Q-homotopy analysis transform technique for two-dimensional fractional diffusion equation

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Abstract

In this paper, we find the solution for two-dimensional fractional diffusion equation using an efficient scheme namely, q-homotopy analysis transform method (q-HATM). To validate the competence and applicability of the proposed scheme, we consider two examples. The proposed algorithm is the joint application of the homotopy analysis method and Laplace transform (LT). For the obtained series solution, the proposed method offers $\hbar$-curves that illustrate convergence region. The numerical behaviour is presented to ensure the reliability and accuracy of the proposed method. Further, the nature of the achieved results is presented for a diverse order of the fractional derivative. The achieved results show that, the considered method is easy to implement and very effective to study the behaviour of nonlinear models.

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Keywords: Laplace transform; diffusion equation; q-homotopy analysis transform method; Caputo fractional operator.

1. INTRODUCTION

The study and analysis of mathematical aspects associated with real-world models has attracted the attention of many young researchers due to its ability to exemplify the corresponding behaviours. The concepts of differential equations are considered as the main core while investigating models related to rate of change. Many senior scholars have proved the importance and effect of describing the complicated phenomena with differential and integral operators. However, when researchers search for the best tool with the deeper investigation for the development of science and technology, they always expect a more accurate and stimulating tool to investigate in an effective manner. Particularly, for phenomena related to history and hereditary-based consequences, the integer-order concept is not appropriate to exemplify complex phenomena. In this regard, many pioneers suggested the most appropriate tool to study these consequences namely, fractional calculus [1-6]. Soon after, it received the most consideration to analyse and study the diverse class of models to predict their corresponding nature. Many authors analysed diverse phenomena with the help of fractional calculus, for instance, the model exemplifying

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deadly disease is analysed by many researchers with the theory and notions of fractional calculus [7-9], with the help of Mittag-Leffler kernel authors in [10, 11] respectively analysed Lakes pollution model and extended Fisher–Kolmogorov equation, fractional Lakshmanan–Porsezian–Daniel model is examined by the help of efficient scheme in [12], the behaviour of the solution for time-fractional Phi-four equation is presented by many authors in [13], numerical solutions for diverse classes of fractional-order nonlinear equations are exemplified by scholars in [14-24]. Moreover, the concept of fractional calculus can provide more degrees of freedom associated with hereditary and memory-based consequences. The study of the diffusion equation and its generalised family has attracted researchers due to its wide applicability in the diverse areas including biology to examine the diffusion of cells, chemistry to exemplify the reaction-diffusion mechanism and most importantly physics with most of the mechanisms, and also many other branches essential to the concept and phenomena of diffusion. Further, it attracts numerous young researchers due to its mathematical aspects to exemplify nature in a systematic manner. In this paper, we consider the two–dimensional diffusion equation in the form [25]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \tag{1}$$

subject to the initial and boundary conditions

$$\begin{aligned} &u(x,y,0) = \phi(x,y), \qquad (x,y) \in \Omega,\ 0 \le t \le T, \\ &u(0,y,t) = f_1(t), \qquad\qquad u(1,y,t) = f_2(t), \\ &u(x,0,t) = f_3(t), \qquad\qquad u(x,1,t) = f_4(t), \end{aligned} \tag{2}$$

On the other hand, as much as it is important to formulate or propose real-world problems or processes with mathematical models, it is precise and vital to find the corresponding accurate solution to investigate its corresponding behaviours of the system. Many senior scholars have suggested distinct advanced algorithms to find solution for integral and differential equations with classical and fractional orders. However, each one has its limitations and requires modification to increase exactness and decrease the dense simplifications. In this connection, based on topological concept Liao Shijun proposed a powerful method called homotopy analysis method (HAM) [26]. Recently, it has been effectively and successfully employed to study the nature of nonlinear models without linearization or perturbation. However, this traditional algorithm necessitates enormous computer memory and time for computational work. Thereafter, many scholars nurture consolidation of this scheme with formerly existing transform methods. Here, to find the solution for the fractional diffusion equation, we evaluate it with the aid of q-HATM within the framework of fractional calculus (FC). With the help of the LT and q-homotopy analysis technique, this scheme was suggested by the authors in [27]. Here, we consider following fractional diffusion (FD) equation of dimension two as follows, The fractional diffusion equation is given by

$$D_t^{\gamma} u(x,y,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \tag{3}$$

