *, \diamond, \circledast)$ is called a Neutrosophic Normed Linear Space ($\mathfrak{NNLS}$) if $\mathfrak{Y}$ is a linear space, μ, ν and ω are fuzzy sets on $\mathfrak{Y} \times (0,\infty)$, $*$ is a continuous t-norm, $\diamond$ and $\circledast$ are continuous t-conorm including the following: for every $\mathfrak{x}, \sigma \in \mathfrak{Y}$ and $\mathfrak{s}, \lambda > 0$;

(i) $0 \leqslant \mu(\mathfrak{x},\lambda) \leqslant 1; 0 \leqslant \nu(\mathfrak{x},\lambda) \leqslant 1; 0 \leqslant \omega(\mathfrak{x},\lambda) \leqslant 1$,

(ii) $\mu(\mathfrak{x},\lambda) + \nu(\mathfrak{x},\lambda) + \omega(\mathfrak{x},\lambda) \leqslant 3$,

(iii) $\mu(\mathfrak{x},\lambda) > 0$,

(iv) $\mu(\mathfrak{x},\lambda) = 1$ if and only if $\mathfrak{x} = 0$,

(v) $\mu(\alpha\mathfrak{x},\lambda) = \mu\left(\mathfrak{x}, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$,

(vi) $\mu(\mathfrak{x},\lambda) * \mu(\sigma,\mathfrak{s}) \leqslant \mu(\mathfrak{x}+\sigma, \lambda+\mathfrak{s})$,

(vii) $\mu(\mathfrak{x},\circ) : (0,\infty) \to [0,1]$ is continuous,

(viii) $\lim\limits_{\lambda\to\infty} \mu(\mathfrak{x},\lambda) = 1$ and $\lim\limits_{\lambda\to 0} \mu(\mathfrak{x},\lambda) = 0$,

(ix) $\nu(\mathfrak{x},\lambda) < 1$,

(x) $\nu(\mathfrak{x},\lambda)=0$ if and only if $\mathfrak{x}=0$,

(xi) $\nu(\alpha\mathfrak{x},\lambda)=\nu\left(\mathfrak{x},\frac{\lambda}{|\alpha|}\right)$ for each $\alpha\neq 0$,

(xii) $\nu(\mathfrak{x},\lambda)\diamond\nu(\sigma,\mathfrak{s})\geqslant\nu(\mathfrak{x}+\sigma,\lambda+\mathfrak{s})$,

(xiii) $\nu(\mathfrak{x},\circ):(0,\infty)\to[0,1]$ is continuous,

(xiv) $\lim\limits_{\lambda\to\infty}\nu(\mathfrak{u},\lambda)=0$ and $\lim\limits_{\lambda\to 0}\nu(\mathfrak{u},\lambda)=1$,

(xv) $\omega(\mathfrak{x},\lambda)<1$,

(xvi) $\omega(\mathfrak{x},\lambda)=0$ if and only if $\mathfrak{x}=0$,

(xvii) $\omega(\alpha\mathfrak{x},\lambda)=\omega(\mathfrak{x},\frac{\lambda}{|\alpha|})$ for each $\alpha\neq 0$,

(xviii) $\omega(\mathfrak{x},\lambda)\circledast\omega(\sigma,\mathfrak{s})\geqslant\omega(\mathfrak{x}+\sigma,\lambda+\mathfrak{s})$,

(xix) $\omega(\mathfrak{x},\circ):(0,\infty)\to[0,1]$ is continuous,

(xx) $\lim\limits_{\lambda\to\infty}\omega(\mathfrak{x},\lambda)=0$ and $\lim\limits_{\lambda\to 0}\omega(\mathfrak{x},\lambda)=1$.

Then (μ,ν,ω) is known as Neutrosophic Norm and we write this norm shortly by $[\mathfrak{NN}](\mu,\nu,\omega)$.

Definition 2.4 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNLS}$. A sequence $\mathfrak{x}=(\mathfrak{x}_\mathfrak{p})$ in $\mathfrak{Y}$ is convergent to $\xi\in\mathfrak{Y}$ with respect to the $\mathfrak{NN}\,(\mu,\nu,\omega)$ for each $\varepsilon\in(0,1)$ and $\lambda>0$, there is $\mathfrak{p}_0\in\mathbb{N}$ so that $\mu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)>1-\varepsilon,\nu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)<\varepsilon$ and $\omega(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)<\varepsilon$, for all $\mathfrak{p}\geqslant\mathfrak{p}_0$. It is indicated by $(\mu,\nu,\omega)-\lim\mathfrak{x}=\xi$.

Definition 2.5 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNLS}$. A sequence $\mathfrak{x}=(\mathfrak{x}_\mathfrak{p})$ in $\mathfrak{Y}$ is referred as a Cauchy sequence with regard to the $\mathfrak{NN}\,(\mu,\nu,\omega)$, if for every $\varepsilon\in(0,1)$ and $\lambda>0$, there is $\mathfrak{p}_0\in\mathbb{N}$ so that
$\mu(\mathfrak{x}_\mathfrak{p}-\mathfrak{x}_\mathfrak{w},\lambda)>1-\varepsilon,\nu(\mathfrak{x}_\mathfrak{p}-\mathfrak{x}_\mathfrak{w},\lambda)<\varepsilon$ and $\omega(\mathfrak{x}_\mathfrak{p}-\mathfrak{x}_\mathfrak{w},\lambda)<\varepsilon$, for all $\mathfrak{p},\mathfrak{w}\geqslant\mathfrak{p}_0$.

