

A new two-parametric weighted generalized inaccuracy measure

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Abstract

In this article, we present a novel approach to measuring inaccuracy, introducing a two-parametric weighted generalized inaccuracy measure of order α and type β , along with its residual version. Our proposed measure depends on the proportional hazard rate model (PHRM) to uniquely determine the survival function, and we have derived a characterization result for this measure. Through our analysis under the PHRM framework, we have studied various properties of the proposed measure and their interrelationships.

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1. INTRODUCTION

In recent years, there has been a growing interest among researchers in the field of uncertainty measurement. One of the foundational concepts in information theory is Shannon's entropy, which was introduced by famous researcher (Claude Shannon 1948). This measure has gained universal recognition and is often referred to as Shannon entropy. Additionally, other researchers have developed various generalizations of entropy, including Renyi's entropy (Renyi 1961) and (Kapur's entropy et al. 1968).

If X be a continuous non-negative random variable having probability density function $f(x)$. Then the Shannon's entropy is defined as

$$S(x) = - \int_0^{\infty} f(x) \log(f(x)) dx \quad (1)$$

Furthermore, it can be stated as

$$S(x) = -E[\log(f(x))] \quad (2)$$

Kerridge inaccuracy is a measure developed by (Kerridge 1961) that is commonly used in experimental research to quantify the degree of error in results. This measure can be used to assess the accuracy of the data collected during an experiment. In this paper we have developed a new two parametric generalized inaccuracy measure based on Kerridge Inaccuracy measure Suppose X and Y are two continuous, non-negative random variables having density functions $f(x)$ and $g(x)$, respectively. Then Kerridge's inaccuracy measure is defined as

$$I(X, Y) = - \int_0^{\infty} f(x) \log(g(x)) dx \quad (3)$$

This can also be written as

$$I(X, Y) = -E[\log(g(x))] \quad (4)$$

REMARKS

1. If $g(x) = f(x)$, then the Kerridge inaccuracy measure becomes the Shannon entropy.

The measurement of information, as defined above, is primarily concerned with the analysis of probability density functions of lifetime distributions. This approach is independent of shift work and does not take into account the values of random variables or preferences. However, in some real-life situations, it may be necessary to consider the values of these variables, which are often referred to as weights or preferences. The selection of these weights or preferences may depend on the specific goals of the researcher, as highlighted in (Rao 1965). Therefore, when analysing information in real-world scenarios, it may be necessary to take into account the values of these variables and how they relate to the overall goals of the analysis. The Shannon entropy measure, which is commonly used in information theory and statistics, is defined as the sum of the probabilities of all possible outcomes multiplied by the logarithm of those probabilities. This measure is shift-invariant, which means that it is not affected by changes in the origin of the probability distribution. However, in some situations, it may be necessary to take into account the shift-dependency of the probabilities. For example, in signal processing, the probability distribution of a signal may change depending on the time or frequency domain, and the Shannon entropy measure may not capture this shift-dependency. To deal with the problem, (Belis and Guisasu 1968) introduced the concept of weighted entropy. The idea behind this measure is to assign different weights to each possible outcome based on its shift-dependent probability distribution. Specifically, the weighted entropy is defined as:

$$S^w(X) = - \int_0^{\infty} x f(x) \log(f(x)) dx \quad (5)$$

REMARKS

1. If $x = 1$, then it becomes simply Shannon's entropy (Shannon 1948)

Based on (3), the dynamic version of the inaccuracy measure is given by

$$I^w(X, Y) = - \int_0^{\infty} x f(x) \log(g(x)) dx \quad (6)$$

REMARKS

1. If $x = 1$, then it becomes simply (Kerridge 1961)

Where the x in the integral denotes the weighted function of the simple events. x is said to be a size-biased random variable when the weight function depends on the length of units. The measures (1) and (2) cannot be applied to a system that exists in units of time (t). The measure of uncertainty for the residual lifetime of a random variable as $X_t = [X - t | X > t]$ introduced by (Ebrahimi 2010), known as the measure of residual entropy, is given by

$$S(X; t) = - \int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx \quad (7)$$

Similarly, the residual inaccuracy is defined by Taneja et al. (Taneja 2009)

$$I(X, Y; t) = - \int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx \quad (8)$$

The weighted form of (5) was introduced by (Di Crescenzo and Longobardi 2007) and defined as

$$S^w(X; t) = - \int_t^{\infty} x \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx \quad (9)$$

The weighted version of (6) is defined by Kumar et al. (2010) as

$$I^w(X, Y; t) = - \int_t^{\infty} x \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx \quad (10)$$

Recently, researchers have developed a new length-biased measure of inaccuracy. For more information in this direction, see (Kumar and Taneja 2009), (Kundu et al. 2016), (Cali et al. 2020).