where γ is the fractional order. This method does not require any of the above-said requirements and deals with many parameters which can help regulate the convergence region. Recently, the proposed algorithm has been employed by many scholars to study and investigate the more stimulating behaviour of the models. The proficiency of the proposed scheme is illustrated by Veeresha et al. in [28] while analyzing the Swift-Hohenberg equation, some stimulating results are derived for the fractional vibration equation by Srivastava et al. in [29], the behaviour of the Klein-Gordon-Schrödinger equation is captured by researchers in [30], a mathematical model of HIV infection with fractional order is analysed by Bulut and his co-authors in [31], authors in [32] presented the convergence analysis for the q-HATM, and many authors have employed q-HATM to present their viewpoints [33-37]. The considered equation has been studied by many researchers due to its applications in different areas. For instance, authors in [38] employed implicit solution algorithm for diffusion equation and also presented stability analysis, some results are derived for fractional diffusion equation by researchers in [39], the Fourier technique is applied in [40] to find the solution for diffusion equation, alternating direction implicit methods are employed by authors in [41] to find the solution for the considered model and many other numerical and semi-analytical schemes are applied [42, 43].

2. PRELIMINARIES

Here, the fundamental notions of Laplace transform (LT) and fractional calculus are recalled [1-6]

DEFINITION 1. The Riemann–Liouville fractional integral of a function $f(t) \in C_\delta$ $(\delta \geq -1)$ of order $\gamma > 0$ is defined as

$$J^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\vartheta)^{\gamma-1} f(\vartheta)\, d\vartheta, \qquad J^0 f(t) = f(t). \tag{4}$$

DEFINITION 2. The Caputo fractional derivative of order γ of a function $f \in C^n_{(-1)}$ is defined as

$$D_t^\gamma f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n}, & \gamma = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\gamma)} \displaystyle\int_0^t (t-\vartheta)^{n-\gamma-1} f^{(n)}(\vartheta)\, d\vartheta, & n-1 < \gamma < n,\ n \in \mathbb{N}. \end{cases} \tag{5}$$

DEFINITION 3. Let $D_t^\gamma f(t)$ be the Caputo fractional derivative. Then its Laplace transform is given by

$$\mathscr{L}\left[D_t^\gamma f(t)\right] = s^\gamma F(s) - \sum_{r=0}^{n-1} s^{\gamma-r-1} f^{(r)}(0^+), \qquad (n-1 < \gamma \leq n), \tag{6}$$

where $F(s)$ denotes the Laplace transform of $f(t)$.

3. BASIC IDEA OF Q-HATM

Consider the nonlinear fractional differential equation

$$D_t^\gamma u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \qquad 0 < \gamma \leq 1, \tag{7}$$

where R and N denote linear and nonlinear differential operators, and $f(x,t)$ is a given source term. Applying the Laplace transform (LT) to Eq. (7), we obtain

$$s^\gamma \mathscr{L}[u(x,t)] - \sum_{k=0}^{n-1} s^{\gamma-k-1} u^{(k)}(x,0) + \mathscr{L}[Ru(x,t)] + \mathscr{L}[Nu(x,t)] = \mathscr{L}[f(x,t)]. \tag{8}$$

Using Eq. (6), Eq. (8) becomes

$$\mathscr{L}[u(x,t)] - \frac{1}{s^\gamma} \sum_{k=0}^{n-1} s^{\gamma-k-1} u^{(k)}(x,0) + \frac{1}{s^\gamma} \left\{\mathscr{L}[Ru(x,t)] + \mathscr{L}[Nu(x,t)] - \mathscr{L}[f(x,t)]\right\} = 0. \tag{9}$$

For $\phi(x,t;q)$, the nonlinear operator $N[\phi]$ is defined as

$$\begin{aligned} N[\phi(x,t;q)] = \mathscr{L}[\phi(x,t;q)] - \frac{1}{s^\gamma} \sum_{k=0}^{n-1} s^{\gamma-k-1} \phi^{(k)}(x,t;q)\big|_{0^+} \\ + \frac{1}{s^\gamma} \left\{\mathscr{L}[R\phi(x,t;q)] + \mathscr{L}[N\phi(x,t;q)] - \mathscr{L}[f(x,t)]\right\}, \qquad q \in \left[0, \frac{1}{n}\right]. \end{aligned} \tag{10}$$

We define the homotopy

$$(1-nq)\mathscr{L}[\phi(x,t;q) - u_0(x,t)] = \hbar q H(x,t) N[\phi(x,t;q)], \tag{11}$$

where $\mathscr{L}$ denotes the Laplace transform, $u_0(x,t)$ is an initial guess, $\hbar \neq 0$ is an auxiliary parameter, and $\phi(x,t;q)$ is an unknown function. For $q=0$ and $q=1/n$, one has

$$\phi(x,t;0)=u_0(x,t), \qquad \phi(x,t;1/n)=u(x,t). \tag{12}$$

As q increases from 0 to $1/n$, $\phi(x,t;q)$ deforms from $u_0(x,t)$ to $u(x,t)$. Expanding $\phi(x,t;q)$ in a Taylor series about $q=0$, we obtain

$$\phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\,q^m, \tag{13}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;q)}{\partial q^m}\Big|_{q=0}. \tag{14}$$