Definition 2.6 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNLS}$. A sequence $\mathfrak{x}=(\mathfrak{x}_\mathfrak{p})\in\mathfrak{Y}$ is a bounded sequence if there exists η, a positive real number, so that $\mu(\mathfrak{x}_\mathfrak{p},\lambda)>1-\eta,\nu(\mathfrak{x}_\mathfrak{p},\lambda)<\eta$ and $\omega(\mathfrak{x}_\mathfrak{p},\lambda)<\eta$, for all $\mathfrak{p}\in\mathbb{N}$.
Take a $\mathfrak{NNLS}\,(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$. A sequence $(\mathfrak{x}_\mathfrak{p})$ is called statistically convergent with respect to $\mathfrak{NN}(\mu,\nu,\omega)$, if there is a number $\xi\in\mathfrak{Y}$ so that for each $\varepsilon>0$ and $\lambda>0$, the natural density of set

$$\mathfrak{K}_\varepsilon=\{\mathfrak{p}\leqslant\mathfrak{n}:\mu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)\leqslant 1-\varepsilon,\nu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)\geqslant\varepsilon\,\text{and}\,\omega(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)\geqslant\varepsilon\}$$

has zero., i.e., $d(\mathfrak{K}_\varepsilon)=0$. That is,

$$\lim_{\mathfrak{n}\to\infty}\frac{1}{\mathfrak{n}}|\mathfrak{p}\leqslant\mathfrak{n}:\mu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)\leqslant 1-\varepsilon,\nu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)\geqslant\varepsilon\,\text{and}\,\omega(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)\geqslant\varepsilon|=0.$$

In this case, we write $\mathfrak{x}_\mathfrak{p}\to\xi(S(\mu,\nu,\omega))$.

Definition 2.7 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNLS}$. A sequence $\mathfrak{x}=(\mathfrak{x}_\mathfrak{p})$ of $\mathfrak{Y}$ is called $\mathfrak{Q}_{\beta,\gamma}$ - summable to ξ with respect to the $\mathfrak{NN}\,(\mu,\nu,\omega)$ if every $\varepsilon\in(0,1)$ and $\lambda>0$, there exists $\mathfrak{n}_0\in\mathbb{N}$ such that

$$\frac{1}{\mathfrak{H}(\mathfrak{n})-\mathfrak{P}(\mathfrak{n})}\sum_{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1}^{\mathfrak{H}(\mathfrak{n})}\mu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)>1-\varepsilon,$$
$$\frac{1}{\mathfrak{H}(\mathfrak{n})-\mathfrak{P}(\mathfrak{n})}\sum_{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1}^{\mathfrak{H}(\mathfrak{n})}\nu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)<\varepsilon\,\text{and}$$
$$\frac{1}{\mathfrak{H}(\mathfrak{n})-\mathfrak{P}(\mathfrak{n})}\sum_{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1}^{\mathfrak{H}(\mathfrak{n})}\omega(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)<\varepsilon,$$

hold for all $\mathfrak{n}>\mathfrak{n}_0$.

Definition 2.8 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNLS}$. A sequence $\mathfrak{x}=(\mathfrak{x}_\mathfrak{p})$ of $\mathfrak{Y}$ is deferred statistically convergent to $\xi\in\mathfrak{Y}$ with respect to the $\mathfrak{NN}\,(\mu,\nu,\omega)$, if for each $\varepsilon\in(0,1)$ and $\lambda>0$,

$$\delta_{\mathfrak{P}}^{\mathfrak{H}}(\mathfrak{p}\in\mathbb{N}:\mu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)\leqslant 1-\varepsilon\,\text{or}\,\nu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)\geqslant\varepsilon,\omega(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)\geqslant\varepsilon)=0,$$

or equivalently

$$\delta_{\mathfrak{P}}^{\mathfrak{H}}(\mathfrak{p}\in\mathbb{N}:\mu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)>1-\varepsilon\,\text{or}\,\nu(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)<\varepsilon,\omega(\mathfrak{x}_\mathfrak{p}-\xi,\lambda)<\varepsilon)=1.$$

In this case, we write $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_{\mathfrak{P}}^{\mathfrak{H}}S(\mu,\nu,\omega))$.

3. MAIN RESULTS

Definition 3.1 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNLS}$. A double sequence $\mathfrak{x}=(\mathfrak{x}_{\mathfrak{p}\zeta})$ of $\mathfrak{Y}$ is convergent to $\xi\in X$ with respect to the $\mathfrak{NN}\,(\mu,\nu,\omega)$, if for each $\varepsilon\in(0,1)$ and $\lambda>0$, there exist $\mathfrak{n}_0,\mathfrak{w}_0\in\mathbb{N}$ such that $\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)>1-\varepsilon,\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon\,\text{and}\,\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon$, for $\mathfrak{n}>\mathfrak{n}_0,\mathfrak{w}>\mathfrak{w}_0$. We denote this by $(\mu,\nu,\omega)-\lim\mathfrak{x}=\xi$ or $\mathfrak{x}_{\mathfrak{p}\zeta}\xrightarrow{(\mu,\nu,\omega)}\xi$.

Definition 3.2 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNLS}$. A double sequence $\mathfrak{x}=(\mathfrak{x}_{\mathfrak{p}\zeta})$ of $\mathfrak{Y}$

is statistically convergent to $\xi \in \mathfrak{Y}$ with respect to the $\mathfrak{N}\mathfrak{N}(\mu,\nu,\omega)$, if for each $\varepsilon \in (0,1)$ and $\lambda > 0$,

$$\delta((\mathfrak{p},\zeta) \in \mathbb{N}\times\mathbb{N} : \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \leqslant 1-\varepsilon, \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \geqslant \varepsilon \,\text{and}\, \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \geqslant \varepsilon) = 0,$$

or equivalently,

$$\delta((\mathfrak{p},\zeta) \in \mathbb{N}\times\mathbb{N} : \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) > 1-\varepsilon, \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) < \varepsilon \,\text{and}\, \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) < \varepsilon) = 1.$$

In this case, we write $\mathfrak{x}_{\mathfrak{p}\zeta} \to \xi(S(\mu,\nu,\omega))$.

Definition 3.3 Let $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{p}\zeta})$ be a double sequence and $\beta(\mathfrak{n}) = \mathfrak{H}(\mathfrak{n}) - \mathfrak{P}(\mathfrak{n}), \gamma(\mathfrak{w}) = \rho(\mathfrak{w}) - \lambda(\mathfrak{w})$. Let $\{\mathfrak{P}(\mathfrak{n})\},\{\mathfrak{H}(\mathfrak{n})\},\{\rho(\mathfrak{w})\}$ and $\{\lambda(\mathfrak{w})\}$ be sequences of nonnegative integers fulfilling the conditions $\mathfrak{P}(\mathfrak{n}) < \lambda(\mathfrak{w}), \lambda(\mathfrak{w}) < \rho(\mathfrak{w})$ and $\lim_{\mathfrak{n}\to\infty} \mathfrak{H}(\mathfrak{n}) = \infty, \lim_{\mathfrak{w}\to\infty} \rho(\mathfrak{w}) = \infty$.
Then defined deferred Cesaro mean $\mathfrak{Q}_{\beta,\gamma}$ of the double sequence $\mathfrak{x}$ is

$$(\mathfrak{Q}_{\beta,\gamma}\mathfrak{x})_{\mathfrak{n}\mathfrak{w}} = \frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})} \sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})} \mathfrak{x}_{\mathfrak{p}\zeta}.$$

