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Original Study

Application of a generalized heat equation to ultra-fast processes in viscoanelastic isotropic medium

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Abstract

Using a classical irreversible thermodynamics of internal variables (CIT-IV) with V.Ciancio's procedure, viscous-inelastic flow relations have been derived by generalizing the Duhamel-Neumann law for ordinary thermoelastic phenomena in isotropic media and the relations for elastic media and for Maxwell, Jeffreys and Poynting-Thomson bodies. In literature, Fourier and Maxwell-Cattaneo-Vernotte's(MCV) heat transport equations are based on hypothesis that lack physical confirmation while the V.Ciancio model obtains both mathematical and physical applications. In this context, through the simple description of the ultra-fast process of energy transmission from a laser source to metal film, using the principles of thermodynamics, the heat equation is derived in the case of isotropic viscoanelastic media subject to constant strain. The solution, obtained numerically with the finite element method, not only highlights the physical significance of the phenomenological coefficients, but also specifies the limits of the previous theories of the MCV.

Keywords: Non equilibrium Termodynamic, heat flow, viscoanelastic media.
AMS 2020 codes: 80A17; 74A15; 74C05; 35K05; 35L05.

1 Introduction

The classical irreversible thermodynamic with internal variables (CIT-IV) means a general approach to the study of different interactions among irreversible thermodynamic processes using classical extensive variables [1]. The flexibility of the methodologies used in CIT-IV consist in the fact that *"a priori"* the physical meaning of the internal thermodynamical variables is not specified but only their influence on particular types of occurring phenomena in the considered medium is assumed. For this reason, in the following, we will call these thermodynamical variables: *dynamical variables* since they have also been used, successfully, for the study of problems in dielectric and magnetic relaxation phenomena [2–5], diffusion [6] and anelastic deformation [7–15]. It should be noted that the many theoretical results have been confirmed by the experimental data [16–23]. For completeness other applications address the heat equation controllability problems using fractional calculation [24, 25].

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In order to study the behavior of rheological media, in the presence of viscous and anelastic phenomena, besides determining the stress-strain relationships it is important to obtain the heat propagation equation. For this purpose [1] the tensor $\varepsilon^{(1)}_{\alpha\beta}$ (inelastic strain) and the vector $\boldsymbol{\xi}$ are introduced as internal variables which characterize both stress-strain relationships and which generalize the Fourier heat equation (parabolic type equation) [26] and Maxwell-Cattaneo-Vernotte (hyperbolic type equation) [27–29].

The rest of this paper is organized by the following parts explained in detail.

In Section 2, we summarize the research of [1], introducing a vectorial dynamical variable j and a stress field $\tau^{(eq)}_{\alpha\beta}$ which has a thermoanelastic nature, an explicit form for the entropy production is derived and phenomenological equations, which generalize the Kelvin, Jeffreys and Poynting-Thomson bodies are obtained. By virtue of the dynamical variable, influencing the thermal transport phenomena, the heat equation is obtained. In particular, in the isotropic case, when the medium has symmetry properties that, under orthogonal transformations, are invariant with respect to all rotations and inversions of the frame of axes, it was obtained that the heat flux can be split in two parts: a first contribution $J^{(0)}$, governed by Fourier law and a second contribution $J^{(1)}$, obeying Maxwell-Cattaneo-Vernotte equation (M-C-V) [27–29] in which a relaxation time is present.

In Section 3, we obtain a general heat equation that generalizes the analogous equations of Fourier and M-C-V, from the phenomenological coefficients associated with the characteristic variables of the visco-anelastic medium. We observe the presence of two relaxation times, t_1 associated with the heat flux of free electrons heated by a beam of pulsed laser or femto-laser, while t_2 associated with the coupling of electron and phonon. The internal variable $\boldsymbol{\xi}$ characterizes the coupling between electron and phonon through cross phenomenological coefficients and therefore leads to the microscopic behavior and thermal properties of material under exame.

