



Original Study

## An approximate approach for fractional relaxation-oscillation equation based on Taylor expansion

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### Abstract

This work presents an application of the efficient method to solve the fractional relaxation-oscillation equation. The method is based on employing Taylor polynomial of degree  $m$  for unknown function and repeated integration technique coupled with the Riemann-Liouville fractional derivative such that the fractional relaxation-oscillation equation is transformed into a linear equations system involving an unknown function and its derivatives which is then solved. An error analysis is given along with several numerical applications to demonstrate the simplicity and the effectiveness of the suggested technique for which the exact solutions are known beforehand.

**Keywords:** Fractional differential equation, fractional relaxation-oscillation equation, RL fractional derivative, RL fractional integral, Taylor expansion, error analysis.

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### 1 Introduction

Fractional differential equations (FDEs) are applied across a broad range of fields due to their ability to model complex systems with memory, hereditary, and non-local behaviors. These equations result from many physical or chemical problems like heat conduction, motion of a large thin plate in a Newtonian fluid, viscoelasticity, diffusion wave, process of cooling a semi-infinite body by radiation, phenomena in electromagnetic, electrochemistry, material science, acoustic and so on [1–4]. Fractional-order differential equations are more appropriate for modeling physical and chemical processes than integer-order differential equations and most realistic FDEs do not have exact solutions, so approximation and numerical techniques must be used. In particular, FDEs with  $1/2$ -order derivative or  $3/2$ -order derivative are especially important since they describe the frequency-dependent damping materials quite satisfactorily.

In literature, different numerical or analytical approaches have been applied for FDEs or other functional equations which can be used for FDEs and the interested reader can refer to [5–29] in order to know these kinds of methods, their modifications, their applications and their importance.

A relaxation oscillator is defined as a type of oscillator which is in fact based on the behavior of a physical

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system's return to equilibrium after being disturbed [5,30,31]. The relaxation-oscillation equation is the primary equation of both relaxation and oscillation processes [5]. Fractional relaxation-oscillation equation broadens the scope of classical relaxation and oscillation models by capturing systems with inherent memory, delayed responses, or anomalous damping and diffusion behaviors.

The relaxation equation in standard form can be expressed as [5]

$$\frac{d\psi}{dx} + A\psi = f(x), \quad (1)$$

in which  $A$  indicates  $E/c$  where  $E$  is the elastic modulus,  $c$  the viscosity coefficient, and  $f(x)$  indicates  $E$  multiplying the strain rate. As  $f(x) = 0$ , the analytic solution can be obtained as

$$\psi(x) = Ce^{-Ax}, \quad (2)$$

in which  $C$  is a constant obtained by the given initial condition.

The oscillation equation in standard form is given by [5]

$$\frac{d^2\psi}{dx^2} + A\psi = f(x), \quad (3)$$

in which  $A$  is equal to  $k/m = w^2$  where  $k$  is stiffness coefficient,  $m$  the mass and  $w$  the angular frequency. As  $f(x) = 0$ , we obtain the following analytical solution as

$$\psi(x) = C \cos \sqrt{Ax} + D \sin \sqrt{Ax}, \quad (4)$$

in which  $C$  and  $D$  are constants obtained by the given initial conditions.

Now, we consider the following fractional differential equation with initial conditions which is occurred in applied problems

$$D^\alpha \psi(x) + A\psi(x) = f(x), \quad x > 0 \quad (5)$$

$$\psi^{(k)}(0) = 0, \quad (k = 0, 1, \dots, n-1) \quad (6)$$

in which  $A$  is a constant and  $n-1 < \alpha \leq n$ ,  $n \in \mathbb{N}$ . When  $0 < \alpha \leq 2$ , this equation is known as relaxation-oscillation equation [1]. In case  $0 < \alpha \leq 1$ , the model characterizes the relaxation with the power law attenuation [32,33]. In case  $1 < \alpha \leq 2$ , the model describes the damped oscillation with viscoelastic intrinsic damping of oscillator [32,33]. Applications of this model can be found in the electrical model of the heart, spruce-budworm interactions, modeling cardiac pacemakers, predator-prey system [32–37]. Below are some of the other key fields and specific applications of this model.

## A. Physics [38,39]

**Quantum Mechanics:** The fractional relaxation-oscillation equation is applied to quantum systems that exhibit non-Markovian behavior, where memory effects are critical. This includes quantum oscillators and phenomena such as anomalous diffusion and quantum tunneling.

**Nonlinear Dynamics and Chaos:** In systems where nonlinear oscillations and chaotic dynamics occur, fractional relaxation-oscillation models are used to describe more complex temporal evolution, capturing both relaxation and oscillatory behavior.

## B. Biological Systems and Medicine [40,41]

**Cardiac Dynamics:** The fractional relaxation-oscillation equation is used to model the heart's electrical activity, particularly in the case of abnormal rhythms or arrhythmias. The fractional approach captures the memory-like behavior in cardiac tissue.

**Neuroscience:** In modeling the dynamics of neurons, fractional equations describe the propagation of action potentials and oscillatory behavior in neuronal circuits, particularly in the brain's response to stimuli.

### C. Electrical and Electronic Engineering [42, 43]

**Circuit Design:** In electrical circuits that exhibit fractional behavior, such as those involving supercapacitors or materials with fractional-order impedance, the fractional relaxation-oscillation equation helps model transient behavior more accurately.

**Signal Processing:** The equation is employed in filter designs where signals exhibit relaxation-oscillation characteristics, especially in biomedical signal processing, such as modeling EEG or ECG signals.

### D. Mechanical Engineering [1, 44]

**Vibration Analysis:** Fractional relaxation-oscillation equations are used to model systems with viscoelastic damping, such as mechanical systems with rubber components or smart materials. These models capture the relaxation and oscillation behavior with memory effects.

**Control Systems:** In control systems where feedback loops involve memory or hereditary effects, such as in robotics and automation, fractional relaxation-oscillation models provide more accurate control system performance.

### E. Civil Engineering [44, 45]

**Structural Dynamics:** The fractional relaxation-oscillation equation is used to model structures that exhibit viscoelastic behavior, such as buildings with seismic dampers. It helps capture both the relaxation and oscillatory response of the structure under dynamic loads like earthquakes.

**Fracture Mechanics:** In fracture mechanics, the fractional equation models the propagation of cracks in materials with memory effects, such as concrete or composites.