Definition 3.4 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{L}\mathfrak{S}$. A double sequence $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{p}\zeta})$ is called $\mathfrak{Q}_{\beta,\gamma}$- summable to ξ with respect to $\mathfrak{N}\mathfrak{N}(\mu,\nu,\omega)$ if for each $\varepsilon \in (0,1)$ and $\lambda > 0$, there exist $\mathfrak{n}_0, \mathfrak{w}_0 \in \mathbb{N}$ such that

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})} \sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})} \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) > 1-\varepsilon,$$

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})} \sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})} \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) < \varepsilon \,\text{and}$$

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})} \sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})} \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) < \varepsilon,$$

for all $\mathfrak{n} > \mathfrak{n}_0, \mathfrak{w} > \mathfrak{w}_0$.

Definition 3.5 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{L}\mathfrak{S}$ and $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{p}\zeta})$ be a double sequence of $\mathfrak{Y}$. A double sequence $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{p}\zeta})$ is deferred statistically convergent to $\xi \in \mathfrak{Y}$ with respect to $\mathfrak{N}\mathfrak{N}(\mu,\nu,\omega)$ if for each $\varepsilon \in (0,1)$ and $\lambda > 0$,

$$\delta_\beta^\gamma((\mathfrak{p},\zeta) \in \mathbb{N}\times\mathbb{N} : \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \leqslant 1-\varepsilon, \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \geqslant \varepsilon,\ \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \geqslant \varepsilon) = 0,$$

$$\delta_\beta^\gamma((\mathfrak{p},\zeta)\in\mathbb{N}\times\mathbb{N}:\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)>1-\varepsilon,\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon,\ \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon)=1.$$

In this case, we write $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega))$.
$\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega)$-Convergence in $\mathfrak{N}\mathfrak{N}\mathfrak{L}\mathfrak{S}$. Next, we are going to give the main results about $\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega)$.

Theorem 3.6 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{L}\mathfrak{S}$ and $\mathfrak{x}=(\mathfrak{x}_{\mathfrak{p}\zeta})$ be a sequence in $\mathfrak{Y}$. Then $\mathfrak{x}_{\mathfrak{p}\zeta}\xrightarrow{(\mu,\nu,\omega)}\xi$ implies $\mathfrak{x}_{\mathfrak{p}\zeta}\xrightarrow{(\mu,\nu,\omega)}\xi(\mathfrak{Q}_\beta^\gamma(\mu,\nu,\omega))$.

Proof. For every $\varepsilon\in(0,1)$ and $\lambda>0$, there exist $\mathfrak{n}_0,\mathfrak{w}_0\in\mathbb{N}$ such that

$$\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)>1-\varepsilon\,\text{or}\,\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon,\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon \tag{2}$$

for every $\mathfrak{n}>\mathfrak{n}_0,\mathfrak{w}>\mathfrak{w}_0$.
If the inequalities in (2) are assumed from $\mathfrak{H}(\mathfrak{n})$ and $\phi(\mathfrak{w})+1$ to $\rho(\mathfrak{w})$, then the following inequalities are obtained:

$$\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\phi(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)>(1-\varepsilon)\beta(\mathfrak{n})\gamma(\mathfrak{w}),$$
$$\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\phi(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon\beta(\mathfrak{n})\gamma(\mathfrak{w})\,\text{and}$$
$$\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\phi(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon\beta(\mathfrak{n})\gamma(\mathfrak{w}). \tag{3}$$

If both sides of (3) are divided by $\beta(\mathfrak{n})\gamma(\mathfrak{w})$, then the desired result will be obtained.

Theorem 3.7 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{L}\mathfrak{S}$ and $\mathfrak{x}=(\mathfrak{x}_{\mathfrak{p}\zeta})$ be a sequence in $\mathfrak{Y}$. Then, $\mathfrak{x}_{\mathfrak{p}\zeta}\xrightarrow{(\mu,\nu,\omega)}\xi$ implies $\mathfrak{x}_{\mathfrak{p}\zeta}\xrightarrow{(\mu,\nu,\omega)}\xi(\mathfrak{Q}_\beta^\gamma\mathfrak{S}(\mu,\nu,\omega))$.

Proof. By hypothesis, for each $\varepsilon\in(0,1)$ and $\lambda>0$, there exist $\mathfrak{y}_0,\mathfrak{w}_0\in\mathbb{N}$ such that

$$\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)>1-\varepsilon,\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon\,\text{and}\,\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon,$$

hold for all $\mathfrak{n}>\mathfrak{n}_0,\mathfrak{w}>\mathfrak{w}_0$. This guarantees that the cardinality of the set

$$\{(\mathfrak{p},\zeta)\in\mathbb{N}\times\mathbb{N}:\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant 1-\varepsilon,\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\,\text{and}\,\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\},$$

is finite. So, immediately we see that

$$\delta_{\beta}^{\gamma}((\mathfrak{p},\zeta)\in\mathbb{N}\times\mathbb{N}:\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant 1-\varepsilon,\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\,\text{and}\,\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon)=0.$$

This gives the proof.

Remark 3.8 The converse of above Theorem is not true, in general.