Series (13) converges at $q=1/n$ for appropriate choices of $u_0(x,t)$, n, and$\hbar$. Thus the solution of Eq. (7) is

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{15}$$

From Eq. (11), we obtain

$$\mathscr{L}\left[u_m(x,t)-k_m\,u_{m-1}(x,t)\right]=\hbar H(x,t)\,R_m(\vec{u}_{m-1}), \tag{16}$$

where

$$\vec{u}_m=\{u_0(x,t),\,u_1(x,t),\,\ldots,\,u_m(x,t)\}. \tag{17}$$

Thus,

$$u_m(x,t)=k_m u_{m-1}(x,t)+\hbar\mathscr{L}^{-1}\left[H(x,t)\,R_m(\vec{u}_{m-1})\right], \tag{18}$$

where

$$\begin{aligned}R_m(\vec{u}_{m-1})=\mathscr{L}[u_{m-1}(x,t)]-\left(1-\frac{k_m}{n}\right)\left(\sum_{k=0}^{n-1}s^{\gamma-k-1}u^{(k)}(x,0)+\frac{1}{s^\gamma}\mathscr{L}[f(x,t)]\right)\\ +\frac{1}{s^\gamma}\mathscr{L}[Ru_{m-1}(x,t)+H_{m-1}(x,t)].\end{aligned} \tag{19}$$

The coefficients k_m are given by

$$k_m=\begin{cases}0, & m\le 1,\\ n, & m>1,\end{cases} \tag{20}$$

and H_m is the homotopy polynomial defined by

$$H_m = \frac{1}{m!}\frac{\partial^m}{\partial q^m}\phi(x,t;q)\Big|_{q=0}, \qquad \phi(x,t;q) = \phi_0 + q\phi_1 + q^2\phi_2 + \cdots. \tag{21}$$

Finally, the series solution of Eq. (7) is

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t). \tag{22}$$

4. Q-HATM SOLUTION FOR FRACTIONAL DIFFUSION EQUATIONS

To demonstrate the applicability and efficiency of the proposed scheme, we consider two distinct examples of fractional diffusion equations in this section.

Example 4.1

Consider the fractional diffusion equation (1) for $\alpha = -1$, $\gamma = 0$, $\eta = 1$ [35]:

$$D_t^{\gamma} u(x,y,t) - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \left(\Gamma(2+\gamma)t - 2t^{1+\gamma}\right)e^{x+y} = 0, \tag{23}$$

with the initial condition

$$u(x,y,0) = 0. \tag{24}$$

Applying the Laplace transform to Eq. (23) and using (24), we obtain

$$\mathscr{L}[u(x,y,t)] - \frac{1}{s}\cdot 0 - \frac{1}{s^{\gamma}}\mathscr{L}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(\Gamma(2+\gamma)t - 2t^{1+\gamma}\right)e^{x+y}\right) = 0. \tag{25}$$

Thus, the nonlinear operator $N[\phi]$ is defined as

$$\begin{aligned} N[\phi(x,y,t;q)] = \mathscr{L}[\phi(x,y,t;q)] - \frac{1}{s}\cdot 0 \\ - \frac{1}{s^{\gamma}}\mathscr{L}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \left(\Gamma(2+\gamma)t - 2t^{1+\gamma}\right)e^{x+y}\right). \end{aligned} \tag{26}$$

The corresponding deformation equation becomes

$$\mathscr{L}[u_m(x,y,t) - k_m u_{m-1}(x,y,t)] = \hbar R_m(\vec{u}_{m-1}), \tag{27}$$

where

$$\begin{aligned} R_m(\vec{u}_{m-1}) = \mathscr{L}[u_{m-1}(x,y,t)] - \left(1 - \frac{k_m}{n}\right)\frac{1}{s}\cdot 0 \\ - \frac{1}{s^{\gamma}}\mathscr{L}\left(\frac{\partial^2 u_{m-1}}{\partial x^2} + \frac{\partial^2 u_{m-1}}{\partial y^2} + \left(\Gamma(2+\gamma)t - 2t^{1+\gamma}\right)e^{x+y}\right). \end{aligned} \tag{28}$$

Taking inverse Laplace transform of Eq. (27), we obtain

$$u_m(x,y,t) = k_m u_{m-1}(x,y,t) + \hbar \mathscr{L}^{-1}\{R_m(\vec{u}_{m-1})\}. \tag{29}$$