Various researchers have generalized Kerridge's measure of inaccuracy in different ways in the same direction. (Nath 1968) developed a new generalized inaccuracy

measure of order α is defined as

$$I_{\alpha}(X, Y) = \frac{1}{(1-\alpha)} \log \int_0^{\infty} f(x)g(x)^{\alpha-1} dx, \quad \alpha \neq 1, \alpha > 0 \quad (11)$$

A new two-parameter inaccuracy measure of order α and type β that we developed as

$$I_{\alpha, \beta}(X, Y) = \frac{1}{\beta(\beta-\alpha)} \log \int_0^{\infty} f(x)g(x)^{\alpha-\beta} dx, \quad \alpha \neq \beta, \alpha > 0, \beta > 0 \quad (12)$$

REMARKS

1. If $\lim_{\alpha \rightarrow 1} I_{\alpha}(X, Y) = - \int_0^{\infty} f(x) \log g(x) dx$ which is the (Kerridge inaccuracy 1961).
2. If $\beta = 1$, then it simply becomes Nath inaccuracy measure measure of inaccuracy of order α
3. If $g(x) = f(x)$, $\beta = 1$, then it becomes Renyi entropy of order α
4. If $g(x) = f(x)$, $\beta = 1$, $\alpha \rightarrow 1$ then it becomes Shannon's entropy.

The residual version of our proposed inaccuracy measure is defined as

$$I_{\alpha, \beta}(X, Y; t) = \frac{1}{\beta(\beta-\alpha)} \log \left[\int_t^{\infty} \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \right] \quad \alpha \neq \beta, \alpha > 0, \beta > 0 \quad (13)$$

REMARKS

1. If $\beta = 1$ then it became generalized inaccuracy of order α
2. If $t=0$ it simply becomes (9)

EXAMPLE 1.1. Suppose X and Y are two random variables having density functions $f(x)$ and $g(x)$ defined as

$$1. f(x) = 1; \quad 0 < x < 1, \quad g(x) = 2x; \quad 0 < x < 1$$

Upon simplification, we obtain

$$I_{\alpha}(X, Y) = \frac{1}{1-\alpha} \log \frac{2^{(\alpha-1)}}{\alpha} \quad (14)$$

and

$$I_{\alpha,\beta}(X, Y) = \frac{1}{\beta(\beta-\alpha)} \log \frac{2^{(\alpha-\beta)}}{\alpha-\beta+1} \quad (15)$$

Table I: Comparison between known measure and proposed measure

α	β	$\zeta_{\alpha}(X, Y)$	$\zeta_{\alpha,\beta}(X, Y)$
0.1	1	1.86	1.51
0.2	1	1.31	0.84
0.3	1	1.02	0.50
0.4	1	0.83	0.30
0.5	1	0.69	0.17
0.6	1	0.58	0.09
0.7	1	0.49	0.04
0.8	1	0.42	0.01
0.9	1	0.36	0.003

The comparison between the Nath inaccuracy measure of order α and a new proposed model of order α and type β suggests that the proposed model is more accurate than the Nath inaccuracy measure when a particular parameter is added. Specifically, when the parameter is changed, the inaccuracy of the proposed measure is reduced compared to the Nath (Nath 1968) inaccuracy measure. Many other also shows interest in the field

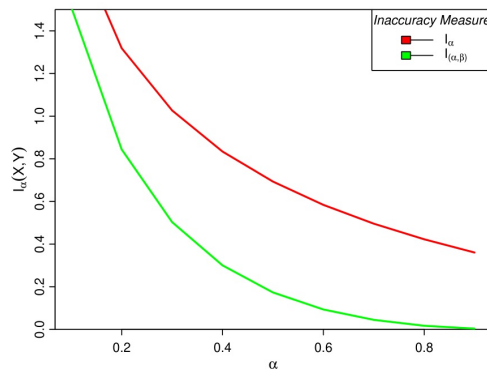


Fig. 1: This graph shows how our suggested measure's error has decreased when compared to the existing measure

