A new method to compute the determinantal polynomial coefficients of a matrix (Symmetric Normalized Kirchhoff matrix) of a complete graph

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Abstract

In this article, we have calculated the coefficients of the characteristic (determinantal) polynomial for a complete graph by considering the vertex number and using the eigenvalues of the normalized matrix. Additionally, we have obtained upper and lower bounds for the normalized eigenvalue of a complete graph. Furthermore, we present algorithms for these calculations.

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1. INTRODUCTION

For the terminologies and notations of graph theory, readers should consult [6], where only simple and finite graphs are taken into consideration.

Various methods can be employed to compute the graph's determinantal polynomial ([13], [14]). According to [14], the Faddeev-Le Verrier-Frame method ([12],[15],[16]) stands out as the most efficient approach for this task. Balasubramanian ([17],[18]) commonly utilized a computerized adaptation of this technique's variant.

An example of a graph G's determinantal polynomial coefficients is

$P_g(\rho) = \rho^z + a_1\rho^{(z-1)} + a_2\rho^{(z-2)} + \cdots + a_z$ are directly related to its structure. For illustration, the number of edges of G is -a_2 and the number of triangles of G is $-a_3 2$ in [20] following theorem determines the coefficients of the determinantal polynomial of matrix of a graph. According to Kelmans [1], *Symmetric Kirchhoff* matrix coefficients can be represented in terms of

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subtree structures of G.

THEOREM 1. Let F be a spanning forest of G with components T_i, $i = 1, 2, ..., k$, having n_i vertices each, and let $\gamma(F) = \prod_{i=1}^{k} n_i$. The Laplacian coefficient q_{n-k} of a graph G is given by

$$(-1)^{n-k} q_n - k = \sum_{F \in \mathscr{F}_k} \gamma(F),$$

where $\mathscr{F}_k$ is the set of all spanning forests of G with exactly k components.

In [9] using the trace of the *Kirchhoff matrix*, Ivailo M. Mladenov et al. have provided a very elegant approach to identify the coefficients of a determinantal polynomial. In [3] Samuel Jurkiewicz et al. has determined the Laplacian (Kirchhoff) coefficients of a determinantal polynomial. We provide a new technique to discover Laplacian (Kirchhoff) coefficients of a determinantal polynomial using the trace of a Laplacian matrix of complete graph G, based on the aforementioned theorem and using Faddeev-LeVerrier algorithm [12].

The first instance of the Normalized Laplacian matrix (Kirchhoff matrix) was presented by F.R.K. Chung [21]. Grossman[10] examined the lower bounds of the normalised laplacian, and Chenet. al. in [8] established its properties. Also, they went into great detail regarding the normalised laplacian and eigenvalues bounds.

The entries of *Symmetric Normalized Kirchhoff* matrix of a graph having vertices i and j, is given by

$$\mathsf{L}_{ij} = \begin{cases} 1, \text{ if } j = i \textit{ and } d_j \neq 0 \\ -\dfrac{1}{\sqrt{d_j d_i}}, \text{ if } i \text{ and } j \text{ are adjacent} \\ 0, \quad \text{otherwise.} \end{cases}$$

Let $\rho_1, \rho_2, \rho_3, \ldots \rho_z$ be the normalized eigenvalues of $\mathsf{L}(G)$, with z vertices. Also $\mathsf{L}(G) = H^{-1/2} L(G) H^{-1/2}$, where $d_i and d_j$ are the degree of vertices i and j respectively, H is the diagonal matrix and $L(G)$ is the Laplacian (Kirchhoff) matrix of G.

We recall the following results on $\mathsf{L}(G)$. [11]

1) The eigenvalues belonging to $\mathsf{L}(G)$ are situated within the range of [0,2].
2) The smallest eigenvalue of the Laplacian matrix of a graph G is 0 if and

only if the graph G is connected.

3) 2 is the eigenvalue of $\mathsf{L}(G)$ if and only if a connected component of G is bipartite and nontrivial.

We provide more evidence for the above results.

THEOREM 2. Suppose G is a complete graph. The smallest eigenvalue, denoted as ρ, of the *Symmetric Normalized Kirchhoff matrix* $\mathsf{L}(G)$ equals zero.

PROOF. If G is complete graph then $\rho(\mathsf{L}(G)) = \rho(H^{-1/2}\ L(G)\ H^{-\frac{1}{2}})$

$$=\rho(H^{-\frac{1}{2}}\ (H(G) - A(G))\ H^{-\frac{1}{2}})$$

$$=\rho(H^{-\frac{1}{2}}\ (H(G))H^{-\frac{1}{2}} - H^{-\frac{1}{2}}(A(G))\ H^{-\frac{1}{2}})$$

$$=\rho(H^{-\frac{1}{2}}\ (H(G))H^{-\frac{1}{2}}) - \rho(H^{-\frac{1}{2}}(A(G))\ H^{-\frac{1}{2}})$$

$$= 1 + (\text{-}1)$$

$$= 0.$$

THEOREM 3. Let G represent a complete graph with z vertices, $\rho_k = \frac{z}{z-1}$, for $1 < k \leq z$.