Let us consider the $\mathfrak{NNLG}$ $(\mathbb{R},\mu_0,\nu_0,\omega_0,*,\diamond,\circledast)$ and $\mathfrak{x}_{\mathfrak{p}\zeta}=\begin{cases}1, & \mathfrak{p}=\mathfrak{w}^2,\zeta=\mathfrak{n}^2\\ 0, & \text{otherwise.}\end{cases}$

For any $\varepsilon>0$ and $\lambda>0$, consider the following set:

$$\mathfrak{K}_{\beta}^{\gamma}(\varepsilon,\lambda)=\left\{\begin{array}{c}\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\\ \mu_0(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant 1-\varepsilon,\nu_0(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon,\,\text{and}\,\omega_0(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\end{array}\right\}.$$

It is clear that

$$\begin{aligned}\mathfrak{K}_{\beta}^{\gamma}(\varepsilon,\lambda)&=\left\{\begin{array}{c}\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\\ |\mathfrak{x}_{\mathfrak{p}\zeta}|\geqslant\frac{\varepsilon t}{1-\varepsilon}\,\text{or}\,\nu_0(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon,\omega_0(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\end{array}\right\}\\ &=\left\{\begin{array}{c}\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\\ \mathfrak{p}=\mathfrak{w}^2,\zeta=\mathfrak{n}^2,\mathfrak{w},\mathfrak{n}\in\mathbb{N}\end{array}\right\}\end{aligned}$$

and we have $\delta_{\beta}^{\gamma}(\mathfrak{K}_{\beta}^{\gamma}(\varepsilon,\lambda))\leqslant\lim_{\mathfrak{n},\mathfrak{w}\to\infty}\dfrac{\sqrt{\beta(\mathfrak{n}}\sqrt{\gamma(\mathfrak{w})}}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}=0.$

The last inequality gives that $\mathfrak{x}_{\mathfrak{p}\zeta}\to 0(\mathfrak{Q}_{\beta}^{\gamma}\mathfrak{S}(\mu_0,\nu_0,\omega_0))$. The sequence $(\mathfrak{x}_{\mathfrak{p}\zeta})$ is not (μ_0,ν_0,ω_0) convergent to zero because it is not convergent to zero in $(\mathbb{R},|\cdot|)$.

Theorem 3.9 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be an $\mathfrak{NNLG}$ and $\mathfrak{x}=(\mathfrak{x}_{\mathfrak{p}\zeta})$ be a double sequence in $\mathfrak{Y}$. Then, the $\mathfrak{Q}_{\beta}^{\gamma}\mathfrak{S}(\mu,\nu,\omega)$ limit of $(\mathfrak{x}_{\mathfrak{p}\zeta})$ is unique.

Proof. Choose two distinct elements $\xi_1,\xi_2\in\mathfrak{Y}$ so that

$\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi_1(\mathfrak{Q}_{\beta}^{\gamma}S(\mu,\nu,\omega))$ and $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi_2(\mathfrak{Q}_{\beta}^{\gamma}S(\mu,\nu,\omega))$.

Given $\varepsilon\in(0,1)$, choose $\rho>0$ such that $(1-\rho)*(1-\rho)>1-\varepsilon,\rho\diamond\rho<\varepsilon$ and $\rho\circledast\rho<\varepsilon$.

From the assumption, for every $\lambda>0$,

$\delta_{\beta}^{\gamma}(\mathfrak{K}_{\mu,1}(\varepsilon,\lambda))=\delta_{\beta}^{\gamma}(\mathfrak{K}_{\nu,1}(\varepsilon,\lambda))=\delta_{\beta}^{\gamma}(\mathfrak{K}_{\omega,1}(\varepsilon,\lambda))=0$ and

$\delta_{\beta}^{\gamma}(\mathfrak{K}_{\mu,2}(\varepsilon,\lambda))=\delta_{\beta}^{\gamma}(\mathfrak{K}_{\nu,2}(\varepsilon,\lambda))=\delta_{\beta}^{\gamma}(\mathfrak{K}_{\omega,2}(\varepsilon,\lambda))=0,$

where

$$\mathfrak{K}_{\mu,1}(\varepsilon,\lambda)=\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi_1,\lambda)\leqslant 1-\varepsilon\},$$

$$\mathfrak{K}_{\mu,2}(\varepsilon,\lambda)=\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi_2,\lambda)\leqslant 1-\varepsilon\},$$

$$\mathfrak{K}_{\nu,1}(\varepsilon,\lambda) = \{\mathfrak{P}(\mathfrak{n})+1 \leqslant \mathfrak{p} \leqslant \mathfrak{H}(\mathfrak{n}), \lambda(\mathfrak{w})+1 \leqslant \zeta \leqslant \rho(\mathfrak{w}) : \nu(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_1, \lambda) \geqslant \varepsilon\},$$

$$\mathfrak{K}_{\nu,2}(\varepsilon,\lambda) = \{\mathfrak{P}(\mathfrak{n})+1 \leqslant \mathfrak{p} \leqslant \mathfrak{H}(\mathfrak{n}), \lambda(\mathfrak{w})+1 \leqslant \zeta \leqslant \rho(\mathfrak{w}) : \nu(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_2, \lambda) \geqslant \varepsilon\},$$

$$\mathfrak{K}_{\omega,1}(\varepsilon,\lambda) = \{\mathfrak{P}(\mathfrak{n})+1 \leqslant \mathfrak{p} \leqslant \mathfrak{H}(\mathfrak{n}), \lambda(\mathfrak{w})+1 \leqslant \zeta \leqslant \rho(\mathfrak{w}) : \omega(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_2, \lambda) \geqslant \varepsilon\} \text{ and}$$

$$\mathfrak{K}_{\omega,2}(\varepsilon,\lambda) = \{\mathfrak{P}(\mathfrak{n})+1 \leqslant \mathfrak{p} \leqslant \mathfrak{H}(\mathfrak{n}), \lambda(\mathfrak{w})+1 \leqslant \zeta \leqslant \rho(\mathfrak{w}) : \omega(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_2, \lambda) \geqslant \varepsilon\}.$$

Define the set

$$\mathfrak{K}_{\mu,\nu,\omega}(\varepsilon,\lambda) = (\mathfrak{K}_{\mu,1}(\varepsilon,\lambda) \cup \mathfrak{K}_{\mu,2}(\varepsilon,\lambda)) \cap (\mathfrak{K}_{\nu,1}(\varepsilon,\lambda) \cup \mathfrak{K}_{\nu,2}(\varepsilon,\lambda)) \cap (\mathfrak{K}_{\omega,1}(\varepsilon,\lambda) \cup \mathfrak{K}_{\omega,2}(\varepsilon,\lambda)).$$