Table II: Expressions of generalized inaccuracy for some life time distribution

Distribution	$f(x)$	$g(x)$	x	$\zeta_{\alpha,\beta}(X,Y)$
Uniform	$\frac{1}{m}$	$\frac{\theta(m-x)^{\theta-1}}{m^\theta}$	$0 < x < m; \theta > 0$	$Ulog\theta^t n^\beta \frac{\Gamma(t+\beta+1)\Gamma(w-t)}{\Gamma(\beta+w+1)}$
Exponential	ne^{-nx}	$n\theta e^{-n\theta x}$	$x \geq 0; n, \theta > 0$	$Ulog\theta^t n^t \left(\frac{1}{w}\right)$
Finite range	$r(1-x)^{r-1}$	$r\theta(1-x)^{r\theta-1}$	$0 \leq x \leq 1; \theta, r > 0$	$Ulog \frac{\theta^t r^{t+1}}{rw-1}$
Beta	sx^{s-1}	$s\theta x^{s\theta-1}$	$0 \leq x \leq 1; s, \theta > 0$	$Ulog \frac{\theta^t s^t}{(sw-t)}$
Power	$\frac{bx^{b-1}}{c^b}$	$\frac{b\theta x^{b\theta}}{c^{b\theta}}$	$0 < x < c; \theta > 0$	$Ulog \frac{\theta^t b^{t+1}}{c^t(bw-t)}$

Where,

$$U = \frac{1}{\beta(\beta-\alpha)}, t = \alpha - \beta, w = \theta t + 1, B(u,v) = \int_0^1 x^{u-1}(1-x)^{v-1} dx = \int_0^1 \frac{x^{u-1}}{(1+x)^{u+v}} = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

is known as complete beta function.

This article aims to introduce a novel two-parametric weighted generalized inaccuracy measure and its corresponding residual version. The subsequent sections of the article are structured as follows: Section 2 provides a definition of the weighted generalized inaccuracy measure and presents examples that illustrate the differences between the weighted and non-weighted versions of the measure. Section 3 introduces the weighted generalized residual inaccuracy measure, which is based on a proportional hazard rate model. The section also presents several key results that help to characterize this dynamic inaccuracy measure. By the end of this article, author will have a thorough understanding of the proposed weighted generalized inaccuracy measure and its residual version, as well as their potential applications in various fields.

2. DYNAMIC GENERALIZED INACCURACY (DGI)

The dynamic generalized inaccuracy measure is a modification of the generalized inaccuracy measure that takes into account the importance of different classes or categories in a dataset. In the original generalized inaccuracy measure, all classes are treated equally in terms of their contribution to the overall inaccuracy. By incorporating weight factors, the dynamic generalized inaccuracy measure can provide a more nuanced and accurate assessment of the performance of a classification system. It allows for the identification of classes that are more important or critical than others, and can help to prioritize efforts to improve the accuracy of those classes.

DEFINITION 2.1. Similar to (2) and based on (9), the dynamic generalized inaccuracy measure defined as

$$I_{\alpha,\beta}^w(X,Y) = \frac{1}{\beta(\beta-\alpha)} \log \int_0^\infty x^\alpha f(x) g(x)^{\alpha-\beta} dx, \quad \alpha \neq \beta, \alpha > 0, \beta > 0 \quad (16)$$

This can be further expressed as

$$I_{\alpha,\beta}^w(X,Y) = \frac{1}{\beta(\beta-\alpha)} \log [E(x^\alpha (g(x))^{\alpha-\beta})] \quad (17)$$

REMARKS

1. If $\beta = 1$ in (11), then it becomes Nath (Nath 1968) weighted generalized inaccuracy (WGI) of order α
2. If $g(x) = f(x)$ in (11), then it becomes Renyi's (Renyi 1961) entropy of order α

The distinction between GI (9) and its dynamic version (11) is shown by the following pairs of examples.

EXAMPLE 2.1. Consider X and Y as two non-negative random variables with the following density functions.

1. $f_1(x) = 1; \quad 0 < x < 1, \quad g_1(x) = 2x; \quad 0 < x < 1$
2. $f_2(x) = 1; \quad 0 < x < 1, \quad g_2(x) = 2(1-x); \quad 0 < x < 1$

Then, with a simple calculation, we get

$$I_{1(\alpha,\beta)}(X,Y) = \frac{1}{\beta(\beta-\alpha)} \log \frac{2^{(\alpha-\beta)}}{(\alpha-\beta+1)} \quad (18)$$

and

$$I_{2(\alpha,\beta)}(X,Y) = \frac{1}{\beta(\beta-\alpha)} \log \frac{2^{(\alpha-\beta)}}{(\alpha-\beta+1)} \quad (19)$$