PROOF. $\rho_k(\mathsf{L}(G)) = \rho_k(H^{-\frac{1}{2}}(L(G))\ H^{-\frac{1}{2}})$

$$=\rho_k(H^{-\frac{1}{2}}\ (H(G) - A(G))\ H^{-\frac{1}{2}})$$

$$=\rho_k(H^{-\frac{1}{2}}\ (H(G))H^{-\frac{1}{2}} - H^{-\frac{1}{2}}(A(G))\ H^{-\frac{1}{2}})$$

$$=\rho_k(H^{-\frac{1}{2}}\ (H(G))H^{-\frac{1}{2}}) - \rho_k(H^{-\frac{1}{2}}(A(G))\ H^{-\frac{1}{2}})$$

$$=\rho_k(H^{-\frac{1}{2}}\ (H(G))H^{-\frac{1}{2}}) + \rho_k(H^{-\frac{1}{2}}(-A(G))\ H^{-\frac{1}{2}})$$

$$= 1 + \frac{1}{z-1}$$

$$= \frac{z}{z-1}.$$

Using the above theorem and motivated by the Faddeev LeVerrier algorithm [12], [9] and [19], we propose a new method to obtain the normalized Laplacian coefficients of a determinantal polynomial by using the trace of a *symmetric normalized Kirchhoff matrix* of the entire graph (G).

PROPOSITION 4. Let G represent a complete graph and z be the number of vertices of a graph G then,

$$tr(\mathsf{L}^{\ell}) = \frac{z^{\ell}}{(z-1)^{(\ell-1)}}$$

.

PROOF. Here, proof is by induction.

$tr(\mathsf{L}) = \rho_1 + \rho_2 + \rho_3 + ... + \rho_{z-1} + \rho_z$

$$= (z-1)\frac{z}{(z-1)}$$

We get

$$tr(\mathsf{L}^2) = (z-1)\frac{z^2}{(z-1)^2}$$

Similarly for an integer k,

$$tr(\mathsf{L}^k) = (z-1)\frac{z^k}{(z-1)^k}$$

$$tr(\mathsf{L}^{k+1}) = \rho_1^{k+1} + \rho_2^{k+1} + \rho_3^{k+1} + ... + \rho_{z-1}^{k+1}.$$

$$= z^{k+1}/(z-1)^{k+1} + z^{k+1}/(z-1)^{k+1} + \cdots + z^{k+1}/(z-1)^{k+1}$$

$$= (z-1)z^{k+1}/(z-1)^{k+1}$$

Hence by induction, $tr(\mathsf{L}^{\ell}) = \dfrac{z^{\ell}}{(z-1)^{(\ell-1)}}$.

Example:

If we put $\ell = 1, 2, 3, 4$ in the above expression we have the following.

$tr(\mathsf{L}) = z$

$tr(\mathsf{L}^2) = z^2/(z-1).$

$tr(\mathsf{L}^3) = z^3/(z-1)^2.$

$tr(\mathsf{L}^4) = z^4/(z-1)^3.$

Coefficients of determinantal polynomial of a *Symmetric Normalized Kirchhoff* matrix of complete graph G, a_1, a_2, a_3, a_4 are calculated as follows.

$$a_1 = -tr(\mathsf{L}) = -z.$$

$$a_2 = \frac{-1}{2} tr(B_1\mathsf{L}). \quad \text{where } B_1 = \mathsf{L} + a_1 I$$

$$= \tfrac{-1}{2}(tr(\mathsf{L}^2) - z tr(\mathsf{L}))$$

$$= \frac{-1}{2}\left(\frac{2z^2 - z^3}{z-1}\right).$$

$$a_3 = \tfrac{-1}{3} tr(B_2\mathsf{L}) \quad \text{where } B_2 = B_1\mathsf{L} + a_2 I$$

$$= \tfrac{-1}{3} tr(B_1\mathsf{L}^2 + a_2\mathsf{L})$$

$$= \tfrac{-1}{3}(tr(\mathsf{L}^3) + a_1 tr(\mathsf{L}^2) + a_2 tr(\mathsf{L}))$$

$$= \frac{-1}{3}\left(z^3/(z-1)^2 - z^3/(z-1) - \frac{1}{2}\left((2z^3 - z^4)/(z-1)\right)\right)$$