Then, it is enough to show that $\delta_\beta^\gamma(\mathfrak{K}_{\mu,\nu,\omega}(\varepsilon,\lambda)) = 0$. It suggests that $\delta_\beta^\gamma(\mathbb{N} - \mathfrak{K}_{\mu,\nu,\omega}(\varepsilon,\lambda)) = 1$.
Now, let $\mathfrak{p} \in \mathbb{N} - \mathfrak{K}_{\mu,\nu,\omega}(\varepsilon,\lambda)$, then there are three cases:
$\mathfrak{p} \in \mathbb{N} - \{\mathfrak{K}_{\mu,1}(\varepsilon,\lambda) \cup \mathfrak{K}_{\mu,2}(\varepsilon,\lambda)\}, \mathfrak{p} \in \mathbb{N} - \{\mathfrak{K}_{\nu,1}(\varepsilon,\lambda) \cup \mathfrak{K}_{\nu,2}(\varepsilon,\lambda)\}$ and
$\mathfrak{p} \in \mathbb{N} - \{\mathfrak{K}_{\omega,1}(\varepsilon,\lambda) \cup \mathfrak{K}_{\omega,2}(\varepsilon,\lambda)\}$.
Consider the above cases,

$$\mu(\xi_1 - \xi_2, \lambda) \geqslant \mu\left(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_1, \frac{\lambda}{2}\right) * \mu\left(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_2, \frac{\lambda}{2}\right) > (1-\rho) * (1-\rho) > 1 - \varepsilon \tag{4}$$

$$\nu(\xi_1 - \xi_2, \lambda) < \nu\left(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_1, \frac{\lambda}{2}\right) \diamond \nu\left(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_2, \frac{\lambda}{2}\right) < \rho \diamond \rho < \varepsilon. \tag{5}$$

$$\omega(\xi_1 - \xi_2, \lambda) < \omega\left(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_1, \frac{\lambda}{2}\right) \circledast \omega\left(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi_2, \frac{\lambda}{2}\right) < \rho \circledast \rho < \varepsilon. \tag{6}$$

are obtained. Since $\varepsilon > 0$ is arbitrary in (4),(5) and (6), $\xi_1 = \xi_2$ is obtained.
Therefore, the limit of the sequence is unique.

Example 3.10 Uniqueness of $\mathfrak{Q}_\beta^\gamma\mathfrak{S}(\mu,\nu,\omega)$-limit.
Let $\mathfrak{Y} = \mathbb{R}$, and define the $\mathfrak{N}\mathfrak{N}$ components by:

$$\mu(\mathfrak{x}) = \frac{|\mathfrak{x}|}{1+|\mathfrak{x}|}, \quad \nu(\mathfrak{x}) = \frac{1}{1+|\mathfrak{x}|}, \quad \omega(\mathfrak{x}) = \frac{1}{|\mathfrak{x}|}$$

Consider the double sequence:

$$\mathfrak{x}_{\mathfrak{p}\zeta} = \frac{1}{\mathfrak{p}+\zeta}, \quad (\mathfrak{p},\zeta) \in \mathbb{N}^2$$

Suppose $\mathfrak{x}_{\mathfrak{p}\zeta} \to a$ and $\mathfrak{x}_{\mathfrak{p}\zeta} \to b$ in the $\mathfrak{Q}_\beta^\gamma\mathfrak{S}(\mu,\nu,\omega)$ sense for some $a \neq b \in \mathbb{R}$.

Then for any $\varepsilon > 0$, the sets:

$$A_a(\varepsilon) = \left\{(\mathfrak{p},\zeta) : \mu(\mathfrak{x}_{\mathfrak{p}\zeta} - a) \geqslant \varepsilon\right\}, \quad A_b(\varepsilon) = \left\{(\mathfrak{p},\zeta) : \mu(\mathfrak{x}_{\mathfrak{p}\zeta} - b) \geqslant \varepsilon\right\}$$

have $\mathfrak{Q}_\beta^\gamma$-density zero.

Choose $\varepsilon = \frac{\mu(a-b)}{2}$. Then, for sufficiently large $(\mathfrak{p},\zeta)$, we have:

$$\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-a) < \varepsilon \quad \text{and} \quad \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-b) < \varepsilon$$

By the triangle inequality:

$$\mu(a-b) \leqslant \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-a)+\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-b) < \varepsilon+\varepsilon = \mu(a-b)$$

which is a contradiction. Hence, $a=b$. Therefore, the limit in the $\mathfrak{Q}_{\beta}^{\gamma}\mathfrak{S}(\mu,\nu,\omega)$ sense is unique.

Theorem 3.11 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNLS}$. Then a double sequence $\mathfrak{x}=(\mathfrak{x}_{\mathfrak{p}\zeta})$ in $\mathfrak{Y}$. If a sequence $(\mathfrak{x}_{\mathfrak{p}\zeta})$ is $\mathfrak{Q}_{\beta}^{\gamma}S(\mu,\nu,\omega)$- convergent, then it is $\mathfrak{Q}_{\beta}^{\gamma}S(\mu,\nu,\omega)$-Cauchy sequence in $\mathfrak{NNLS}$.

Proof. Assume that the sequence $\mathfrak{x}=(\mathfrak{x}_{\mathfrak{p}\zeta})$ is $\mathfrak{Q}_{\beta}^{\gamma}S(\mu,\nu,\omega)$- convergent to $\xi\in\mathfrak{Y}$.
Let us choose $\mathfrak{s}>0$ so that $(1-\varepsilon)*(1-\varepsilon)>1-\mathfrak{s}, \varepsilon\diamond\varepsilon<\mathfrak{s}$ and $\varepsilon\circledast\varepsilon<\mathfrak{s}$ hold for any $\varepsilon>0$.
Then, for any $\lambda>0$, we have,

$$\delta_{\beta}^{\gamma}\left(\left\{\begin{array}{c}(\mathfrak{p},\zeta)\in\mathbb{N}\times\mathbb{N}:\mu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)\leqslant 1-\varepsilon,\\ \nu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)\geqslant\varepsilon, \text{and}\,\omega\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)\geqslant\varepsilon\end{array}\right\}\right)=0$$

and this implies that

$$\delta_{\beta}^{\gamma}\left(\left\{\begin{array}{c}(\mathfrak{p},\zeta)\in\mathbb{N}\times\mathbb{N}:\mu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)> 1-\varepsilon,\\ \nu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)<\varepsilon, \text{and}\,\omega\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)<\varepsilon\end{array}\right\}\right)=0.$$