Similarly for weight version

$$I_{1(\alpha,\beta)}^w(X,Y) = \frac{1}{\beta(\beta-\alpha)} \log \frac{2^{(\alpha-\beta)}}{(2\alpha-\beta+1)} \quad (20)$$

and

$$I_{2(\alpha,\beta)}^w(X,Y) = \frac{1}{\beta(\beta-\alpha)} \log 2^{(\alpha-\beta)} \frac{\Gamma(\alpha+1)\Gamma(\alpha-\beta+1)}{\Gamma(2\alpha-\beta+2)} \quad (21)$$

According to the computation above the generalized inaccuracy measure for the two distributions is equal when no weight is applied. We can therefore draw the conclusion that the distribution is determined only by the weighted form of the inaccuracy measure.

THEOREM 2.1. Let X and Y be two random variables having weighted generalized inaccuracy (WGI) $I_{(\alpha,\beta)}^w(X,Y)$ and $\alpha \neq \beta$, $\alpha, \beta > 0$ then we have

$$I_{(\alpha,\beta)}^w(X,Y) \leq \frac{1}{\beta(\beta-\alpha)} \left(\int_0^\infty x^\alpha f(x)(g(x))^{\alpha-\beta} dx - 1 \right), \alpha \neq \beta, \alpha, \beta > 0 \quad (22)$$

PROOF. Since we know that for any positive number (for any $x > 0$) then by using this inequality $\log x \leq x-1$ we get

$$I_{(\alpha,\beta)}^w(X,Y) = \frac{1}{\beta(\beta-\alpha)} \log \int_0^\infty x^\alpha f(x)(g(x))^{\alpha-\beta} dx, \alpha \neq \beta, \alpha, \beta > 0 \quad (23)$$

Further simplification, we obtained the result.

$$I_{(\alpha,\beta)}^w(X,Y) \leq \frac{1}{\beta(\beta-\alpha)} \left(\int_0^\infty x^\alpha f(x)(g(x))^{\alpha-\beta} dx - 1 \right), \alpha \neq \beta, \alpha, \beta > 0 \quad (24)$$

THEOREM 2.2. Suppose two random variables having weighted generalized inaccuracy (WGI) $I_{(\alpha,\beta)}^w(X,Y)$ and $\alpha \neq \beta$, $\alpha, \beta > 0$ then we have

$$I_{(\alpha,\beta)}^w(X,Y) \geq \frac{1}{\beta(\beta-\alpha)} \left((\alpha-\beta)\zeta(X,Y) - \alpha \int_0^\infty f(x) \log x dx \right) \quad (25)$$

PROOF. By using log sum inequality.

$$\int_0^\infty f(x) \log \frac{f(x)}{x^\alpha f(x)(g(x))^{\alpha-\beta}} dx \geq \int_0^\infty f(x) dx \log \frac{f(x) dx}{x^\alpha f(x)(g(x))^{\alpha-\beta} dx} \quad (26)$$

$$\int_0^\infty f(x) \log \frac{f(x)}{x^\alpha f(x)(g(x))^{\alpha-\beta}} dx = - \int_0^\infty f(x) dx \log \frac{x^\alpha f(x)(g(x))^{\alpha-\beta} dx}{f(x) dx} \quad (27)$$

$$\beta(\beta-\alpha)I_{(\alpha,\beta)}^w(X,Y) = -\log \int_0^\infty x^\alpha f(x)(g(x))^{\alpha-\beta} dx \quad (28)$$

After simplification, we obtained the result.

3. DYNAMIC GENERALIZED RESIDUAL INACCURACY

Dynamic Generalized Residual Inaccuracy (WGRI) is a statistical measure that is used to evaluate the accuracy of a predictive model. It is a modification of the Generalized Inaccuracy (GRI) measure, which is a normalized measure of the difference between observed and predicted values.

DEFINITION 3.1. Similar to (4) and weighted form of (10) called weighted generalized residual inaccuracy (WGRI) is given by

$$I_{\alpha,\beta}^w(X,Y;t) = \frac{1}{\beta(\beta-\alpha)} \log \left[\int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \right] \quad \alpha \neq \beta, \alpha > 0, \beta > 0 \quad (29)$$

REMARK

1. If $t=0$, then it simply becomes a weighted generalized inaccuracy(WGI).

DEFINITION 3.2. The mathematical relationship between two random variables, X and Y, has the form

$$\lambda_G(x) = \theta \lambda_F(x)$$

or

$$\bar{G}(x) = [\bar{F}(x)]^\theta,$$

then this relationship is known as the proportional hazard rate model (PHRM). $\lambda_G(x)$, $\lambda_F(x)$ is known as hazard rate and θ is a constant proportionality.