In Section 4, we describe the behavior of a thin metal film, from a physical point of view, to the heat transfer due to a laser source [30]. Considering the heat transfer on metals, without defects or for which it can reasonably be considered isotropic, due to a laser source with pulse duration proxime to zero, case 1-D can be analysed both from an analytical and numerical point of view [31]. By resorting to experimental values and using the method of internal variables, a typical approach of physics-mathematics, results are obtained in accordance with physical behavior

In Section 5, by means of the dimensional analysis of the quantities involved in the general heat equation, we arrive at a dimensionless form, incorporating the effects of any external sources in the flow due to the internal variable. This equation is used to describe the behavior of a thin metal to the heat transfer due to a laser source. The values of the phenomenological coefficients depend on the material considered.

In Section 6, using a procedure similar to Section 5, we obtain the dimensionless form by specifying the amount of heat absorbed by a gold film and evaluating the temperature distribution during the transitional phase of heat transfer and achievement of the thermodynamic equilibrium which however implies the production of entropy. Section 6 shows the results obtained.

Section 7 highlights the importance of the method of internal variables using the principles of thermodynamics. For this reason, a direct description of the physical phenomena associated with the transfer of energy in the form of heat in relation to the medium in question is evident without the need to resort to ad hoc hypotheses.

Finally, Section 8, concludes the paper by giving the main contribution of this paper.

2 Method and material

The simulations are obtained by applying the finite element method. We used the scientific software COMSOL version 5.6, and in particular its MUltifrontal Massively Parallel Sparse direct solver (MUMPS) with a Newton automatic termination method. For more details, see references [32–38]. The hardware used is a processor Intel(R) Core(TM) i7-8550U CPU @ 1.80GHz, 1.99 GHz, RAM 16 GB processor with paralell programming up to eight threads.

3 The survey of Ciancio's theory

In the context of irreversible processes an important role is played by the flow of heat which, classically, is not considered to be a state variable. Therefore we will suppose that the specific entropy s, depends not only on the specific internal energy u and the total strain $\varepsilon_{\alpha\beta}$, the tensor $\varepsilon^{(1)}_{\alpha\beta}$ describing the inelastic strain and also on a vectorial dynamic variable, $\boldsymbol{\xi}$, that is an odd function of microscopic particles velocities that have influence on the propagation phenomena which occur in an isotropic medium.

3.1 The balance equation of entropy

In the contest of irreversible process an important role is played by the flow of heat which, classically, is not considered to be a state variable. The specific entropy s, depends not only on the specific internal energy u and total strain $\varepsilon_{\alpha\beta}$, the tensor $\varepsilon^{(1)}_{\alpha\beta}$ describing the inelastic strain and also on a vectorial dynamic variable, $\boldsymbol{\xi}$, that is an odd function of microscopic particles velocities that have influence on the propagation phenomena which occur in the medium.

$$s = s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{1}$$

where ξ_α $(\alpha = 1,2,3)$ is the α-component of the vector $\boldsymbol{\xi}$.
We define the *absolute temperature*

$$T^{-1} \stackrel{\text{def}}{=} \frac{\partial}{\partial u} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{2}$$

the *equilibrium-stress tensor*

$$\tau^{(eq)}_{\alpha\beta} \stackrel{\text{def}}{=} -\rho T \frac{\partial}{\partial \varepsilon_{\alpha\beta}} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{3}$$

the *affinity-stress* conjugate to $\varepsilon^{(1)}_{\alpha\beta}$

$$\tau^{(1)}_{\alpha\beta} \stackrel{\text{def}}{=} \rho T \frac{\partial}{\partial \varepsilon^{(1)}_{\alpha\beta}} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{4}$$

and the vector j conjugate to the internal vector variable $\boldsymbol{\xi}$

$$j_\alpha \stackrel{\text{def}}{=} \rho T \frac{\partial}{\partial \xi_\alpha} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{5}$$

where $\rho \stackrel{\text{def}}{=} v^{-1}$ is the mass density.
By using (1) we can obtain the following Gibbs relation

$$T ds = du - v\,\tau^{(eq)}_{\alpha\beta} d\varepsilon_{\alpha\beta} + v\,\tau^{(1)}_{\alpha\beta} d\varepsilon^{(1)}_{\alpha\beta} + v\, j_\alpha\, d\xi_\alpha\,, \tag{6}$$

where v is the specific volume.
From (6) we have:

$$\rho T \frac{ds}{dt} = \rho \frac{du}{dt} - \tau^{(eq)}_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} + \tau^{(1)}_{\alpha\beta} \frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} + j_\alpha \frac{d\xi_\alpha}{dt}\,, \tag{7}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla \tag{8}$$

is the substantial derivative with respect to time and v is the velocity field and ∇ is the gradient.