### F. Chemistry and Materials Science [41, 46]

**Viscoelastic Materials:** In chemistry and materials science, the fractional relaxation-oscillation equation is used to describe the behavior of viscoelastic materials, such as polymers or gels, which exhibit both relaxation and oscillation in response to stress.

**Diffusion in Porous Media:** Fractional models are used to study relaxation and oscillatory behaviors in chemical reactions or diffusion processes within porous media, capturing the anomalous diffusion seen in materials like catalysts or membranes.

### G. Finance and Economics [45, 47]

**Economic Cycles:** The fractional relaxation-oscillation equation is applied to model economic cycles, capturing both the oscillatory nature of market trends and the memory effects that influence economic behavior.

**Option Pricing and Financial Derivatives:** In financial modeling, this equation helps in better representing markets that exhibit memory and non-Markovian behaviors, which are critical in modeling financial derivatives and options.

### H. Ecology and Environmental Science [46, 48]

**Population Dynamics:** Fractional relaxation-oscillation models describe the growth and oscillations in population sizes over time, incorporating the effects of memory and hereditary behaviors in species interactions.

**Pollution Dispersion:** These models are used to describe the oscillatory and relaxation effects seen in the dispersion of pollutants, particularly in complex environments like air or water systems with feedback loops.

In this paper, we propose a simple and effective approach to solve the fractional relaxation-oscillation equation founded on a new type of Taylor expansion (See [14, 23–29, 49–54]). In this method, the fractional

relaxation-oscillation equation is transformed into a Volterra integral equation and then the Taylor polynomial of degree  $m$  for unknown function and repeated integration are employed to reduce the resulting integral equation into a linear equations system with respect to unknown function and its derivatives. The significant advantage of this method aside from its dependability and practicality is that an approximation of order  $m$  matches the precise solution if the precise solution is a polynomial function of degree at most  $m$ .

This paper is constructed as follows. Some basic concepts of fractional calculus are reviewed in Section 2. The method is presented in Section 3. An error analysis is provided by Section 4. Eight illustrative applications are made in Section 5. Some preliminary conclusions are given in Section 6.

## 2 Review of some basic concepts

We review some fundamental ideas and characteristics of the fractional calculus that we will be using later.

**Definition 1.** A real function  $\phi(x)$ ,  $x > 0$ , is said to be in the space  $C_\mu$ ,  $\mu \in \mathbf{R}$  if there exists a real number  $p (> \mu)$ , such that  $\phi(x) = x^p \phi_1(x)$ , where  $\phi_1(x) \in C[0, \infty)$ , and it is said to be in the space  $C_\mu^n$  if  $\phi^{(n)} \in C_\mu$ ,  $n \in \mathbf{N}$ .

**Definition 2.** The Riemann-Liouville fractional integral operator of order  $\alpha \geq 0$ , of a function  $\phi \in C_\mu$ ,  $\mu \geq -1$ , is considered as follows

$$J^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \phi(t) dt, \quad \alpha > 0, \quad x > 0, \quad (7)$$

$$J^0 \phi(x) = \phi(x). \quad (8)$$

**Definition 3.** The Caputo fractional derivative of  $\phi(x)$  is considered as follows

$$D_*^\alpha \phi(x) = J^{n-\alpha} \left( \frac{d^n}{dx^n} \phi(x) \right) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \phi^{(n)}(t) dt, \quad (9)$$

for  $n-1 < \alpha \leq n$ ,  $n \in \mathbf{N}$ ,  $x > 0$ ,  $\phi \in C_{-1}^n$ .

**Definition 4.** The Riemann-Liouville fractional derivative of  $\phi(x)$  is considered as follows

$$D^\alpha \phi(x) = \frac{d^n}{dx^n} (J^{n-\alpha} \phi(x)), \quad (10)$$

for  $n-1 < \alpha \leq n$ ,  $n \in \mathbf{N}$ ,  $x > 0$ ,  $\phi \in C_{-1}^n$ .

## 3 Description of the method

As mentioned in Section 1, we consider the fractional relaxation-oscillation equation as follows:

$$D^\alpha \psi(x) + A\psi(x) = f(x), \quad x > 0 \quad (11)$$

$$\psi(0) = 0, \quad \psi'(0) = 0, \quad (12)$$

where  $A$  is a constant and  $0 < \alpha \leq 2$ . In equation (11),  $D^\alpha \psi(x)$  indicates Riemann-Liouville fractional derivative of order  $\alpha$ . In order to describe the method, we consider two cases as  $0 < \alpha \leq 1$  and  $1 < \alpha \leq 2$ , for simplicity.

**Case I:**  $0 < \alpha \leq 1$ .

According to definition (10), equation (11) is rewritten as

$$\frac{d}{dx} (J^{1-\alpha} \psi(x)) + A\psi(x) = f(x). \quad (13)$$

By using definition (8), the aforementioned equation is equivalent to

$$\frac{d}{dx} \left( \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} \psi(t) dt \right) + A\psi(x) = f(x). \quad (14)$$

In the following, integrating both side of equation (14) from 0 to  $x$ , leads to

$$\frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} \psi(t) dt + A \int_0^x \psi(t) dt = f_{(0)}(x), \quad (15)$$

where

$$f_{(0)}(x) = \int_0^x f(t) dt. \quad (16)$$

Hence we convert the fractional relaxation-oscillation equation (11) under initial condition (12) into a Volterra integral equation. To approximately solve the obtained Volterra integral equation, we reduce it into a linear system of equations with respect to unknown function and its derivatives. To realize this goal, it is supposed that the solution  $\psi(t)$  is  $m+1$  times continuously differentiable on the interval  $I$ , that is to say  $\psi \in C^{m+1}(I)$ . Therefore, for  $\psi \in C^{m+1}(I)$ , we can express unknown function  $\psi(t)$  in terms of the Taylor expansion of order  $m$  at an arbitrary point  $x \in I$  as follows.