The Cox proportional hazards model is a statistical model used to analyze survival data, which is also known as time-to-event data. This model is named after its creator, Sir David Cox. The model assumes that the hazard rate, or the instantaneous risk of an event occurring, is proportional to a set of predictor variables. The Cox proportional hazards model is widely used in medical research, particularly in oncology and epidemiology, to study the effect of different risk factors on survival outcomes. The model can also be used to estimate the effect of interventions on survival, and to identify subgroups of patients who are more likely to benefit from specific treatments. (Di Crescenzo 2000), (Gupta and Gupta 2007), (Sankaran and Gleeja 2008), (Kundu and Gupta 2010), (Sengupta and Nanda 2011), (Asokan and Sankaran 2014), (Unnikrishnan et.al. 2018), (Di Crescenzo et.al. 2019)

THEOREM 3.1. Suppose X and Y be two random variables having probability density function $f(x)$ and $g(x)$ and survival functions $\bar{F}(x)$ and $\bar{G}(x)$ respectively, $t > 0$, then for $\beta \leq \alpha + 1$ $I_{\alpha,\beta}^w(X,Y;t)$ obtained a lower bound as

$$I_{\alpha,\beta}^w(X,Y;t) \geq \frac{1}{\beta(\beta-\alpha)} \left[(\alpha-\beta)I(X,Y;t) + \alpha \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log x \right] \quad (30)$$

PROOF. By using log-sum inequality we get

$$\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{\frac{f(x)}{\bar{F}(t)}}{x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta}} dx \geq \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{\frac{f(x)}{\bar{F}(t)}}{x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta}} dx \quad (31)$$

and

$$\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{\frac{f(x)}{\bar{F}(t)}}{x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta}} dx = -\log \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \quad (32)$$

After simplification, we obtained the result.

THEOREM 3.2. Suppose X and Y be two random variables having weighted generalized inaccuracy (WGRI) $I_{\alpha,\beta}(X,Y;t)$ and $\beta \leq \alpha + 1$, we have

$$I_{\alpha,\beta}^w(X,Y;t) \geq \frac{1}{\beta(\beta-\alpha)} \left[1 - \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \right] \quad (33)$$

PROOF. Since we know that for any positive number for any $x > 0$ by using this inequality we get $-\log x \geq 1-x$

$$I_{\alpha,\beta}^w(X,Y;t) = \frac{1}{\beta(\beta-\alpha)} \log \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \quad (34)$$

$$I_{\alpha,\beta}^w(X,Y;t) = -\frac{1}{\beta(\beta-\alpha)} \log \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \quad (35)$$

After simplification, we get the desired result.

THEOREM 3.3. If two non negative random variables X and Y satisfied the PHRM with proportionality constant $\theta > 0$ for all $t > 0$ then, equality given by

$$\left[\frac{1}{\beta(\beta-\alpha)} \left[t^\alpha \exp(\beta(\beta-\alpha)) I_{\alpha,\beta}(X,Y;t) + \alpha \int_{z=t}^{\infty} z^{\alpha-1} \left(\frac{\bar{F}(z)}{\bar{F}(t)} \right)^{\theta(\alpha-\beta)+1} \exp(\beta(\beta-\alpha)) I_{\alpha,\beta}(X,Y;z) dz \right] \right] \quad (36)$$

PROOF.

$$\int_t^{\infty} x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx = \int_t^{\infty} (\alpha z^{\alpha-1} dz) \left(\frac{f(z)}{\bar{F}(t)} \right) \left(\frac{g(z)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \quad (37)$$

$$\int_t^{\infty} x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx = \alpha \int_t^{\infty} (z^{\alpha-1} dz) \left(\frac{f(z)}{\bar{F}(t)} \right) \left(\frac{g(z)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \quad (38)$$

$$\left[\int_t^{\infty} x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx = t^\alpha \int_t^{\infty} \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx + \alpha \int_{z=t}^{\infty} z^{\alpha-1} \left(\int_{x=z}^{\infty} \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \right) dz \right] \quad (39)$$

From (10) we get

$$\left(\int_{x=z}^{\infty} \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \right) = \exp(\beta(\beta-\alpha)) I_{\alpha,\beta}^w(X,Y;t) \quad (40)$$

$$\int_t^{\infty} f(x) (g(x))^{\alpha-\beta} = \bar{F}(t) \bar{G}(t)^{\alpha-\beta} \exp(\beta(\beta-\alpha)) I_{\alpha,\beta}^w(X,Y;t) \quad (41)$$