To analyze phenomena due to viscous flows (analogous to those which occur during flows in ordinary viscous liquids and gases) we introduce the following viscous stress tensor

$$\tau_{\alpha\beta}^{(vi)} \stackrel{\text{def}}{=} \tau_{\alpha\beta} - \tau_{\alpha\beta}^{(eq)}, \tag{9}$$

where $\tau_{\alpha\beta}$ is the mechanical stress tensor which occurs in the equation of motion

$$\rho \frac{dv_\alpha}{dt} = \rho F_\alpha + \frac{\partial \tau_{\alpha\beta}}{\partial x_\beta}, \tag{10}$$

and in the first law of thermodynamics

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{J}^{(q)} + \tau_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt}. \tag{11}$$

In (10), the force F_α is the volume force per unit of mass and in (11) the vector $\mathbf{J}^{(q)}$ is the heat flux. The equation (11), by virtue of (9) becomes

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{J}^{(q)} + \left(\tau_{\alpha\beta}^{(eq)} + \tau_{\alpha\beta}^{(vi)}\right) \frac{d\varepsilon_{\alpha\beta}}{dt}, \tag{12}$$

which substituted in the equation (7) gives

$$\rho \frac{ds}{dt} = T^{-1} \left(-\nabla \cdot \mathbf{J}^{(q)} + \tau_{\alpha\beta}^{(vi)} \frac{d\varepsilon_{\alpha\beta}}{dt} + \tau_{\alpha\beta}^{(1)} \frac{d\varepsilon_{\alpha\beta}^{(1)}}{dt} + j_\alpha \frac{d\xi_\alpha}{dt}\right), \tag{13}$$

and we have

$$\rho \frac{ds}{dt} = -\text{div}\left(\frac{\mathbf{J}^{(q)}}{T}\right) + \sigma^{(s)} \tag{14}$$

and the entropy production

$$\sigma^{(s)} = T^{-1}\left[\mathbf{J}^{(q)} \cdot \left(-T^{-1} \text{grad}\, T\right) + \tau_{\alpha\beta}^{(vi)} \frac{d\,\varepsilon_{\alpha\beta}}{dt} + \tau_{\alpha\beta}^{(1)} \frac{d\varepsilon_{\alpha\beta}^{(1)}}{dt} + j_\alpha \frac{d\xi_\alpha}{dt}\right] \geq 0. \tag{15}$$

3.2 Phenomenological equations

According to the usual procedure of non-equilibrium thermodynamics, by virtue of the form (15) for the entropy production, we have for anisotropic media the following phenomenological equations:

$$J_\alpha^{(q)} = L_{\alpha\beta}^{(q)(q)} \left(-T^{-1} \frac{\partial T}{\partial x^\beta}\right) + L_{\alpha(\mu\nu)}^{(q)(0)} \frac{d\varepsilon_{\mu\nu}}{dt} + L_{\alpha(\mu\nu)}^{(q)(1)} \tau_{\mu\nu}^{(1)} + L_{\alpha\mu}^{(q)(\xi)} \frac{d\xi_\mu}{dt}, \tag{16}$$

$$\tau_{\alpha\beta}^{(vi)} = L_{(\alpha\beta)\mu}^{(0)(q)} \left(-T^{-1} \frac{\partial T}{\partial x^\mu}\right) + L_{(\alpha\beta)(\mu\nu)}^{(0)(0)} \frac{d\varepsilon_{\mu\nu}}{dt} + L_{(\alpha\beta)(\mu\nu)}^{(0)(1)} \tau_{\mu\nu}^{(1)} + L_{(\alpha\beta)\mu}^{(0)(\xi)} \frac{d\xi_\mu}{dt}, \tag{17}$$

$$\frac{d\varepsilon_{\alpha\beta}^{(1)}}{dt} = L_{(\alpha\beta)\mu}^{(1)(q)} \left(-T^{-1} \frac{\partial T}{\partial x^\mu}\right) + L_{(\alpha\beta)(\mu\nu)}^{(1)(0)} \frac{d\varepsilon_{\mu\nu}}{dt} + L_{(\alpha\beta)(\mu\nu)}^{(1)(1)} \tau_{\mu\nu}^{(1)} + L_{(\alpha\beta)\mu}^{(1)(\xi)} \frac{d\xi_\mu}{dt}, \tag{18}$$