$$= \frac{-1}{6}\left(\frac{z^5 - 5z^4 + 6z^3}{(z-1)^2}\right).$$

$$a_4 = \tfrac{-1}{4} tr(B_3\mathsf{L}) \quad \text{where } B_3 = B_2\mathsf{L} + a_3 I$$

$$= \tfrac{-1}{4} tr(B_2\mathsf{L}^2 + a_3\mathsf{L})$$

$$= \tfrac{-1}{4} tr((B_1\mathsf{L} + a_2)\mathsf{L}^2 + a_3\mathsf{L})$$

$$= \tfrac{-1}{4} tr((\mathsf{L} + a_1)\mathsf{L}^3 + a_2\mathsf{L}^2 + a_3\mathsf{L})$$

$$= \tfrac{-1}{4} (tr(\mathsf{L}^4) + a_1 tr(\mathsf{L}^3) + a_2 tr(\mathsf{L}^2) + a_3 tr(\mathsf{L}))$$

$$= \frac{-1}{4}\left(z^4/(z-1)^3 - z^4/(z-1)^2 - \frac{1}{2}\left(2z^4 - z^5/(z-1)^2\right) - \right.$$
$$\left. -\frac{1}{6}\left((z^6 - 5z^5 + 6z^4)/(z-1)^2\right)\right)$$

$$= \frac{-1}{24}\left(\frac{24z^4 - 26z^5 + 9z^6 - z^7}{(z-1)^3}\right).$$

Theorem 5. The determinantal polynomial of *Symmetric Normalized Kirchhoff* matrix of a complete graph G with $a_0 = 1$ is $\zeta(G,t) = t^z + a_1 t^{z-1} + ... + a_{z-1}t + a_z$ then

$$a_\Psi = \frac{-1}{\Psi} \sum_{p=0}^{\Psi-1} a_p tr(\mathsf{L}^{\Psi-p})$$

.

where, a_Ψ is the coefficient of determinantal polynomial and $\Psi \neq 0$.

Proof. Proof is by Leverriers Algorithm

Corollary 6. The *Symmetric Normalized Kirchhoff* matrix having determinantal polynomial of a complete graph G is $\zeta(G,t) = a_0 t^z + a_1 t^{z-1} + ... + a_{z-1}t + a_z$ with $a_0 = 1$ then,

$$a_\Psi = \frac{-1}{\Psi} \sum_{p=0}^{\Psi-1} a_p \left(\frac{z^\Psi}{(z-1)^{\Psi-p-1}}\right)$$

where, $\Psi \neq 0$.

PROOF. Proof is from proposition 4 and Theorem 5.

PROPOSITION 7. Let G stands for a complete graph, ρ_k, $(1 < k \leq z)$ is the Normalized eigenvalue of $\mathsf{L}(G)$, then $1 \leqslant \rho_k < 2$

PROOF. Since

$$z^{-1} \sum \rho_k^2 - \left(z^{-1} \sum \rho_k \right)^2 \geqslant 0$$

and

$$\sum \rho_k = z$$

$$z^{-1} \sum \rho_k^2 \geqslant 1$$

$$z/(z-1) \ \geqslant \ 1$$

$$\rho_k \geqslant 1.$$

By Cauchy Schwartz inequality,

$$(\rho_1 + \rho_2 + \rho_3 + ... + \rho_n)^2 \leq z \ \ (\rho_1^2 + \rho_2^2 + \rho_3^2 + ... + \rho_z^2)$$

$$\rho_k + z(z-2)/(z-1)^2 \leq z^3(z-1)/(z-1)^2$$

$$\rho_k \leq \sqrt{z^3/(z-1)} - [z(z-2)/(z-1)]$$

.

$$\rho_k \leq \sqrt{\rho_k z^2} - \rho_k(z-2)(z-1)$$

since

$$\rho_k \geq 1$$

$$\rho_k < 2$$

.

2. ALGORITHM

2.1. Algorithm to compute Characteristic polynomial of complete graphs using Normalized Laplacian Matrix.

Input: z (number of vertices of a complete graph)

Output: coefficients of Characteristic polynomial.

1. Initialize the coefficients array coefficients = 1
   $a_0 = 1$

2. For Ψ from 1 to z:
   2.1 Initialize sum-term to 0:
   sum-term = 0

   2.2 For p from 0 to $\Psi - 1$:
   2.2.1 Compute the term:
   term = $(z^{\Psi})/(z-1)^{\Psi-p-1}$.

   2.2.2 Update sum-term:
   sum-term = sum-term + coefficients[p] * term.

   2.3 Compute the coefficient a_Ψ:
   $a_\Psi = (-1/\Psi) * sum - term$

   2.4 Append a_Ψ to the coefficients array:
   Append a_Ψ to coefficients

3. Output the coefficients array:
   Output coefficients

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