Using PHRM we get

$$\int_t^{\infty} f(x) (g(x))^{\alpha-\beta} = \bar{F}(t) [\bar{F}(t)]^{\theta(\alpha-\beta)} \exp(\beta(\beta-\alpha)) I_{\alpha,\beta}^w(X,Y;t) \quad (42)$$

Using (20), (21) and (22) in (14) we get

$$I_{\alpha,\beta}^w(X,Y;t) = \frac{1}{\beta(\beta-\alpha)} \log \left[\int_t^{\infty} x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{g(x)}{\bar{G}(t)} \right)^{\alpha-\beta} dx \right] \quad (43)$$

After simplification, we obtained the result.

THEOREM 3.4. Suppose two continuous non-negative random variables having survival function $\bar{F}(x)$ and $\bar{G}(x)$ respectively, the survival function satisfying the proportional hazard rate model and suppose $I_{\alpha,\beta}^w(X,Y;t) \leq \infty, \forall t \geq 0$ be an increasing function of t , then $I_{\alpha,\beta}^w(X,Y;t)$ determine $\bar{F}(t)$ uniquely.

PROOF. Equation (14) can be written as

$$\exp(\beta(\beta - \alpha))I_{\alpha, \beta}^w(X, Y; t) = \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} dx \quad (44)$$

Diff. w.r.t. t we get

$$\frac{\partial}{\partial t} \exp(\beta(\beta - \alpha))I_{\alpha, \beta}^w(X, Y; t) = -t^\alpha \lambda_F(t) (\lambda_G)^{\alpha - \beta} + (\lambda_F(t) + (\alpha - \beta) \lambda_G(t)) \exp(\beta(\beta - \alpha))I_{\alpha, \beta}^w(X, Y; t) \quad (45)$$

$\lambda_F(x)$ and $\lambda_G(x)$ is defined in the introduction part.

By using PHRM

$$\lambda_G(x) = \theta \lambda_F(x)$$

$$t^\alpha \theta^{\alpha - \beta} (\lambda_F(t))^{\alpha - \beta + 1} - \lambda_F(t) [1 + (\alpha - \beta) \theta] \exp(\beta(\beta - \alpha))I_{\alpha, \beta}^w(X, Y; t) + \frac{\partial}{\partial t} \exp(\beta(\beta - \alpha))I_{\alpha, \beta}^w(X, Y; t) = 0 \quad (46)$$

Hence for fixed $t > 0$ then $\lambda_F(t)$ is a solution of $\tau(x_t) = 0$ then we have,

$$t^\alpha \theta^{\alpha - \beta} (x_t)^{\alpha - \beta + 1} - x_t [1 + (\alpha - \beta) \theta] \exp(\beta(\beta - \alpha))I_{\alpha, \beta}^w(X, Y; t) + \frac{\partial}{\partial t} \exp(\beta(\beta - \alpha))I_{\alpha, \beta}^w(X, Y; t) = 0 \quad (47)$$

$$\tau(0) = \frac{\partial}{\partial t} \exp(\beta(\beta - \alpha))I_{\alpha, \beta}^w(X, Y; t) \geq 0 \quad (48)$$

because it was assumed that $I_{\alpha, \beta}^w(X, Y; t)$ is increasing in t

If $x_t \rightarrow \infty$, then $\tau(\infty) = \infty$

Differentiating (26) both sides w.r.t. x_t , we get

$$\frac{\partial}{\partial x_t} \tau(x_t) = (\alpha - \beta + 1) t^\alpha \theta^{\alpha - \beta} (x_t)^{\alpha - \beta + 1 - 1} - [1 + (\alpha - \beta) \theta] \exp(\beta(\beta - \alpha))I_{\alpha, \beta}^w(X, Y; t) \quad (49)$$

and

$$\frac{\partial^2}{\partial x_t^2} \tau(x_t) = (\alpha - \beta)(\alpha - \beta + 1) t^\alpha \theta^{\alpha - \beta} (x_t)^{\alpha - \beta - 1} \quad (50)$$

$$\frac{\partial}{\partial x_t} \tau(x_t) = 0 \quad (51)$$

After simplification, we obtained the result.

$$x_t = \left[\frac{(1 + (\alpha - \beta)\theta) \exp(\beta(\beta - \alpha)) I_{\alpha, \beta}^w(X, Y; t)}{(\alpha - \beta + 1) t^\alpha \theta^{\alpha - \beta}} \right] = x_0 \quad (52)$$