Solving iteratively, we get the following terms:

$$u_0(x,y,t) = 0,$$

$$u_1(x,y,t) = \frac{\hbar t^{\gamma}}{\Gamma(\gamma+1)}\left[-e^{x+y}\left(-2t^{1+\gamma} + t\,\Gamma(2+\gamma)\right)\right],$$

$$\begin{aligned} u_2(x,y,t) = {} & \frac{\hbar(n+\hbar)t^{\gamma}}{\Gamma(\gamma+1)}\left[-e^{x+y}\left(-2t^{1+\gamma} + t\,\Gamma(2+\gamma)\right)\right] \\ & + \hbar t^{\gamma}\frac{e^{x+y}t\left(-2t^{\gamma}\hbar + \Gamma(1+\gamma)\right)\left(2t^{\gamma} - \Gamma(2+\gamma)\right)}{\Gamma(1+2\gamma)}. \end{aligned}$$

Higher-order terms can be computed similarly. Thus, the q-HATM series solution of Eq. (23) is

$$u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t)\left(\frac{1}{n}\right)^m. \tag{30}$$

The exact solution is given by

$$u(x,y,t) = e^{x+y}t^{1+\gamma}. \tag{31}$$

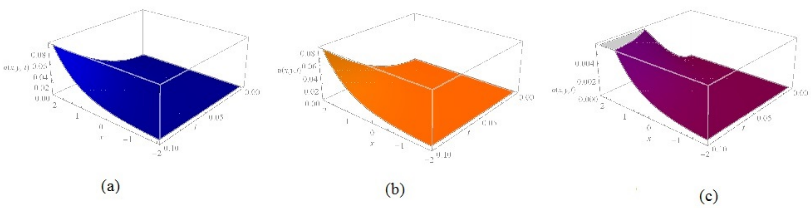

Fig. 1: (a) Nature of the q-HATM result; (b) Behaviour of the exact solution; (c) Surface of the absolute error $|u_{\text{Exact}} - u_{\text{q-HATM}}|$ at $\hbar = -1$, $n = 1$, $y = 0.1$, and $\gamma = 1$ for Example 4.1.

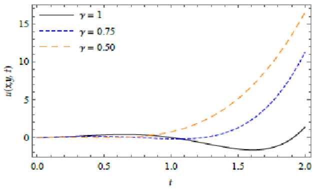

Fig. 2: Behaviour of the q-HATM solution $u(x,y,t)$ with respect to t at $\hbar=-1$, $n=1$, and $x=y=0.01$ with distinct values of γ for Example 4.1.

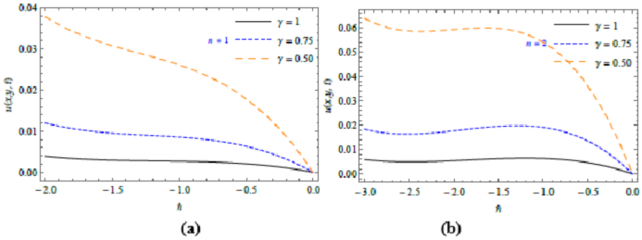

Fig. 3: $\hbar$-curves for $u(x,y,t)$ of Example 4.1 with diverse values of γ at $x=y=1$, $t=0.01$ for (a) $n=1$ and (b) $n=2$.

Example 4.2

Consider the fractional diffusion equation (1):

$$D_t^{\gamma}u(x,y,t)-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-\left(\frac{2}{\Gamma(3-\gamma)}t^{2-\gamma}+2t^2\right)\sin(x)\sin(y)=0, \tag{32}$$

with the initial condition

$$u(x,y,0)=0. \tag{33}$$

Applying the Laplace transform to Eq. (32), we get

$$\mathscr{L}[u(x,y,t)]-\frac{1}{s}\cdot 0-\frac{1}{s^{\gamma}}\mathscr{L}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\left(\frac{2}{\Gamma(3-\gamma)}t^{2-\gamma}+2t^2\right)\sin(x)\sin(y)\right]=0. \tag{34}$$

Define the nonlinear operator $N[\phi]$ as

$$\begin{aligned} N[\phi(x,y,t;q)] &= \mathscr{L}[\phi(x,y,t;q)]-\frac{1}{s}\cdot 0 \\ &\quad -\frac{1}{s^{\gamma}}\mathscr{L}\left[\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\left(\frac{2}{\Gamma(3-\gamma)}t^{2-\gamma}+2t^2\right)\sin(x)\sin(y)\right]. \end{aligned} \tag{35}$$