Let $(\mathfrak{w},\mathfrak{n})\in\left\{\begin{array}{c}(\mathfrak{p},\zeta)\in\mathbb{N}\times\mathbb{N}:\mu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)\leqslant q1-\varepsilon,\\ \nu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)\geqslant\varepsilon, \text{and}\\ \omega\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)\geqslant\varepsilon\end{array}\right\}$ be an arbitrary element.
Let

$$B(\varepsilon,\lambda):=\left\{\begin{array}{c}(\mathfrak{p},\zeta)\in\mathbb{N}\times\mathbb{N}:\mu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\frac{\lambda}{2}\right)\leqslant 1-\varepsilon,\\ \nu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\frac{\lambda}{2}\right)\geqslant\varepsilon\,\text{and}\,\omega\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\frac{\lambda}{2}\right)\geqslant\varepsilon\end{array}\right\}.$$

It suffices to demonstrate that

$$B(\varepsilon,\lambda)\subset\left\{\begin{array}{c}(\mathfrak{p},\zeta)\in\mathbb{N}\times\mathbb{N}:\mu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\frac{\lambda}{2}\right)\leqslant 1-\varepsilon,\\ \nu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\frac{\lambda}{2}\right)\geqslant\varepsilon\,\text{and}\,\omega\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\frac{\lambda}{2}\right)\geqslant\varepsilon\end{array}\right\}.$$

Let

$$(\mathfrak{p},\zeta)\in B(\varepsilon,\lambda)-\left\{\begin{array}{c}(\mathfrak{p},\zeta)\in\mathbb{N}\times\mathbb{N}:\mu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\frac{\lambda}{2}\right)\leqslant 1-\varepsilon,\\ \nu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\frac{\lambda}{2}\right)\geqslant\varepsilon\,\text{and}\,\omega\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\frac{\lambda}{2}\right)\geqslant\varepsilon\end{array}\right\}.$$

Then, we have $\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\lambda)\leqslant 1-\varepsilon$ and $\mu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)\leqslant 1-\varepsilon$.
In particular $\mu\left(\mathfrak{x}_{\mathfrak{wn}}-\xi,\frac{\lambda}{2}\right)>1-\varepsilon$.
Hence, $1-\mathfrak{s}\geqslant\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\lambda)\geqslant\mu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)*\mu\left(\mathfrak{x}_{\mathfrak{wn}}-\xi,\frac{\lambda}{2}\right)>(1-\varepsilon)*(1-\varepsilon)>1-\mathfrak{s}$, which is impossible.
On the other hand, $\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\lambda)\geqslant\mathfrak{s}$ and $\nu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)<\varepsilon$.
Then, $\mathfrak{s}\leqslant\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\lambda)\leqslant\nu\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)\diamond\nu\left(\mathfrak{x}_{\mathfrak{wn}}-\xi,\frac{\lambda}{2}\right)<\varepsilon\diamond\varepsilon<\mathfrak{s}$, which is impossible.
Also, $\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\lambda)\geqslant\mathfrak{s}$ and $\omega\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)<\varepsilon$.

Then, $\mathfrak{s}\leqslant\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\mathfrak{x}_{\mathfrak{wn}},\lambda)\leqslant\omega\left(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\frac{\lambda}{2}\right)\circledast\omega\left(\mathfrak{x}_{\mathfrak{wn}}-\xi,\frac{\lambda}{2}\right)<\varepsilon\circledast\varepsilon<\mathfrak{s}$, which is impossible. This proves our claim.

Example 3.12 Let $\mathfrak{F}=\mathbb{R}$ be a real linear space equipped with the $\mathfrak{NN}$ $\mathfrak{f}:\mathbb{R}\to[0,1]^3$ defined by:

$$\mathfrak{f}(\mathfrak{x})=(\mu(\mathfrak{x}),\nu(\mathfrak{x}),\omega(\mathfrak{x}))=\left(\frac{|\mathfrak{x}|}{1+|\mathfrak{x}|},\frac{1}{1+|\mathfrak{x}|},\frac{1}{|\mathfrak{x}|}\right)$$

This satisfies the axioms of a neutrosophic normed linear space.

Define the double sequence $\{\mathfrak{x}_{m,n}\}\subset\mathbb{R}$ by:

$$\mathfrak{x}_{m,n}=\frac{1}{m+n}$$

Let the deferred index set be:

$$D=\{(m,n)\in\mathbb{N}\times\mathbb{N}:m,n\text{ are even}\}$$

We first observe that the sequence converges to $\mathfrak{x}_0=0$ in the deferred statistical sense. That is, for any $\varepsilon>0$, the set

$$\mathfrak{A}_\varepsilon=\{(m,n)\in D:\mu(\mathfrak{x}_{m,n}-0)\geqslant\varepsilon\}$$

has deferred density zero:

$$\delta_D(\mathfrak{A}_\varepsilon)=0$$

since

$$\mu(\mathfrak{x}_{m,n}) = \frac{\left|\frac{1}{m+n}\right|}{1+\left|\frac{1}{m+n}\right|} = \frac{1}{(m+n)+1} \to 0 \quad \text{as } m,n \to \infty$$

Therefore, $\{\mathfrak{x}_{m,n}\}$ is deferred statistically convergent to 0.

Now, we show that $\{\mathfrak{x}_{m,n}\}$ is a Cauchy sequence. Given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m,n \geqslant N$, $(m,n) \in D$, we have:

$$\mu(\mathfrak{x}_{m,n}) < \frac{\varepsilon}{2}$$

Consider $(m_1,n_1),(m_2,n_2) \in D$ with $m_1,n_1,m_2,n_2 \geqslant N$. Then:

$$|\mathfrak{x}_{m_1,n_1} - \mathfrak{x}_{m_2,n_2}| \leqslant |\mathfrak{x}_{m_1,n_1}| + |\mathfrak{x}_{m_2,n_2}|$$

and hence

$$\mu(\mathfrak{x}_{m_1,n_1} - \mathfrak{x}_{m_2,n_2}) \leqslant \mu(\mathfrak{x}_{m_1,n_1}) + \mu(\mathfrak{x}_{m_2,n_2}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, for sufficiently large indices, the difference between any two terms becomes arbitrarily small in the neutrosophic sense. Therefore, $\{\mathfrak{x}_{m,n}\}$ is a Cauchy sequence in the $\mathfrak{NNLS}$.

Comparison of $\mathfrak{Q}_\beta^\gamma(\mu,\nu,\omega)$ and $\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega)$:
In this subsection we deal with the relation between $\mathfrak{Q}_\beta^\gamma(\mu,\nu,\omega)$ and $\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega)$ in a $\mathfrak{NNLS}$.