$$j_\alpha = L_{\alpha\beta}^{(\xi)(q)} \left(-T^{-1} \frac{\partial T}{\partial x^\beta}\right) + L_{\alpha(\mu\nu)}^{(\xi)(0)} \frac{d\varepsilon_{\mu\nu}}{dt} + L_{\alpha(\mu\nu)}^{(\xi)(1)} \tau_{\mu\nu}^{(1)} + L_{\alpha\beta}^{(\xi)(\xi)} \frac{d\xi_\beta}{dt}. \tag{19}$$

The tensors L are called phenomenological tensors and the indices of these tensors enclosed in round brackets mean that they are symmetrical because the tensors $\varepsilon_{\mu\nu}$, $\tau_{\alpha\beta}^{(vi)}$ are symmetric. The first of these equations may be regarded as a generalization of Fourier's law. Equation (17) describes the viscous flow phenomenon and it may be considered to be a generalization of Stockes-Navier's law. Finally, the equations (18) and (19) are the phenomenological equations for the irreversible process of the dynamic degrees of freedom. By (16)-(19) from the (15) one has:

$$\begin{aligned} T\sigma^{(s)} =& L^{(q)(q)}_{\alpha\beta}\left(T^{-2}\frac{\partial T}{\partial x^{\alpha}}\frac{\partial T}{\partial x^{\beta}}\right)+\left[L^{(\xi)(q)}_{\alpha\beta}+L^{(q)(\xi)}_{\alpha\beta}\right]\frac{d\xi_{\alpha}}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^{\beta}}\right)+L^{(0)(0)}_{(\alpha\beta)(\mu\nu)}\frac{d\varepsilon_{\alpha\beta}}{dt}\frac{d\varepsilon_{\mu\nu}}{dt}+ \\ & \left[L^{(1)(0)}_{(\alpha\beta)(\mu\nu)}+L^{(0)(1)}_{(\mu\nu)(\alpha\beta)}\right]\tau^{(1)}_{\alpha\beta}\frac{d\varepsilon_{\mu\nu}}{dt}+L^{(1)(1)}_{(\alpha\beta)(\mu\nu)}\tau^{(1)}_{\alpha\beta}\,\tau^{(1)}_{\mu\nu}+L^{(\xi)(\xi)}_{\alpha\beta}\frac{d\xi_{\alpha}}{dt}\frac{d\xi_{\beta}}{dt}\,. \end{aligned} \tag{20}$$

If the phenomenologic tensors of rank three are null, i.e. in the case of the constant volume. If the media are isotropic the equations (16)-(19) become:

$$J^{(q)}_{\alpha} = -\lambda^{(q,q)}\frac{\partial T}{\partial x^{\alpha}}+\lambda^{(q,\xi)}\frac{d\xi_{\alpha}}{dt}\,, \tag{21}$$

$$\tau^{(vi)}_{\alpha\beta} = \eta^{(0,0)}_{s}\frac{d\varepsilon_{\alpha\beta}}{dt}+(\eta^{(0,0)}_{v}-\eta^{(0,0)}_{s})\frac{d\varepsilon}{dt}\,\delta_{\alpha\beta}+\eta^{(0,1)}_{s}\tau^{(1)}_{\alpha\beta}+(\eta^{(0,1)}_{v}-\eta^{(0,1)}_{s})\,\tau^{(1)}\,\delta_{\alpha\beta}\,, \tag{22}$$

$$\frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} = \eta^{(1,0)}_{s}\frac{d\varepsilon_{\alpha\beta}}{dt}+(\eta^{(1,0)}_{v}-\eta^{(1,0)}_{s})\frac{d\varepsilon}{dt}\,\delta_{\alpha\beta}+\eta^{(1,1)}_{s}\tau^{(1)}_{\alpha\beta}+(\eta^{(1,1)}_{v}-\eta^{(1,1)}_{s})\,\tau^{(1)}\,\delta_{\alpha\beta}\,, \tag{23}$$

$$j_{\alpha} = -\lambda^{(\xi,q)}\frac{\partial T}{\partial x^{\alpha}}+\lambda^{(\xi,\xi)}\frac{d\xi_{\alpha}}{dt}\,, \tag{24}$$

where $\delta_{\alpha\beta}$ is the Kronecker's delta function.