The deformation equation is

$$\mathscr{L}\left[u_m(x,y,t)-k_m u_{m-1}(x,y,t)\right]=\hbar R_m\left(\vec{u}_{m-1}\right), \tag{36}$$

where

$$\vec{u}_m=\{u_0(x,y,t),u_1(x,y,t),\ldots,u_m(x,y,t)\}, \tag{37}$$

and

$$\begin{aligned}R_m\left(\vec{u}_{m-1}\right)=\mathscr{L}[u_{m-1}(x,y,t)]-\left(1-\frac{k_m}{n}\right)\frac{1}{s}\cdot 0\\ -\frac{1}{s^\gamma}\mathscr{L}\left[\frac{\partial^2 u_{m-1}}{\partial x^2}+\frac{\partial^2 u_{m-1}}{\partial y^2}+\left(\frac{2}{\Gamma(3-\gamma)}t^{2-\gamma}+2t^2\right)\sin(x)\sin(y)\right].\end{aligned} \tag{38}$$

Applying the inverse Laplace transform gives

$$u_m(x,y,t)=k_m u_{m-1}(x,y,t)+\hbar\mathscr{L}^{-1}\left[R_m\left(\vec{u}_{m-1}\right)\right]. \tag{39}$$

The first few iterative terms are

$$u_0(x,y,t)=0,$$

$$u_1(x,y,t)=\frac{\hbar t^\gamma}{\Gamma(\gamma+1)}\left[2t^2\left(-1-\frac{t^{-\gamma}}{\Gamma(3-\gamma)}\right)\sin(x)\sin(y)\right],$$

$$\begin{aligned}u_2(x,y,t)=\frac{\hbar(n+\hbar)t^\gamma}{\Gamma(\gamma+1)}\left[2t^2\left(-1-\frac{t^{-\gamma}}{\Gamma(3-\gamma)}\right)\sin(x)\sin(y)\right]\\ -\frac{\hbar t^\gamma 2t^{2-\gamma}\left(1+t^\gamma\Gamma(3-\gamma)\right)\left(2t^\gamma\hbar+\Gamma(1+\gamma)\right)\sin(x)\sin(y)}{\Gamma(3-\gamma)\Gamma(1+2\gamma)}.\end{aligned}$$

Higher-order terms can be computed similarly. Thus, the q-HATM series solution of Eq. (32) is

$$u(x,y,t)=u_0(x,y,t)+\sum_{m=1}^{\infty}u_m(x,y,t)\left(\frac{1}{n}\right)^m. \tag{40}$$

The exact solution is

$$u(x,y,t)=t^2\sin(x)\sin(y). \tag{41}$$

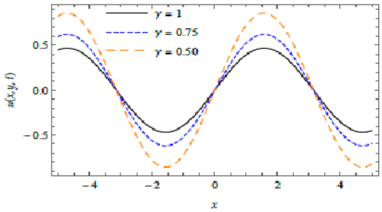

Fig. 4: Behaviour of the q-HATM solution $u(x,y,t)$ with respect to t at $\hbar = -1$, $n = 1$, $t = 1$, $y = 0.1$ with distinct values of γ for Example 4.2.

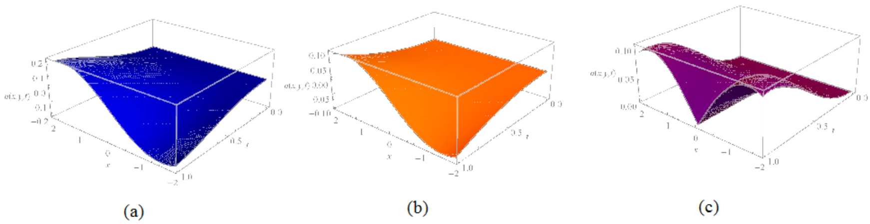

Fig. 5: (a) Nature of the q-HATM result; (b) Behaviour of the exact solution; (c) Surface of the absolute error $|u_{\text{Exact}} - u_{\text{q-HATM}}|$ at $\hbar = -1$, $n = 1$, $y = 0.1$, and $\gamma = 1$ for Example 4.2.

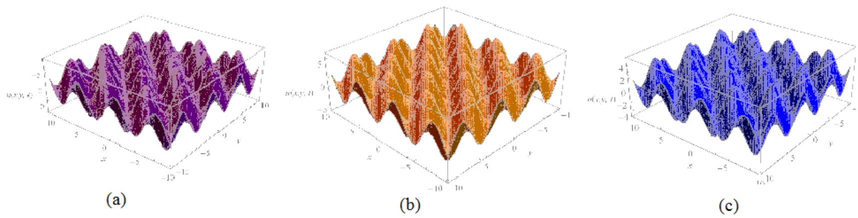

Fig. 6: Response of the q-HATM solution at $\hbar = -1$, $n = 1$, and $t = 1$ with (a) $\gamma = 0.50$, (b) $\gamma = 0.75$, and (c) $\gamma = 1$ for Example 4.2.