$$\psi(t) = \psi(x) + \psi'(x)(t-x) + \cdots + \frac{1}{m!} \psi^{(m)}(x)(t-x)^m + E_m(t, x), \quad (17)$$

in which  $E_m(t, x)$  is the bound of Lagrange error as

$$E_m(t, x) = \frac{\psi^{(m+1)}(\delta)}{(m+1)!} (t-x)^{m+1}, \quad (18)$$

for some point  $\delta$  between  $x$  and  $t$ . Generally, the Lagrange error bound  $E_m(t, x)$  gets sufficiently small as  $m$  grows enough provided that  $\psi^{(m+1)}(x)$  is a uniformly bounded function. It is important to note that the Lagrange error bound becomes zero for a polynomial function of maximum degree  $m$ , thus the above Taylor expansion of order  $m$  is equal to exact solution. With due attention to aforesaid hypothesis, by omitting the last bound of Lagrange error, we approximately expand  $\psi(t)$  as

$$\psi(t) \approx \sum_{k=0}^m \psi^{(k)}(x) \frac{(t-x)^k}{k!}. \quad (19)$$

Replacing the truncated series (19), for unknown function  $\psi(t)$ , into (15) leads to

$$\frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^m \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_0^x (x-t)^{k-\alpha} dt + A \sum_{k=0}^m \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_0^x (x-t)^k dt = f_{(0)}(x), \quad (20)$$

which can be expressed simply as follows

$$v_{00}(x)\psi(x) + v_{01}(x)\psi'(x) + \cdots + v_{0m}(x)\psi^{(m)}(x) = f_{(0)}(x), \quad (21)$$

where

$$v_{0k}(x) = \frac{(-1)^k x^{k-\alpha+1}}{(k-\alpha+1)\Gamma(1-\alpha)k!} + \frac{(-1)^k A x^{k+1}}{k!(k+1)}, \quad k = 0, \dots, m. \quad (22)$$

In essence, equation (11) was converted into an  $m$ th-order linear ordinary differential equation involving  $\psi(x)$  and its derivations up to order  $m$ . Next, our goal is to solve a linear system of equations to determine  $\psi(x), \dots, \psi^{(m)}(x)$ . To accomplish this aim, we require other  $m$  additional linear equations involving  $\psi(x)$  and its derivatives up to order  $m$  which obtain by repeated integration of both sides of equation (15)  $m$  times as

$$\frac{1}{\Gamma(1-\alpha)} \int_0^x \int_t^x \frac{(x-s)^{i-1}}{(i-1)!} \frac{\psi(t)}{(s-t)^\alpha} ds dt + A \int_0^x \frac{(x-t)^i}{(i)!} \psi(t) dt = f_{(i)}(x), \quad (23)$$

where

$$f_{(i)}(x) = \int_0^x \frac{(x-t)^i}{(i)!} f(t) dt, \quad i = 1, \dots, m. \quad (24)$$

Taylor expansion is applied again and after replacing (19), for unknown function  $\psi(t)$ , into (23) we obtain

$$\begin{aligned} & \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^m \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_0^x \int_t^x \frac{(x-s)^{i-1}}{(i-1)!} (x-t)^k (s-t)^{-\alpha} ds dt \\ & + A \sum_{k=0}^m \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_0^x \frac{(x-t)^{k+i}}{(i)!} dt = f_{(i)}(x), \quad i = 1, \dots, m, \end{aligned} \quad (25)$$

or equivalently

$$v_{i0}(x)\psi(x) + v_{i1}(x)\psi'(x) + \dots + v_{im}(x)\psi^{(m)}(x) = f_{(i)}(x), \quad i = 1, \dots, m, \quad (26)$$

where

$$v_{ik}(x) = \frac{(-1)^k x^{k+i-\alpha+1}}{(k+i-\alpha+1)\Gamma(i+1-\alpha)k!} + \frac{(-1)^k A x^{k+i+1}}{k!i!(k+i+1)}, \quad k = 0, \dots, m. \quad (27)$$

In this way, equations (21) and (26) make a system of linear equations involving  $\psi(x), \dots, \psi^{(m)}(x)$ . Now, we use the following notation for the resulting system

$$\mathbf{V}(x)\Psi(x) = F(x), \quad (28)$$

where

$$\mathbf{V}(x) = \begin{bmatrix} v_{00}(x) & v_{01}(x) & \dots & v_{0m}(x) \\ v_{10}(x) & v_{11}(x) & \dots & v_{1m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ v_{m0}(x) & v_{m1}(x) & \dots & v_{mm}(x) \end{bmatrix}, \quad (29)$$

$$\mathbf{F}(x) = \begin{bmatrix} f_{(0)}(x) \\ f_{(1)}(x) \\ \vdots \\ f_{(m)}(x) \end{bmatrix}, \quad \Psi(x) = \begin{bmatrix} \psi(x) \\ \psi'(x) \\ \psi''(x) \\ \vdots \\ \psi^{(m)}(x) \end{bmatrix}. \quad (30)$$

In the sequel, an approximate solution of equation (11) can be achieved by applying the conventional approach to the system (28).

**Case II:**  $1 < \alpha \leq 2$ .

According to definition (10), equation (11) is rewritten as

$$\frac{d^2}{dx^2} (J^{2-\alpha} \psi(x)) + A \psi(x) = f(x). \quad (31)$$

By using definition (8), the aforementioned equation is equivalent to

$$\frac{d^2}{dx^2} \left( \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-t)^{1-\alpha} \psi(t) dt \right) + A \psi(x) = f(x). \quad (32)$$

In the following, integrating both side of equation (32) twice from 0 to  $x$ , leads to

$$\frac{1}{\Gamma(2-\alpha)} \int_0^x (x-t)^{1-\alpha} \psi(t) dt + A \int_0^x (x-t) \psi(t) dt = f_{(0)}(x), \quad (33)$$

where

$$f_{(0)}(x) = \int_0^x (x-t) f(t) dt. \quad (34)$$

Hence we convert the fractional relaxation-oscillation equation (11) under initial conditions (12) into a Volterra integral equation. To approximately solve the resulting Volterra integral equation, we reduce it into a linear equations system with respect to  $\psi(x), \dots, \psi^{(m)}(x)$ .