Theorem 3.13 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNLS}$ and a double sequence $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{p}\zeta})$ in $\mathfrak{Y}$.
Then $\mathfrak{x}_{\mathfrak{p}\zeta} \to \xi(\mathfrak{Q}_\beta^\gamma(\mu,\nu,\omega))$ implies $\mathfrak{x}_{\mathfrak{p}\zeta} \to \xi(\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega))$.

Proof. Assume that $\mathfrak{x}_{\mathfrak{p}\zeta} \to \xi(\mathfrak{Q}_\beta^\gamma(\mu,\nu,\omega))$. That is, for any $\varepsilon > 0$ and $\lambda > 0$, there exist $\mathfrak{n}_0, \mathfrak{w}_0 \in \mathbb{N}$ such that

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})} \sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1 \\ \zeta=\phi(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})} \tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi, \lambda) > 1 - \varepsilon,$$

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})} \sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1 \\ \zeta=\phi(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})} \nu(\mathfrak{x}_{\mathfrak{p}\zeta} - \xi, \lambda) < \varepsilon \text{ and}$$

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\phi(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon$$

hold for all $\mathfrak{n}>\mathfrak{n}_0,\mathfrak{w}>\mathfrak{w}_0$.
Then, we have the following facts:

$$\begin{aligned}
&\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\\
&\quad\geqslant\frac{|\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\}|}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\\
&\quad=:A_\beta^\gamma(\mathfrak{n},\mathfrak{w})\\
&\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\\
&\quad\leqslant\frac{|\{\mathfrak{P}(\mathfrak{y})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\tilde{\nu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\}|}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\\
&\quad=:B_\beta^\gamma(\mathfrak{n},\mathfrak{w})\,\text{and}\\
&\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\\
&\quad\leqslant\frac{|\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\tilde{\omega}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\}|}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\\
&\quad=:C_\beta^\gamma(\mathfrak{n},\mathfrak{w}).
\end{aligned}$$

As a consequence of $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_\beta^\gamma(\mu,\nu,\omega))$ and the above inequalities, we have $\delta_\beta^\gamma(A_\beta^\gamma(\mathfrak{n},\mathfrak{w}))=0,\delta_\beta^\gamma(B_\beta^\gamma(\mathfrak{n},\mathfrak{w}))=0$ and $\delta_\beta^\gamma(C_\beta^\gamma(\mathfrak{n},\mathfrak{w}))=0$.
Therefore,
$\delta_\beta^\gamma(\{(\mathfrak{n},\mathfrak{w}):\tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant 1-\varepsilon,\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant\varepsilon,\text{and}\,\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant\varepsilon\})=0.$
It means that $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega))$.

Remark 3.14 The inverse of the above theorem is not true. For example, let $\mathfrak{H}(\mathfrak{n})$ and $\rho(\mathfrak{w})$ be monotonic increasing sequences of positive integers and h_1,h_2 be fixed numbers. Let us define a double sequence for $\mathfrak{n},\mathfrak{w}=1,2,\ldots$

$$\mathfrak{x}_{\mathfrak{p}\zeta}=\begin{cases}\mathfrak{p}^2\zeta^2,\left\{\left\|\sqrt{\mathfrak{H}(\mathfrak{n})}\right\|-h_1<\mathfrak{p}\leqslant\left|\sqrt{\mathfrak{H}(\mathfrak{n})}\right\|\right\}\\ \left\{\left\|\sqrt{\rho(\mathfrak{w})}\right\|-h_2<\zeta\leqslant\left\|\sqrt{\rho(\mathfrak{w})}\right\|\right\}\end{cases}$$

$$\mathfrak{x}_{\mathfrak{p}\zeta}=\begin{cases}\mathfrak{p}^2\zeta^2, & \begin{cases}\left\|\sqrt{\mathfrak{H}(\mathfrak{n})}\right\|-h_1<\mathfrak{p}\leqslant\left|\sqrt{\mathfrak{H}(\mathfrak{n})}\right\| \\ \left\|\sqrt{\rho(\mathfrak{w})}\right\|-h_2<\zeta\leqslant\left\|\sqrt{\rho(\mathfrak{w})}\right\|\end{cases} \\ 0, & \text{Otherwise}\end{cases}$$

If we consider $\mathfrak{Q}_\beta^\gamma$ for the sequence $\mathfrak{P}(\mathfrak{n}),\tau(\mathfrak{w})$ satisfying

$$0<\mathfrak{P}(\mathfrak{y})\leqslant\left\|\sqrt{\mathfrak{H}(\mathfrak{n})}\right\|-h_1, 0<\tau(\mathfrak{w})\leqslant\left\|\sqrt{\rho(\mathfrak{w})}\right\|-h_2$$

then we have

$$\frac{\left|\left\{\begin{array}{c}\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}) \\ \mu_0(\mathfrak{x}_{\mathfrak{p}\zeta},\lambda)\geqslant 1-\varepsilon, \nu_0(\mathfrak{x}_{\mathfrak{p}\zeta},\lambda)\leqslant\varepsilon, \text{ and} \\ \omega_0(\mathfrak{x}_{\mathfrak{p}\zeta},\lambda)\leqslant\varepsilon\end{array}\right\}\right|}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}$$
$$=\frac{\left|\left\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\tau(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\left|\mathfrak{x}_{\mathfrak{p}\zeta}\right|\geqslant\frac{\varepsilon\lambda}{1-\varepsilon}\right\}\right|}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}=\frac{h_1h_2}{\beta(\mathfrak{n})\gamma(\mathfrak{w})},$$

which implies that $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega))$.
Also, we have the following inequalities:

$$\lim_{\mathfrak{n},\mathfrak{w}\to\infty}\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant h_1h_2 \text{ and}$$

$$\lim_{n,m\to\infty}\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant h_1h_2.$$

Since $h_1\neq 0$ and $h_2\neq 0, \mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_\beta^\gamma(\mu,\nu,\omega))$.

Theorem 3.15 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be an $\mathfrak{NNLG}$ and $\mathfrak{x}=(\mathfrak{x}_{\mathfrak{p}\zeta})$ be a bounded sequence in $\mathfrak{Y}$. Then, the convergence $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega))$ implies $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_\beta^\gamma(\mu,\nu,\omega))$.