5. CONVERGENCE ANALYSIS

THEOREM 5.1. Suppose $u_n(x,t)$ and $u(x,t)$ are defined in the inner product space $(B[0,T], \|\cdot\|)$. Let $\{S_n\}$ denote the nth partial sum of the infinite series given in Eq. (23). Then

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0. \tag{42}$$

PROOF Let $\{S_n\}$ be a sequence of partial sums of Eq. (28). We need to show that $\{S_n\}$ is a Cauchy sequence in $(B[0,T], \|\cdot\|)$. Consider

$$\|S_{n+1}(x,t)-S_n(x,t)\| = \|u_{n+1}(x,t)\| \leq \lambda \|u_n(x,t)\| \leq \lambda^2 \|u_{n-1}(x,t)\| \leq \cdots \leq \lambda^{n+1} \|u_0(x,t)\|. \tag{43}$$

For every $n,m \in \mathbb{N}$ with $m \leq n$, we have

$$\begin{aligned} \|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_{m+1} - S_m)\| \\ &\leq \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \cdots + \|S_{m+1} - S_m\| \\ &\leq (\lambda^n + \lambda^{n-1} + \cdots + \lambda^{m+1})\|u_0\| \\ &\leq \lambda^{m+1}(\lambda^{n-m-1} + \lambda^{n-m-2} + \cdots + 1)\|u_0\| \\ &\leq \lambda^{m+1} \frac{1-\lambda^{n-m}}{1-\lambda} \|u_0\|. \end{aligned} \tag{44}$$

Since $0 < \lambda < 1$, it follows that

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0. \tag{45}$$

Hence, $\{S_n\}$ is a Cauchy sequence, which completes the proof.

THEOREM 5.2. The maximum absolute error appearing in the solution (30) is given by

$$\left\| u(x,t) - \sum_{n=0}^{M} u_n(x,t) \right\| \leq \frac{\lambda^{M+1}}{1-\lambda} \|u_0(x,t)\|. \tag{46}$$

PROOF Using Eq. (44), we have

$$\|u(x,t) - S_n\| = \lambda^{m+1} \frac{1-\lambda^{n-m}}{1-\lambda} \|u_0(x,t)\|. \tag{47}$$

Since $0 < \lambda < 1$, it follows that $1 - \lambda^{n-m} < 1$. Therefore,

$$\left\| u(x,t) - \sum_{n=0}^{M} u_n(x,t) \right\| \leq \frac{\lambda^{M+1}}{1-\lambda} \|u_0(x,t)\|. \tag{48}$$

This completes the proof.

6. NUMERICAL RESULTS AND DISCUSSION

In the present section, the behaviour of the obtained results and the analytical solution along with the absolute error, is illustrated in Figure 1. The response of the attained solution for Eq. (24) with distinct values of γ is presented in Figure 2. From these plots, we can notice that for different γ, the solution exhibits stimulating changes in nature, which helps to understand the consequences of the model more clearly.

Similarly, for Example 4.2, the behaviour is presented in Figures 4 and 6 for different fractional orders. This confirms that generalizing the model with a fractional order can offer additional properties and can assist in examining other corresponding characteristics of the problem. The surface of the obtained q-HATM results compared with the exact solution, along with the absolute error, is illustrated for Example 4.2 in Figure 5.

The $\hbar$-curves with distinct γ and n for the fractional diffusion equation are presented in Figure 3. These plots can help to adjust the convergence domain of the method.

Overall, these captured plots can assist in studying more classes of models that exemplify real-world phenomena. Specifically, from Figure 6, we can observe the effect of different fractional orders. Such plots are widely used for the analysis of biological models with diffusion mechanisms. These types of studies can help researchers explore fractional operators and efficient numerical schemes.

7. CONCLUSION

In this study, we effectively employed the q-HATM to find the solution of the two-dimensional fractional diffusion equation. The q-HATM offers parameters $\hbar$ and n, which can assist in controlling the convergence of the obtained solution.

From the numerical results, it is observed that q-HATM is efficient in terms of evaluation time while maintaining high accuracy compared to existing numerical methods. Moreover, the method encompasses solutions of many classical approaches as special cases. The employed scheme allows precise control and manipulation of the obtained solution, which rapidly converges to the analytical results within a sufficiently small domain.

Hence, it can be concluded that the proposed method is highly effective and systematic for studying fractional-order mathematical models in a rigorous and reliable manner.

REFERENCES

[1] LIOUVILLE, J. (1832). Memoire sur quelques questions de geometrie et de mecanique, et sur un nouveau genre de calcul pour resoudre ces questions. *J. Ecole Polytech.*, 13, pp. 1–69.

[2] RIEMANN, G.F.B. (1896). Versuche einer allgemeinen Auffassung der Integration und Differentiation. *Gesammelte Mathematische Werke*, Leipzig.

[3] CAPUTO, M. (1969). Elasticita e Dissipazione. Zanichelli, Bologna.