Toward this end, using a procedure analogous to the case I, replacing the truncated series (19), for unknown function  $\psi(t)$ , into (33) leads to

$$\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^m \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_0^x (x-t)^{k+1-\alpha} dt + A \sum_{k=0}^m \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_0^x (x-t)^{k+1} dt = f_{(0)}(x). \quad (35)$$

or equivalently

$$v_{00}(x) \psi(x) + v_{01}(x) \psi'(x) + \dots + v_{0m}(x) \psi^{(m)}(x) = f_{(0)}(x), \quad (36)$$

where

$$v_{0k}(x) = \frac{(-1)^k x^{k-\alpha+2}}{(k-\alpha+2)\Gamma(2-\alpha)k!} + \frac{(-1)^k A x^{k+2}}{k!(k+2)}, \quad k = 0, \dots, m. \quad (37)$$

In equation (36),  $\psi^{(k)}(x)$ , for  $k = 0, \dots, m$  are unknown functions. To find these unknown functions, we treat equation (36) as a linear equation in terms of  $\psi(x), \dots, \psi^{(m)}(x)$ . As a result, we require  $m$  additional linear equations involving  $\psi(x)$  and its derivatives up to order  $m$ . These required equations are derived by repeated integration of both sides of equation (33)  $m$  times as shown below

$$\frac{1}{\Gamma(2-\alpha)} \int_0^x \int_t^x \frac{(x-s)^{i-1}}{(i-1)!} \frac{\psi(t)}{(s-t)^{\alpha-1}} ds dt + A \int_0^x \frac{(x-t)^{i+1}}{(i+1)!} \psi(t) dt = f_{(i)}(x), \quad (38)$$

where

$$f_{(i)}(x) = \int_0^x \frac{(x-t)^{i+1}}{(i+1)!} f(t) dt, \quad i = 1, \dots, m. \quad (39)$$

Replacing (19) for  $\psi(t)$  into (38), we obtain

$$\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^m \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_0^x \int_t^x \frac{(x-s)^{i-1}}{(i-1)!} (x-t)^k (s-t)^{1-\alpha} ds dt$$

$$+A \sum_{k=0}^m \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_0^x \frac{(x-t)^{k+i+1}}{(i+1)!} dt = f_{(i)}(x), \quad i = 1, \dots, m, \quad (40)$$

or equivalently

$$v_{i0}(x)\psi(x) + v_{i1}(x)\psi'(x) + \dots + v_{im}(x)\psi^{(m)}(x) = f_{(i)}(x), \quad i = 1, \dots, m, \quad (41)$$

where

$$v_{ik}(x) = \frac{(-1)^k x^{k+i-\alpha+2}}{(k+i-\alpha+2)\Gamma(i+2-\alpha)k!} + \frac{(-1)^k A x^{k+i+2}}{k!(i+1)!(k+i+2)}, \quad k = 0, \dots, m. \quad (42)$$

Hence, a system of linear equations with respect to  $\psi(x), \dots, \psi^{(m)}(x)$  is constructed by equations (36) and (41). The system acquired is indicated by

$$\mathbf{V}(x)\Psi(x) = \mathbf{F}(x), \quad (43)$$

where

$$\mathbf{V}(x) = \begin{bmatrix} v_{00}(x) & v_{01}(x) & \dots & v_{0m}(x) \\ v_{10}(x) & v_{11}(x) & \dots & v_{1m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ v_{m0}(x) & v_{m1}(x) & \dots & v_{mm}(x) \end{bmatrix}, \quad (44)$$

$$\mathbf{F}(x) = \begin{bmatrix} f_{(0)}(x) \\ f_{(1)}(x) \\ \vdots \\ f_{(m)}(x) \end{bmatrix}, \quad \mathbf{Y}(x) = \begin{bmatrix} \psi(x) \\ \psi'(x) \\ \psi''(x) \\ \vdots \\ \psi^{(m)}(x) \end{bmatrix}. \quad (45)$$

Finally, by solving the resulting system an approximate solution of equation (11) is obtained.

#### 4 Error analysis

The error analysis of the proposed method is fully discussed in [49–61] and reviewed here for  $m$ th-order approximating solution of relaxation-oscillation equation (5).

The exact solution  $\psi(t)$  is supposed to be infinitely differentiable on the interval  $I$ ; thus we can expand  $\psi(t)$  as an uniformly convergent Taylor series in  $I$  as the following.

$$\psi(t) = \sum_{k=0}^{\infty} \psi^{(k)}(x) \frac{(t-x)^k}{k!}. \quad (46)$$

Applying the suggested technique outlined in Section 2, relaxation-oscillation equation (5) is converted into the following equivalent infinitely linear equations system with respect to unknowns  $\psi^{(k)}(x), k = 0, 1, \dots$ ,

$$\mathbf{V}\Psi = \mathbf{F}, \quad (47)$$

where

$$\mathbf{V} = \lim_{n \rightarrow \infty} \mathbf{V}_{nn}, \quad \mathbf{\Psi} = \lim_{n \rightarrow \infty} \mathbf{\Psi}_n, \quad \mathbf{F} = \lim_{n \rightarrow \infty} \mathbf{F}_n, \quad (48)$$



in which  $\mathbf{V}_{nn}$ ,  $\Psi_n$ , and  $\mathbf{F}_n$ , as demonstrated in the part before, are described as

$$\mathbf{V}_{nn} = [v_{ij}(x)]_{(n+1) \times (n+1)} \quad , \quad \Psi_n = [\psi^{(i)}(x)]_{(n+1) \times 1} \quad , \quad \mathbf{F}_n = [f_{(i)}(x)]_{(n+1) \times 1} . \quad (49)$$

Thus, under the solvability conditions of the obtained system (47) and putting  $\mathbf{B} = \mathbf{V}^{-1}$  the solution of this system is uniquely indicated as

$$\Psi = \mathbf{B}\mathbf{F} . \quad (50)$$

The relation (50) is rewritten in a matrix form as

$$\begin{bmatrix} \Psi_n \\ \Psi_\infty \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{nn} & \mathbf{B}_{n\infty} \\ \mathbf{B}_{\infty n} & \mathbf{B}_{\infty\infty} \end{bmatrix} \begin{bmatrix} \mathbf{F}_n \\ \mathbf{F}_\infty \end{bmatrix} . \quad (51)$$

Thus, it can be realized that the vector  $\Psi_n$  which is composed of the first  $n+1$  elements of the exact solution vector  $\Psi$  must satisfy the subsequent relation

$$\Psi_n = \mathbf{B}_{nn}\mathbf{F}_n + \mathbf{B}_{n\infty}\mathbf{F}_\infty . \quad (52)$$

Based on the suggested method in Section 3, the unique solution of system  $\mathbf{V}\Psi = \mathbf{F}$  is represented as

$$\tilde{\Psi}_n = \mathbf{V}_{nn}^{-1}\mathbf{F}_n , \quad (53)$$

where  $\tilde{\Psi}_n$  is a notation for approximate solution which is replaced by  $\Psi_n$  for convenience.