Then we can say that the $I_{\alpha, \beta}^w(X, Y; t)$ weighted generalized inaccuracy uniquely determined $\bar{F}(t)$

Table III: Expression of weighted generalized inaccuracy for some life time distribution

Distribution	f(x)	g(x)	x	$\zeta_{\alpha, \beta}^w(X, Y)$
Uniform	$\frac{1}{m}$	$\frac{\theta(m-x)^{\theta-1}}{m^\theta}$	$0 < x < m; \theta > 0$	$U \log \theta^t m^\beta \left(\frac{\Gamma\alpha + 1 \Gamma vt + 1}{\Gamma\alpha + vt + 2} \right)$
Exponential	ne^{-nx}	$n\theta e^{-n\theta x}$	$x \geq 0; n, \theta > 0$	$U \log \frac{\theta^t}{n^\beta} \left(\frac{\Gamma\alpha + 1}{n^\alpha + 1} \right)$
Finite range	$r(1-x)^{r-1}$	$r\theta(1-x)^{r\theta-1}$	$0 \leq x \leq 1; \theta, r > 0$	$U \log \theta^t r^{t+1} \frac{\Gamma\alpha + 1 \Gamma rw - t}{\Gamma\alpha + rw - t + 1}$
Beta	sx^{s-1}	$s\theta x^{s\theta-1}$	$0 \leq x \leq 1; s, \theta > 0$	$U \log \frac{\theta^t s^{t+1}}{\alpha + sw - t}$
Power	$\frac{bx^{b-1}}{c^b}$	$\frac{b\theta x^{b\theta}}{c^{b\theta}}$	$0 < x < c; \theta > 0, b > 0$	$U \log \frac{\theta^t b^{t+1} c^{bt-t}}{\alpha + bw - t}$

Where,

$$U = \frac{1}{\beta(\beta - \alpha)}, t = \alpha - \beta, w = \theta t + 1, B(u, v) = \int_0^1 x^{u-1} (1-x)^{v-1} dx = \int_0^1 \frac{x^{u-1}}{(1+x)^{u+v}} = \frac{\Gamma u \Gamma v}{\Gamma(u+v)}$$

is known as complete beta function.

THEOREM 3.5. When the hazard rate denoted by $\lambda_G(t)$ is decreasing in (t) then the inequality holds as

$$I_{\alpha, \beta}^w(X, Y; t) \geq \frac{1}{\beta(\beta - \alpha)} \log \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) dx - \log \lambda_G(t) \quad (53)$$

PROOF. From (16) can be rewritten as

$$I_{\alpha, \beta}^w(X, Y; t) = \frac{1}{\beta(\beta - \alpha)} \log \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) (\lambda_G(x)^{\alpha - \beta}) \left(\frac{\bar{G}(x)}{\bar{G}(t)} \right)^{\alpha - \beta} dx \quad (54)$$

because $\bar{G}(x) \leq \bar{G}(t)$ for $x \geq t$, which implies that the hazard rate $\lambda_G(x) \leq \lambda_G(t)$, then we have

$$I_{\alpha, \beta}^w(X, Y; t) \geq \frac{1}{\beta(\beta - \alpha)} \log (\lambda_G(t)^{\alpha - \beta}) \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) \left(\frac{\bar{G}(x)}{\bar{G}(t)} \right)^{\alpha - \beta} dx \quad (55)$$

$$I_{\alpha, \beta}^w(X, Y; t) = \frac{1}{\beta(\beta - \alpha)} \left[(\alpha - \beta) \log \lambda_G(t) + \log \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \right) dx \right] \quad (56)$$

After simplification, we obtained the result.

THEOREM 3.6. Suppose X and Y be two non-negative random variables with interval $(0, b)$ having density function $f(x)$ and $g(x)$ and survival function $\bar{F}(t)$ and $\bar{G}(t)$ respectively then $\beta > \alpha + 1$ then for upper bound of $I_{\alpha, \beta}^w(X, Y; t)$ follows as

$$I_{\alpha, \beta}^w(X, Y; t) \leq \frac{1}{\beta(\beta - \alpha)} \left[\frac{\int_t^b x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} \log x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} dx}{\int_t^b x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} dx} + \log(b - t) \right] \quad (57)$$

PROOF. Using log-sum inequality in (16), we get

$$\begin{aligned} \int_t^b x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} \log x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} dx &\geq \\ &\geq \int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} dx \log \frac{\int_0^\infty x^\alpha f(x) g(x)^{\alpha - \beta} dx}{\int_t^b \bar{F}(t) (\bar{G}(t))^{\alpha - \beta} dx} \quad (58) \end{aligned}$$

$$\begin{aligned} \int_t^b x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} \log x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} dx &= \\ = \int_t^b x^\alpha \left(\frac{f(x)}{\bar{F}(x)} \right) \left(\frac{g(x)}{\bar{G}(x)} \right)^{\alpha - \beta} dx [(\beta(\beta - \alpha) I_{\alpha, \beta}^w(X, Y; t)) - \log(b - t)] \quad (59) \end{aligned}$$

Then, after simplification, we obtained the result.