[4] MILLER, K.S. and ROSS, B. (1993). An introduction to fractional calculus and fractional differential equations. A Wiley, New York.

[5] PODLUBNY, I. (1999). Fractional Differential Equations. Academic Press, New York.

[6] KILBAS, A.A., SRIVASTAVA, H.M., and TRUJILLO, J.J. (2006). Theory and applications of fractional differential equations. Elsevier, Amsterdam.

[7] GAO, W., BASKONUS, H.M., and SHI, L. (2020). New investigation of Bats-Hosts-Reservoir-People coronavirus model and apply to 2019-nCoV system. *Adv. Diff. Equ.*, 391, pp. 1–11.

[8] GAO, W., et al. (2020). Novel dynamic structures of 2019-nCoV with nonlocal operator via powerful computational technique. *Biology*, 9(5). DOI: 10.3390/biology9050107.

[9] KIRAN, M.S., et al. (2020). A mathematical analysis of ongoing outbreak COVID-19 in India through nonsingular derivative. *Numer. Meth. Partial Differ. Equ.*. DOI: 10.1002/num.22579.

[10] PRAKASHA, D.G. and VEERESHA, P. (2020). Analysis of Lakes pollution model with Mittag-Leffler kernel. *J. Ocean Eng. Sci.*. DOI: 10.1016/j.joes.2020.01.004.

[11] VEERESHA, P., PRAKASHA, D.G., SINGH, J., KHAN, I., and KUMAR, D. (2020). Analytical approach for fractional extended Fisher–Kolmogorov equation with Mittag-Leffler kernel. *Adv. Differ. Equ.*, 174. DOI: 10.1186/s13662-020-02617-w.

[12] VEERESHA, P., PRAKASHA, D.G., BASKONUS, H.M., and YEL, G. (2020). An efficient analytical approach for fractional Lakshmanan–Porsezian–Daniel model. *Math. Meth. Appl. Sci.*, 43(7), pp. 4136–4155.

[13] GAO, W., et al. (2020). New numerical results for the time-fractional Phi-four equation using a novel analytical approach. *Symmetry*, 12(3). DOI: 10.3390/sym12030478.

[14] AKINYEMI, L. (2020). A fractional analysis of Noyes–Field model for the nonlinear Belousov–Zhabotinsky reaction. *Comput. Appl. Math.*, 39, pp. 1–34.

[15] AKINYEMI, L. and HUSEEN, S.N. (2020). A powerful approach to study the new modified coupled Korteweg-de Vries system. *Math. Comput. Simul.*, 177, pp. 556–567.

[16] VEERESHA, P. and PRAKASHA, D.G. (2020). Solution for fractional generalized Zakharov equations with Mittag-Leffler function. *Results Eng.*, 5, pp. 1–12. DOI: 10.1016/j.rineng.2019.100085.

[17] ESEN, A., SULAIMAN, T.A., BULUT, H., and BASKONUS, H.M. (2018). Optical solitons and other solutions to the conformable space–time fractional Fokas–Lenells equation. *Optik*, 167, pp. 150–156.

[18] VEERESHA, P., PRAKASHA, D.G., and BASKONUS, H.M. (2019). New numerical surfaces to the mathematical model of cancer chemotherapy effect in Caputo fractional derivatives. *Chaos*, 29, 013119. DOI: 10.1063/1.5074099.

[19] VEERESHA, P. and PRAKASHA, D.G. (2020). Novel approach for modified forms of Camassa–Holm and Degasperis–Procesi equations using fractional operator. *Commun. Theor. Phys.*, 72(10). DOI: 10.1088/1572-9494/aba24b.

[20] GAO, W., et al. (2020). A new study of unreported cases of 2019-nCOV epidemic outbreaks. *Chaos Solitons Fractals*, 138. DOI: 10.1016/j.chaos.2020.109929.

[21] VEERESHA, P., PRAKASHA, D.G., and BALEANU, D. (2020). An efficient technique for fractional coupled system arisen in magneto-thermoelasticity with rotation using Mittag-Leffler kernel. *J. Comput. Nonlinear Dynam.*, 16(1). DOI: 10.1115/1.4048577.

[22] VEERESHA, P., PRAKASHA, D.G., and KUMAR, D. (2020). Fractional SIR epidemic model of childhood disease with Mittag-Leffler memory. *Fractional Calculus in Medical and Health Science*, pp. 229–248.

[23] AL-GHAFRI, K.S. and REZAZADEH, H. (2019). Solitons and other solutions of (3+1)-dimensional space–time fractional modified KdV–Zakharov–Kuznetsov equation. *Appl. Math. Nonlinear Sci.*, 4(2), pp. 289–304.