Subtracting (53) from (52) leads to

$$\Psi_n - \tilde{\Psi}_n = \mathbf{D}_{nn}\mathbf{F}_n + \mathbf{B}_{n\infty}\mathbf{F}_\infty , \quad (54)$$

where  $\mathbf{D}_{nn} = \mathbf{B}_{nn} - \mathbf{V}_{nn}^{-1}$ .

As we now extend the right side of equation (54), we can express the first element of the vector at the left side of (54) as

$$\psi(x) - \tilde{\psi}(x) = \sum_{j=0}^n d_{0,j}(x)f_{(j)}(x) + \sum_{j=n+1}^{\infty} b_{0,j}(x)f_{(j)}(x) , \quad (55)$$

where  $d_{i,j}(x)$  and  $b_{i,j}(x)$  are the elements of  $\mathbf{D}_{nn}$  and  $\mathbf{B}_{n\infty}$ , respectively. So, based on the Cauchy-Schwarz inequality the following relation is resulted

$$|\psi(x) - \tilde{\psi}(x)| \leq \left( \sum_{j=0}^n |d_{0,j}(x)|^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^n |f_{(j)}(x)|^2 \right)^{\frac{1}{2}} + \left( \sum_{j=n+1}^{\infty} |b_{0,j}(x)|^2 \right)^{\frac{1}{2}} \left( \sum_{j=n+1}^{\infty} |f_{(j)}(x)|^2 \right)^{\frac{1}{2}} . \quad (56)$$

It must be noted that as  $\lim_{n \rightarrow \infty} \mathbf{D}_{nn} = 0$  and  $\lim_{n \rightarrow \infty} \mathbf{B}_{n\infty} = 0$ , then  $\lim_{n \rightarrow \infty} |\psi(x) - \tilde{\psi}(x)| = 0$ .

## 5 Applications

This section is devoted to illustrate the efficiency of the technique for some specific applications of the fractional equations for which the exact solutions are known in advance and the procedure leads to an exact solution. Our obtained results are compared with some existing methods and superiority of our proposed method is confirmed.

### 5.1 Application 1.

Consider the following fractional relaxation-oscillation equation [15]

$$D^\alpha \psi(x) + \psi(x) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x, \quad (57)$$

with initial and boundary conditions given as

$$\psi(0) = 0, \quad 0 < \alpha \leq 1. \quad (58)$$

According to definition (10), for  $\alpha = \frac{1}{2}$ , equation (57) is rewritten as

$$\frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_0^x \frac{\psi(t)}{\sqrt{x-t}} dt + \psi(x) = \frac{8x^{\frac{3}{2}}}{3\sqrt{\pi}} - \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} + x^2 - x. \quad (59)$$

Using the proposed process in this paper, we integrate both sides of equation (59) and obtain the following Volterra integral equation

$$\frac{1}{\sqrt{\pi}} \int_0^x \frac{\psi(t)}{\sqrt{x-t}} dt + \int_0^x \psi(t) dt = \frac{16x^{\frac{5}{2}}}{15\sqrt{\pi}} - \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{x^3}{3} - \frac{x^2}{2}, \quad (60)$$

which is solved by the proposed method. Using the first-order Taylor expansion

$$\psi(t) \approx \psi(x) + \psi'(x)(t-x), \quad (61)$$

we have the following first-order approximate solution as

$$\psi_1(x) = \frac{x(175\pi x(5x-6) + 256(32x-35) + 60\sqrt{\pi}\sqrt{x}(87x-98))}{70(128 + 84\sqrt{\pi}\sqrt{x} + 15\pi x)}. \quad (62)$$

Moreover, if the second-order Taylor expansion

$$\psi(t) \approx \psi(x) + \psi'(x)(t-x) + \frac{1}{2}\psi''(x)(t-x)^2, \quad (63)$$

is applied, results in the second-order approximate solution as

$$\psi_2(x) = x^2 - x, \quad (64)$$

which is the exact solution. It is worth mentioning that the Mathematica command 'LinearSolve' is employed to solve the resulting system after transforming equation (57) into a system of linear equations.

The absolute errors between the exact solution, its approximations, and the results from [15] are presented in Table 1. As seen in the table, the accuracy of our results is highly satisfactory. The second-order approximate solution matches the exact solution, as anticipated, since an  $m$ th-order approximation will yield the exact solution when the exact solution is a polynomial of degree up to  $m$ .

**Table 1** Absolute errors of application 1 for  $\alpha = \frac{1}{2}$ .

| $x$ | Method in [15]                  | Suggested method         | Suggested method |
|-----|---------------------------------|--------------------------|------------------|
|     | $ \psi_{Exact} - \psi_{Euler} $ | $m = 1$                  | $m = 2$          |
| 0.1 | $1.96 \times 10^{-4}$           | $9.47835 \times 10^{-4}$ | 0                |
| 0.2 | $2.12 \times 10^{-4}$           | $3.92452 \times 10^{-3}$ | 0                |
| 0.3 | $1.93 \times 10^{-4}$           | $9.04525 \times 10^{-3}$ | 0                |
| 0.4 | $1.58 \times 10^{-4}$           | $1.63858 \times 10^{-2}$ | 0                |
| 0.5 | $1.07 \times 10^{-4}$           | $2.60039 \times 10^{-2}$ | 0                |
| 0.6 | $4.70 \times 10^{-5}$           | $3.79465 \times 10^{-2}$ | 0                |
| 0.7 | $1.90 \times 10^{-5}$           | $5.22532 \times 10^{-2}$ | 0                |
| 0.8 | $9.10 \times 10^{-5}$           | $6.89581 \times 10^{-2}$ | 0                |
| 0.9 | $1.68 \times 10^{-4}$           | $8.80911 \times 10^{-2}$ | 0                |
| 1.0 | $2.49 \times 10^{-4}$           | $1.09679 \times 10^{-1}$ | 0                |

This application was addressed in [15] and resolved by fractional Euler's method as a generalization of the classical Euler's method with  $h = 0.001$ .

## 5.2 Application 2.

Consider the following Bagley-Torvik equation [16, 17]

$$\psi''(x) + D^{\frac{3}{2}}\psi(x) + \psi(x) = 2 + 4\sqrt{\frac{x}{\pi}} + x^2, \quad (65)$$

with initial conditions

$$\psi(0) = 0, \quad \psi'(0) = 0. \quad (66)$$

The exact solution of the equation is  $\psi(x) = x^2$ .