3.3 Linear equations of state

Let f be the specific free energy of the medium

$$f = u - T\,s. \tag{25}$$

With the aid of Gibbs relation (6) we have:

$$df = v\,\tau^{(eq)}_{\alpha\beta}d\varepsilon_{\alpha\beta} - v\tau^{(1)}_{\alpha\beta}d\varepsilon^{(1)}_{\alpha\beta} - v\,j_{\alpha}\,d\xi_{\alpha} - s dT\,, \tag{26}$$

where

$$v\,\tau^{(eq)}_{\alpha\beta} = \frac{\partial f}{\partial\varepsilon_{\alpha\beta}},\qquad v\,\tau^{(1)}_{\alpha\beta} = -\frac{\partial f}{\partial\varepsilon^{(1)}_{\alpha\beta}},\qquad v\,j_{\alpha} = -\frac{\partial f}{\partial\xi_{\alpha}},\qquad s = -\frac{\partial f}{\partial T}. \tag{27}$$

Let us choose a configuration $\Sigma^{(0)}$ with uniform temperature T_0 and in which s_0 is the specific entropy, v_0 is the specific volume and $(\tau^{(eq)}_{\alpha\beta})_0$, $(\tau^{(1)}_{\alpha\beta})_0$, $(\tau_{\alpha\beta})_0$, $(\varepsilon_{\alpha\beta})_0$, $(\varepsilon^{(1)}_{\alpha\beta})_0$ and $(j_{\alpha})_0$ are zero.
Let us suppose that the deviations from $\Sigma^{(0)}$ are sufficiently small (with $\rho \approx \rho_0$) and the free energy for isotropic medium can be written in the following form

$$\begin{aligned} f =\ & v_0\left\{\frac{1}{2}\left[a^{(0)(0)}_{\alpha\beta\mu\nu}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\mu\nu}+a^{(1)(1)}_{\alpha\beta\mu\nu}\,\varepsilon^{(1)}_{\alpha\beta}\,\varepsilon^{(1)}_{\mu\nu}+a^{(\xi)(\xi)}_{\alpha\beta}\,\xi_{\alpha}\,\xi_{\beta}\right]+a^{(0)(1)}_{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta}\,\varepsilon^{(1)}_{\mu\nu}+a^{(0)(\xi)}_{\alpha\beta\mu}\,\varepsilon_{\alpha\beta}\,\xi_{\mu}+\right. \\ & \left.+a^{(1)(\xi)}_{\alpha\beta\mu}\,\varepsilon^{(1)}_{\alpha\beta}\,\xi_{\mu}+(T-T_0)\left[a^{(0)(T)}_{\alpha\beta}\varepsilon_{\alpha\beta}+a^{(1)(T)}_{\alpha\beta}\,\varepsilon^{(1)}_{\alpha\beta}+a^{(\xi)(T)}_{\alpha}\,\xi_{\alpha}\right]\right\}-\Psi(T)\,. \end{aligned} \tag{28}$$

In isotropic case we obtain:

$$\Psi = c_{(\varepsilon)}\,T\log\left(\frac{T}{T_0}\right)+s_0\,T-c_{(\varepsilon)}(T-T_0)-u_0\,, \tag{29}$$

where c_{ε} is the specific heat at constant deformation and

$$\tau^{(eq)}_{\alpha\beta} = a^{(0,0)}\,\varepsilon_{\alpha\beta}+(b^{(0,0)}-a^{(0,0)})\,\varepsilon\,\delta_{\alpha\beta}+a^{(0,1)}\varepsilon^{(1)}_{\alpha\beta}+(b^{(0,1)}-a^{(0,1)})\,\varepsilon^{(1)}\delta_{\alpha\beta}+a^{(0,T)}\,(T-T_0)\,\delta_{\alpha\beta}\,, \tag{30}$$