[24] AGHILI, A. (2020). Complete solution for the time fractional diffusion problem with mixed boundary conditions by operational method. *Appl. Math. Nonlinear Sci.*, pp. 1–12. DOI: 10.2478/amns.2020.2.00002.

[25] BALASIM, A. and ALI, N.H.M. (2015). A rotated Crank-Nicolson iterative method for the solution of two-dimensional time-fractional diffusion equation. *Indian Journal of Science and Technology*, 8(32), pp. 1–7.

[26] LIAO, S.J. (1997). Homotopy analysis method and its applications in mathematics. *Journal of Basic Science and Engineering*, 5(2), pp. 111–125.

[27] SINGH, J., KUMAR, D., and SWROOP, R. (2016). Numerical solution of time- and space-fractional coupled Burgers' equations via homotopy algorithm. *Alexandria Eng. J.*, 55(2), pp. 1753–1763.

[28] VEERESHA, P., PRAKASHA, D.G., and BALEANU, D. (2019). Analysis of fractional Swift-Hohenberg equation using a novel computational technique. *Math. Meth. Appl. Sci.*, 43(4), pp. 1970–1987.

[29] SRIVASTAVA, H.M., KUMAR, D., and SINGH, J. (2017). An efficient analytical technique for fractional model of vibration equation. *Appl. Math. Model.*, 45, pp. 192–204.

[30] VEERESHA, P., PRAKASHA, D.G., SINGH, J., KUMAR, D., and BALEANU, D. (2020). Fractional Klein-Gordon-Schrödinger equations with Mittag-Leffler memory. *Chinese J. Phy.*, 68, pp. 65–78.

[31] BULUT, H., KUMAR, D., SINGH, J., SWROOP, R., and BASKONUS, H.M. (2018). Analytic study for a fractional model of HIV infection of CD4+T lymphocyte cells. *Math. Nat. Sci.*, 2(1), pp. 33–43.

[32] KUMAR, D., AGARWAL, R.P., and SINGH, J. (2018). A modified numerical scheme and convergence analysis for fractional model of Lienard's equation. *J. Comput. Appl. Math.*, 399, pp. 405–413.

[33] VEERESHA, P., PRAKASHA, D.G., and BASKONUS, H.M. (2019). Novel simulations to the time-fractional Fisher's equation. *Math. Sci.*, 13(1), pp. 33–42.

[34] GAO, W., et al. (2020). Iterative method applied to the fractional nonlinear systems arising in thermoelasticity with Mittag-Leffler kernel. *Fractals*, 28(8), 2040040. DOI: 10.1142/S0218348X2040040X.

[35] VEERESHA, P., PRAKASHA, D.G. (2020). A reliable analytical technique for fractional Caudrey-Dodd-Gibbon equation with Mittag-Leffler kernel. *Nonlinear Eng.*, 9(1), pp. 319–328.

[36] SINGH, J., SECER, A., SWROOP, R., and KUMAR, D. (2018). A reliable analytical approach for a fractional model of advection-dispersion equation. *Nonlinear Eng.*, 8(1), pp. 107–116. DOI: 10.1515/nleng-2018-0027.

[37] VEERESHA, P., PRAKASHA, D.G., and HAMMOUCH, Z. (2020). An efficient approach for the model of thrombin receptor activation mechanism with Mittag-Leffler function. *Nonlinear Analysis: Problems, Applications and Computational Methods*, pp. 44–60. DOI: 10.1007/978-3-030-62299-2.

[38] LANGELANDS, T.A.M. and HENRY, B.I. (2005). The accuracy and stability of an implicit solution method for the fractional diffusion equation. *J. Comput. Phys.*, 205, pp. 719–736.

[39] TADJERAN, C. and MEERSCHAERT, M.M. (2007). A second-order accurate numerical method for the two dimensional fractional diffusion equation. *J. Comput. Phys.*, 220, pp. 813–823.

[40] CHEN, C.M., LIU, F., TURNER, I., and ANH, V. (2007). A Fourier method for the fractional diffusion equation describing sub-diffusion. *J. Comput. Phys.*, 227(2), pp. 886–897.

[41] ZHANG, Y.N. and SUN, Z.Z. (2011). Alternating direction implicit schemes for the two-dimensional fractional sub-diffusion equation. *J. Comput. Phys.*, 230(24), pp. 8713–8728.

[42] CUI, M. (2013). Convergence analysis of high-order compact alternating direction implicit schemes for the two-dimensional time fractional diffusion equation. *Numerical Algorithms*, 62(3), pp. 383–409.

[43] WANG, H. and WANG, K. (2011). An O(Nlog^2 N) alternating-direction finite difference method for two dimensional fractional diffusion equations. *J. Comput. Phys.*, 230, pp. 7830–7839.

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