According to definition (10), for  $\alpha = \frac{3}{2}$ , equation (65) is rewritten as

$$\psi''(x) + \frac{1}{\sqrt{\pi}} \frac{d^2}{dx^2} \int_0^x \frac{\psi(t)}{\sqrt{x-t}} dt + \psi(x) = 2 + 4\sqrt{\frac{x}{\pi}} + x^2. \quad (67)$$

Based on the proposed method, by integrating both sides of equation (67) twice, we have

$$\psi(x) + \frac{1}{\sqrt{\pi}} \int_0^x \frac{\psi(t)}{\sqrt{x-t}} dt + \int_0^x (x-t)\psi(t) dt = x^2 + \frac{16x^{\frac{5}{2}}}{15\sqrt{\pi}} + \frac{x^4}{12}. \quad (68)$$

Consequently, to solve the obtained integral equation we apply the first-order Taylor expansion

$$\psi(t) \approx \psi(x) + \psi'(x)(t-x), \quad (69)$$

which results in the following first-order approximate solution as

$$\psi_1(x) = \frac{25200\pi x^2 + 56000\sqrt{\pi}x^{\frac{5}{2}} + 32768x^3 + 2800\pi x^4 + 4400\sqrt{\pi}x^{\frac{9}{2}} + 245\pi x^6}{70(360\pi + 816\sqrt{\pi}\sqrt{x} + 512x + 30\pi x^2 + 68\sqrt{\pi}x^{\frac{5}{2}} + 5\pi x^4)}, \quad (70)$$

and if the second-order Taylor expansion

$$\psi(t) \approx \psi(x) + \psi'(x)(t-x) + \frac{1}{2}\psi''(x)(t-x)^2, \quad (71)$$

is applied, results in the second-order approximate solution as

$$\psi_2(x) = x^2, \quad (72)$$

which is the exact solution.

The obtained absolute errors between the exact values and its approximate solutions at equidistant points in  $[0,1]$  are shown in Table 2. From Table 2, one can find that the accuracy of obtained results is quite satisfactory and the second-order approximation yields the exact solution, as anticipated.

**Table 2** Absolute errors of application 2 for  $\alpha = \frac{3}{2}$ .

| $x$ | $m = 1$                  | $m = 2$ |
|-----|--------------------------|---------|
| 0.1 | $7.96634 \times 10^{-5}$ | 0       |
| 0.2 | $4.32073 \times 10^{-4}$ | 0       |
| 0.3 | $1.13436 \times 10^{-3}$ | 0       |
| 0.4 | $2.21276 \times 10^{-3}$ | 0       |
| 0.5 | $3.67007 \times 10^{-3}$ | 0       |
| 0.6 | $5.49867 \times 10^{-3}$ | 0       |
| 0.7 | $7.68955 \times 10^{-3}$ | 0       |
| 0.8 | $1.02393 \times 10^{-2}$ | 0       |
| 0.9 | $1.31560 \times 10^{-2}$ | 0       |
| 1.0 | $1.64640 \times 10^{-2}$ | 0       |

This application was solved using the collocation method and the collocation-shooting approach, respectively, in [16, 17]. In Table 3, we display the highest absolute errors from Refs. [16, 17].

**Table 3** The maximum of the absolute errors in [16, 17].

| Method in [16]        | Method in [17]         |
|-----------------------|------------------------|
| $0.44 \times 10^{-7}$ | $7.22 \times 10^{-12}$ |

### 5.3 Application 3.

Consider the following fractional relaxation-oscillation equation [18]

$$D^\alpha \psi(x) + 3\psi(x) = 3x^3 + \frac{8}{\Gamma(0.5)}x^{1.5}, \quad (73)$$

with initial conditions

$$\psi(0) = 0, \quad \psi'(0) = 0. \quad (74)$$

For  $\alpha = 1.5$ , the precise solution is  $\psi(x) = x^3$ . By using  $m = 1, 2, 3$ , we may derive the approximations based on the suggested method. Table 4 contains a list of the obtained absolute errors. Furthermore, the results obtained in [18] are shown in Table 5. From Table 4, We note that, as anticipated, the third-order approximate solution provides the precise solution, and the precision of our results is quite good.

**Table 4** Absolute errors of application 3 for  $\alpha = 1.5$ .

| $x$ | $m = 1$                  | $m = 2$                  | $m = 3$ |
|-----|--------------------------|--------------------------|---------|
| 0.1 | $1.61568 \times 10^{-4}$ | $2.16231 \times 10^{-5}$ | 0       |
| 0.2 | $1.29155 \times 10^{-3}$ | $1.72617 \times 10^{-4}$ | 0       |
| 0.3 | $4.37732 \times 10^{-3}$ | $5.80789 \times 10^{-4}$ | 0       |
| 0.4 | $1.04941 \times 10^{-2}$ | $1.37156 \times 10^{-3}$ | 0       |
| 0.5 | $2.08974 \times 10^{-2}$ | $2.66859 \times 10^{-3}$ | 0       |
| 0.6 | $3.71081 \times 10^{-2}$ | $4.59691 \times 10^{-3}$ | 0       |
| 0.7 | $6.09707 \times 10^{-2}$ | $7.28893 \times 10^{-3}$ | 0       |
| 0.8 | $9.46758 \times 10^{-2}$ | $1.08933 \times 10^{-2}$ | 0       |
| 0.9 | $1.40749 \times 10^{-1}$ | $1.55858 \times 10^{-2}$ | 0       |
| 1.0 | $2.02012 \times 10^{-1}$ | $2.15812 \times 10^{-2}$ | 0       |

**Table 5** Absolute errors of application 3 for  $\alpha = 1.5$  in [18].

| $x$ | $N = 3$                 | $N = 5$                   | $N = 7$                   | $N = 9$                   |
|-----|-------------------------|---------------------------|---------------------------|---------------------------|
| 0.1 | $2.2655 \times 10^{-5}$ | $8.68960 \times 10^{-10}$ | $1.67093 \times 10^{-10}$ | $3.65474 \times 10^{-14}$ |
| 0.3 | $3.4118 \times 10^{-4}$ | $1.60361 \times 10^{-6}$  | $1.67093 \times 10^{-10}$ | $4.21589 \times 10^{-14}$ |
| 0.5 | $6.3397 \times 10^{-4}$ | $9.63817 \times 10^{-7}$  | $1.06212 \times 10^{-9}$  | $9.14555 \times 10^{-13}$ |
| 0.7 | $1.6705 \times 10^{-4}$ | $1.71558 \times 10^{-7}$  | $7.03957 \times 10^{-10}$ | $8.21475 \times 10^{-14}$ |
| 0.9 | $5.5043 \times 10^{-4}$ | $8.98678 \times 10^{-7}$  | $1.92731 \times 10^{-10}$ | $1.32512 \times 10^{-13}$ |

This application was used in [18] and has been solved based on the first kind of Bessel functions collocation method.