Proof. Suppose that $(\mathfrak{x}_{\mathfrak{p}\zeta})$ is a bounded sequence and $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_\beta^\gamma S(\mu,\nu,\omega))$.
There exists a positive integer M under the assumption that (x_{kl}) is so that
$\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant 1-M, \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<M$ and $\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<M$ hold for all $\mathfrak{p},\zeta$.

Therefore, the following inequality

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)=\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\left\{\left(\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1,\\ \zeta=\lambda(\mathfrak{w})+1\\ \tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}+\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1,\\ \zeta=\lambda(\mathfrak{w})+1\\ \tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\right)\tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\right\}$$

$$<\varepsilon\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}|\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon\}|$$
$$+M\cdot\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}|\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\}|$$

holds. If we take limit by considering $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_{\beta}^{\gamma}S(\mu,\nu))$, then it is obtained that

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)>\varepsilon.$$

This gives

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\tilde{\mu}(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<1-\varepsilon. \tag{7}$$

Also, the following inequality

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)=\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\left\{\left(\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1,\\ \zeta=\lambda(\mathfrak{w})+1\\ \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}+\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1,\\ \zeta=\lambda(\mathfrak{w})+1\\ \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\right)\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\right\}$$

$$<\varepsilon\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}|\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon\}|$$
$$+M\cdot\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}|\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\}|$$

holds. Now,

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon. \tag{8}$$

Also, the following inequality holds.

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)=\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\left\{\left(\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1,\\ \zeta=\lambda(\mathfrak{w})+1\\ \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}+\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1,\\ \zeta=\lambda(\mathfrak{w})+1\\ \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\right)\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\right\}$$

$$<\varepsilon\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}|\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon\}|$$
$$+M\cdot\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}|\{\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n}),\lambda(\mathfrak{w})+1\leqslant\zeta\leqslant\rho(\mathfrak{w}):\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\}|$$

This gives that

$$\frac{1}{\beta(\mathfrak{n})\gamma(\mathfrak{w})}\sum_{\substack{\mathfrak{p}=\mathfrak{P}(\mathfrak{n})+1\\ \zeta=\lambda(\mathfrak{w})+1}}^{\mathfrak{H}(\mathfrak{n}),\rho(\mathfrak{w})}\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)<\varepsilon. \tag{9}$$

The proof follows from (7), (8) and (9).

Theorem 3.16 Let $(\mathfrak{Y},\mu,\nu,\omega,*,\diamond,\circledast)$ be an $\mathfrak{NNLS}$ and $\mathfrak{x}=(\mathfrak{x}_{\mathfrak{p}\zeta})$ be a bounded sequence in $\mathfrak{Y}$, and the sequences $\left\{\frac{\mathfrak{P}(\mathfrak{n})}{\beta(\mathfrak{n})}\right\},\left\{\frac{\lambda(\mathfrak{w})}{\gamma(\mathfrak{w})}\right\}$ be bounded. Then, $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(S(\mu,\nu,\omega))$ implies $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(\mathfrak{Q}_{\beta}^{\gamma}S(\mu,\nu,\omega))$.

Proof. Since $\lim_{\mathfrak{n}\to\infty}\beta(\mathfrak{n})=\infty$ and $\mathfrak{x}_{\mathfrak{p}\zeta}\to\xi(S(\mu,\nu,\omega))$,

$$\lim_{\mathfrak{n}\to\infty}\frac{1}{\beta(\mathfrak{n})}|\{\mathfrak{p}\leqslant\beta(n):\mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant1-\varepsilon,\nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon,\text{ and }\omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\}|=0.$$

Also, the following inclusion and the inequality hold.

$$\left\{\begin{matrix}\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n})\\ \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant1-\varepsilon,\\ \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon,\text{ and}\\ \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\end{matrix}\right\}\subset\left\{\begin{matrix}\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n})\\ \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant1-\varepsilon,\\ \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon,\text{ and}\\ \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\end{matrix}\right\}$$
$$\left|\left\{\begin{matrix}\mathfrak{P}(\mathfrak{n})+1\leqslant\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n})\\ \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant1-\varepsilon,\\ \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon,\text{ and}\\ \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\end{matrix}\right\}\right|\leqslant\left\{\begin{matrix}\mathfrak{p}\leqslant\mathfrak{H}(\mathfrak{n})\\ \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\leqslant1-\varepsilon,\\ \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon,\text{ and}\\ \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda)\geqslant\varepsilon\end{matrix}\right\}$$

So, we get

$$\frac{1}{\beta(\mathfrak{n})}\left|\left\{\begin{array}{c}\mathfrak{P}(\mathfrak{n})+1 \leqslant \mathfrak{p} \leqslant \mathfrak{H}(\mathfrak{n}) : \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \leqslant 1-\varepsilon, \\ \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \geqslant \varepsilon, \text{and}\, \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \geqslant \varepsilon\end{array}\right\}\right|$$
$$\leqslant \mathfrak{L}(\mathfrak{n}) \cdot \frac{1}{\mathfrak{H}(\mathfrak{n})}\left|\left\{\begin{array}{c}\mathfrak{p} \leqslant \mathfrak{H}(\mathfrak{n}) : \mu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \leqslant 1-\varepsilon, \\ \nu(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \geqslant \varepsilon, \text{and}\, \omega(\mathfrak{x}_{\mathfrak{p}\zeta}-\xi,\lambda) \geqslant \varepsilon\end{array}\right\}\right|.$$

By taking limit when $\mathfrak{n} \to \infty$, we get $\mathfrak{x}_{\mathfrak{p}\zeta} \to \xi\,(\mathfrak{Q}^{\gamma}_{\beta} S(\mu,\nu,\omega))$.

4. CONCLUSION

In this article, we mainly focused on a few key components of deferred statistical convergence in $\mathfrak{NNS}$. We have demonstrated that deferred statistical convergent sequences and deferred statistical Cauchy sequences are identical in a $\mathfrak{NNLS}$. We have looked at a sequence's deferred statistical boundedness and its connections to deferred statistical convergent sequences in $\mathfrak{NNLS}$. The scope of this research can be naturally extended and enriched in several directions: Extending deferred statistical convergence to triple sequences could uncover new structural properties and functional limits within $\mathfrak{NNLS}$, opening pathways to multidimensional approximation and data modeling. The results may be applied to study the continuity and compactness of operators under deferred statistical convergence, particularly in the analysis of uncertain dynamical systems.

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