DEFINITION 3.3. If $f(x)$ and $g(x)$ be the density function associated two non negative random variables X and Y , then under likelihood ratio ordering, the random variable X is said to be less than or equal to Y and is represented as $X \text{ lhr} \leq Y$ because $\frac{f(x)}{g(x)}$ is decreasing in x i.e. if $\frac{f(x)}{g(x)} \leq \frac{f(t)}{g(t)} \forall x > t$

THEOREM 3.7. Suppose the three non-negative random variables X_1, X_2, X_3 with density function f_1, f_2, f_3 . Suppose $\bar{F}_1, \bar{F}_2, \bar{F}_3$ be the associated survival function having the hazard rate function $\lambda_{F_1}, \lambda_{F_2}, \lambda_{F_3}$. Also assuming that $X_1 \leq X_2$ in the likelihood ratio ordering i-e $\frac{f_2}{f_1}$ is increasing in x then

$$I_{\alpha, \beta}(X_1, X_3; t) - I_{\alpha, \beta}(X_2, X_3; t) \leq \frac{1}{\beta(\beta - \alpha)} \log \frac{\lambda_{F_1}(t)}{\lambda_{F_2}(t)} \quad 0 < \alpha < 1, \beta \neq 0 \quad (60)$$

PROOF. From (16) we get

$$I_{\alpha,\beta}(X_1, X_3; t) \leq \frac{1}{\beta(\beta - \alpha)} \log \left[\int_t^\infty x^\alpha \left(\frac{f_1(x)}{\bar{F}_1(t)} \right) \left(\frac{f_3(x)}{\bar{F}_3(t)} \right)^{\alpha - \beta} dx \right] \quad (61)$$

$$I_{\alpha,\beta}(X_1, X_3; t) = \frac{1}{\beta(\beta - \alpha)} \log \left[\int_t^\infty \left(x^\alpha \frac{f(x)}{\bar{F}(t)} \right) \left(\frac{f_3(x)}{\bar{F}_3(t)} \right)^{\alpha - \beta} dx \right] \quad (62)$$

By the given condition, $\lambda_F(x) \leq \lambda_F(t)$

$$I_{\alpha,\beta}(X_1, X_3; t) = \frac{1}{\beta(\beta - \alpha)} \log \left[\int_t^\infty x^\alpha \left(\frac{f(x)}{\bar{F}(t)} \frac{f_2(x)}{f_2(x)} \right) \left(\frac{f_3(x)}{\bar{F}_3(t)} \right)^{\alpha - \beta} dx \right] \quad (63)$$

$$I_{\alpha,\beta}(X_1, X_3; t) = \frac{1}{\beta(\beta - \alpha)} \log \left[\int_t^\infty x^\alpha \left(\frac{f_2(x)}{\bar{F}_2(t)} \right) \left(\frac{f_3(x)}{\bar{F}_3(t)} \right)^{\alpha - \beta} \left(\frac{\lambda_{F_1}(t)}{\lambda_{F_2}(t)} \right) \right] \quad (64)$$

After simplification, the required result is obtained.

COROLLARY

For $\alpha > 1$, $\beta \neq 0$, we have

$$I_{\alpha,\beta}(X_1, X_3; t) - I_{\alpha,\beta}(X_2, X_3; t) \geq \frac{1}{\beta(\beta - \alpha)} \log \frac{\lambda_{F_1}(t)}{\lambda_{F_2}(t)} \quad (65)$$

4. CONCLUSIONS

In this communication, we present a novel two-parametric weighted generalised inaccuracy measure of order α and type β and compare it to the existing measure. It demonstrates that the proposed measure is more accurate than the existing measure. For some life-time distributions, we also examine generalised and weighted generalised inaccuracy. We looked at the outcomes of the categorization and some suggested measures for inequality. Finally, we demonstrate that the survival function is solely determined by the weighted generalised residual inaccuracy.

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