#### 5.4 Application 4.

Consider the fractional differential equation [18]

$$\psi''(x) - 2\psi'(x) + D^\alpha \psi(x) + \psi(x) = x^3 - 6x^2 + 6x + \frac{16}{5\sqrt{\pi}}x^{2.5}, \quad (75)$$

with initial conditions

$$\psi(0) = 0, \quad \psi'(0) = 0. \quad (76)$$

The exact solution is  $\psi(x) = x^3$  for  $\alpha = 0.5$ . Using the process described in Section 3, we determine the approximate solutions of equation (75) by putting  $m = 1, 2, 3$ . As expected, the present approximate result is

identical to the exact value when  $m = 3$ , since the true solution  $\psi(x)$  is a polynomial function of degree 3. The obtained absolute errors are listed in Table 6. Furthermore, the results obtained in [18] are shown in Table 7.

**Table 6** Absolute errors of application 4 for  $\alpha = 0.5$ .

| $x$ | $m = 1$                  | $m = 2$                  | $m = 3$ |
|-----|--------------------------|--------------------------|---------|
| 0.1 | $1.83884 \times 10^{-7}$ | $1.67374 \times 10^{-8}$ | 0       |
| 0.2 | $8.57564 \times 10^{-7}$ | $2.45555 \times 10^{-7}$ | 0       |
| 0.3 | $1.52024 \times 10^{-5}$ | $5.48380 \times 10^{-7}$ | 0       |
| 0.4 | $4.06132 \times 10^{-5}$ | $2.33971 \times 10^{-6}$ | 0       |
| 0.5 | $4.92393 \times 10^{-5}$ | $2.08813 \times 10^{-5}$ | 0       |
| 0.6 | $7.41184 \times 10^{-4}$ | $8.73476 \times 10^{-5}$ | 0       |
| 0.7 | $3.30409 \times 10^{-3}$ | $2.70423 \times 10^{-4}$ | 0       |
| 0.8 | $1.04898 \times 10^{-2}$ | $6.99303 \times 10^{-4}$ | 0       |
| 0.9 | $2.76795 \times 10^{-2}$ | $1.59673 \times 10^{-3}$ | 0       |
| 1.0 | $6.47880 \times 10^{-2}$ | $3.32226 \times 10^{-3}$ | 0       |

**Table 7** Absolute errors of application 4 for  $\alpha = 0.5$  in [18].

| $x$ | $N = 3$                 | $N = 5$                  | $N = 7$                    |
|-----|-------------------------|--------------------------|----------------------------|
| 0.1 | $1.8769 \times 10^{-4}$ | $2.98543 \times 10^{-7}$ | $8.029610 \times 10^{-10}$ |
| 0.3 | $9.7022 \times 10^{-4}$ | $3.84222 \times 10^{-6}$ | $3.057123 \times 10^{-9}$  |
| 0.5 | $2.6184 \times 10^{-3}$ | $7.27991 \times 10^{-6}$ | $6.018712 \times 10^{-9}$  |
| 0.7 | $3.4247 \times 10^{-3}$ | $1.24040 \times 10^{-6}$ | $9.156311 \times 10^{-9}$  |
| 0.9 | $6.9465 \times 10^{-3}$ | $1.75728 \times 10^{-5}$ | $9.754413 \times 10^{-9}$  |

This application was used in [18] and has been solved based on the first kind of Bessel functions collocation method.

## 5.5 Application 5.

Consider the following fractional differential equation [19]

$$\psi''(x) + 0.5D^{0.5}\psi(x) + \psi(x) = 2 + x \left( \frac{1}{\Gamma(2.5)}x^{0.5} + x \right), \quad (77)$$

with initial and boundary conditions given as

$$\psi(0) = 0, \quad \psi'(0) = 0, \quad (78)$$

where the exact solution is  $\psi(x) = x^2$ . Employing the approach described in Section 3, we determine the approximate solutions of equation (77) by putting  $m = 1, 2$ . The obtained absolute errors and the results obtained in [19] are listed in Table 8. It is evident that our findings are quite satisfying and that, as predicted, the exact answer is obtained from the second-order approximate solution.

**Table 8** Absolute errors of application 5 for  $\alpha = \frac{1}{2}$ .

| $x$   | Method in [19]        | Suggested method $m = 1$ | Suggested method $m = 2$ |
|-------|-----------------------|--------------------------|--------------------------|
| 0.125 | $2.78 \times 10^{-3}$ | $1.39 \times 10^{-5}$    | 0                        |
| 0.250 | $8.95 \times 10^{-3}$ | $1.81 \times 10^{-4}$    | 0                        |
| 0.375 | $2.06 \times 10^{-3}$ | $7.90 \times 10^{-4}$    | 0                        |
| 0.500 | $3.81 \times 10^{-3}$ | $2.21 \times 10^{-3}$    | 0                        |
| 0.625 | $3.79 \times 10^{-3}$ | $4.72 \times 10^{-3}$    | 0                        |
| 0.750 | $8.62 \times 10^{-3}$ | $8.50 \times 10^{-3}$    | 0                        |
| 0.875 | $7.19 \times 10^{-3}$ | $1.34 \times 10^{-2}$    | 0                        |
| 1.000 | $2.90 \times 10^{-3}$ | $1.92 \times 10^{-2}$    | 0                        |

The cubic splines method was used to solve this case, which was mentioned in [19].

### 5.6 Application 6.

Consider the following fractional relaxation-oscillation equation [20]

$$D^\alpha \psi(x) = \frac{8}{3} \sqrt{\frac{x^3}{\pi}} - 2 \sqrt{\frac{x}{\pi}}, \quad (79)$$

with initial and boundary conditions given as

$$\psi(0) = 0, \quad 0 < \alpha \leq 1. \quad (80)$$

When  $\alpha = 0.5$ , the exact solution is  $\psi(x) = x^2 - x$ . The approximate solutions of equation (79) are found in the following by setting  $m = 1, 2$  and applying the procedure outlined in Section 3. Table 9 lists the absolute errors that were acquired as well as the findings from [20]. It is evident that our results are quite satisfactory and that, as predicted, the exact answer is obtained from the second-order approximate solution.

**Table 9** Absolute errors of application 6 for  $\alpha = \frac{1}{2}$ .

| $x$ | (Method in [20]) $ \psi_{Exact} - \psi_{FFDM} $ | Suggested method $m = 1$ | Suggested method $m = 2$ |
|-----|-------------------------------------------------|--------------------------|--------------------------|
| 0.1 | $1.49 \times 10^{-4}$                           | $8.57143 \times 10^{-4}$ | 0                        |
| 0.2 | $2.19 \times 10^{-4}$                           | $3.42857 \times 10^{-3}$ | 0                        |
| 0.3 | $2.72 \times 10^{-4}$                           | $7.71429 \times 10^{-3}$ | 0                        |
| 0.4 | $3.17 \times 10^{-4}$                           | $1.37143 \times 10^{-2}$ | 0                        |
| 0.5 | $3.57 \times 10^{-4}$                           | $2.14286 \times 10^{-2}$ | 0                        |
| 0.6 | $3.93 \times 10^{-4}$                           | $3.08571 \times 10^{-2}$ | 0                        |
| 0.7 | $4.26 \times 10^{-4}$                           | $4.20000 \times 10^{-2}$ | 0                        |
| 0.8 | $4.56 \times 10^{-4}$                           | $5.48571 \times 10^{-2}$ | 0                        |
| 0.9 | $4.85 \times 10^{-4}$                           | $6.94286 \times 10^{-2}$ | 0                        |
| 1.0 | $5.12 \times 10^{-4}$                           | $8.57143 \times 10^{-2}$ | 0                        |

This application is given in [20] and solved by fractional finite difference method (FFDM) with  $h = 0.01$ .

### 5.7 Application 7.

Consider the following fractional relaxation-oscillation equation [20]

$$D^\alpha \psi(x) + \psi(x) = x^2 + \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}, \quad (81)$$

with initial and boundary conditions given as

$$\psi(0) = 0, \quad 0 < \alpha \leq 1. \quad (82)$$

For  $\alpha = 0.5$ , the precise solution is  $\psi(x) = x^2$ . By putting  $m = 1, 2$ , we apply the procedure outlined in this study to derive the approximate solutions of equation (81). Table 10 lists the absolute errors that were acquired as well as the findings from [20]. It is evident that, as would be expected, the exact answer comes from the second-order approximate solution.

**Table 10** Absolute errors of application 7 for  $\alpha = \frac{1}{2}$ .

| $x$ | (Method in [20]) $ \psi_{Exact} - \psi_{FFDM} $ | Suggested method $m = 1$ | Suggested method $m = 2$ |
|-----|-------------------------------------------------|--------------------------|--------------------------|
| 0.1 | $1.16 \times 10^{-4}$                           | $9.47835 \times 10^{-4}$ | 0                        |
| 0.2 | $1.56 \times 10^{-4}$                           | $3.92452 \times 10^{-3}$ | 0                        |
| 0.3 | $1.81 \times 10^{-4}$                           | $9.04525 \times 10^{-3}$ | 0                        |
| 0.4 | $2.00 \times 10^{-4}$                           | $1.63858 \times 10^{-2}$ | 0                        |
| 0.5 | $2.15 \times 10^{-4}$                           | $2.60039 \times 10^{-2}$ | 0                        |
| 0.6 | $2.27 \times 10^{-4}$                           | $3.79465 \times 10^{-2}$ | 0                        |
| 0.7 | $2.37 \times 10^{-4}$                           | $5.22532 \times 10^{-2}$ | 0                        |
| 0.8 | $2.46 \times 10^{-4}$                           | $6.89581 \times 10^{-2}$ | 0                        |
| 0.9 | $2.54 \times 10^{-4}$                           | $8.80911 \times 10^{-2}$ | 0                        |
| 1.0 | $2.61 \times 10^{-4}$                           | $1.09679 \times 10^{-1}$ | 0                        |

This application is given in [20] and solved by the fractional finite difference method (FFDM) with  $h = 0.01$ .

### 5.8 Application 8.

Consider the following fractional relaxation-oscillation equation [21]

$$D^{\frac{1}{2}} \psi(x) + \psi(x) = \sqrt{x} + \frac{\sqrt{\pi}}{2}, \quad \psi(0) = 0. \quad (83)$$

The exact solution of this problem is  $\psi(x) = \sqrt{x}$ . We employ the proposed method in this paper to obtain the approximate solutions of equation (83) by setting  $m = 5, \dots, 10$ . The maximum of the absolute errors is shown on  $[0, 1]$  in Table 11. This problem has been solved in [21] by using Legendre-collocation scheme that was considered in [22] and we present the results in Table 12.



**Table 11** The maximum of the absolute errors for application 8.

| m  | Maximal Errors           |
|----|--------------------------|
| 5  | $2.27804 \times 10^{-3}$ |
| 6  | $1.51433 \times 10^{-3}$ |
| 7  | $1.06370 \times 10^{-3}$ |
| 8  | $7.79340 \times 10^{-4}$ |
| 9  | $5.90323 \times 10^{-4}$ |
| 10 | $4.59348 \times 10^{-4}$ |

**Table 12** Approximation errors of Legendre-collocation scheme proposed in [22] for application 8.

| N  | Maximal Errors |
|----|----------------|
| 5  | 0.0221         |
| 10 | 0.0067         |
| 15 | 0.0033         |
| 20 | 0.0019         |

## 6 Conclusion

In this paper, we introduced an approximate method to solve the fractional relaxation-oscillation equation. The proposed approach transformed the relaxation-oscillation equation into a Volterra integral equation. By applying the Taylor expansion for the unknown function and repeated integration method, the integral equation was reduced to a linear system of equations involving the unknown function and its derivatives. The resulting system was then solved using a standard method. The main contribution of the proposed technique is that the approximation of order  $m$  is exact if the exact solution is a polynomial of degree up to  $m$ .

## 7 Declarations

### 7.1 Conflicts of interest:

The authors declare no conflicts of interest.

### 7.2 Funding:

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### 7.3 Author's contribution:

M.D.-Conceptualization, Methodology, Software, Writing-Original Draft. F.E.-Investigation, Writing, Review and Editing. All authors read and approved the final submitted version of this manuscript.

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#### 7.5 Data availability statement:

All data that support the findings of this study are included within the article.

#### 7.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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