$$\tau^{(1)}_{\alpha\beta} = a^{(1,1)}\,\varepsilon^{(1)}_{\alpha\beta}+(b^{(1,1)}-a^{(1,1)})\,\varepsilon^{(1)}\,\delta_{\alpha\beta}+a^{(0,1)}\varepsilon_{\alpha\beta}+(b^{(0,1)}-a^{(0,1)})\,\varepsilon\,\delta_{\alpha\beta}+a^{(1,T)}\,(T-T_0)\,\delta_{\alpha\beta}\,, \tag{31}$$

$$j_{\alpha} = -\,a^{(\xi,\xi)}\,\xi_{\alpha}\,. \tag{32}$$

3.4 Heat flows in visco-anelastic processes

The results obtained are based on the decomposition theorem of the flow vector [1] using the thermodynamic theory of irreversible nonlinear processes with the method of internal variables. From the equations (24) and (32), eliminating the vector j_α, we have:

$$\frac{d\xi_\alpha}{dt} = -\frac{a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\,\xi_\alpha + \frac{\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}\frac{\partial T}{\partial x^\alpha}. \tag{33}$$

By replacing the (33) in the (21) expressing the heat flow vector, we obtain:

$$J^{(q)}_\alpha = -\left[\lambda^{(q,q)} - \frac{\lambda^{(\xi,q)}\lambda^{(q,\xi)}}{\lambda^{(\xi,\xi)}}\right]\frac{\partial T}{\partial x^\alpha} - \frac{\lambda^{(q,\xi)}a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\,\xi_\alpha. \tag{34}$$

Setting:

$$\lambda^{(0)} = \lambda^{(q,q)} - \frac{\lambda^{(\xi,q)}\lambda^{(q,\xi)}}{\lambda^{(\xi,\xi)}} \tag{35}$$

the (34), in vector form, becomes:

$$J^{(q)} = -\lambda^{(0)}\nabla T - \frac{\lambda^{(q,\xi)}a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\,\boldsymbol{\xi}\,. \tag{36}$$

Therefore $J^{(q)} = J^{(0)} + J^{(1)}$ vector decomposes into two vectors:

$$J^{(0)} = -\lambda^{(0)}\nabla T \tag{37}$$

and

$$J^{(1)} = -\frac{\lambda^{(q,\xi)}a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\,\boldsymbol{\xi}. \tag{38}$$

The (37) expresses Fourier's law whereas the (38) represents the constitutive M-C-V equation. We rewrite the (33) in vector form in the following way:

$$\lambda^{(\xi,\xi)}\frac{d\boldsymbol{\xi}}{dt} = -a^{(\xi,\xi)}\,\boldsymbol{\xi} + \lambda^{(\xi,q)}\nabla T. \tag{39}$$

We derive the (38) with respect to time, thus obtaining:

$$\frac{dJ^{(1)}}{dt} = -\frac{\lambda^{(q,\xi)}a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\frac{d\boldsymbol{\xi}}{dt} \tag{40}$$

from which:

$$\frac{d\boldsymbol{\xi}}{dt} = -\frac{\lambda^{(\xi,\xi)}}{\lambda^{(q,\xi)}a^{(\xi,\xi)}}\frac{dJ^{(1)}}{dt}. \tag{41}$$

Replacing the (41) in the (39) and simplifying, we obtain:

$$-\frac{\lambda^{(\xi,\xi)}}{a^{(\xi,\xi)}}\frac{dJ^{(1)}}{dt} = J^{(1)} + \frac{\lambda^{(q,\xi)}\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}\nabla T. \tag{42}$$

From a dimensional point of view, the second member of the (42) has the dimensions of a heat flow or a specific energy flow per unit of time. Therefore also the first member given by the product of the scalar quantity $\frac{\lambda^{(\xi,\xi)}}{a^{(\xi,\xi)}}$ for the vector $\frac{dJ^{(1)}}{dt}$ having the dimensions of a heat flow rate must have the dimensions of a heat flow:

$$\left[\frac{\lambda^{(\xi,\xi)}}{a^{(\xi,\xi)}}\right] \times \frac{[Watt]}{[meters]^2} \times [seconds]^{-1} = \frac{[Watt]}{[meters]^2}.$$

Simplifying, we get:

$$\left[\frac{\lambda^{(\xi,\xi)}}{a^{(\xi,\xi)}}\right] = [seconds]\,.$$

Therefore the scalar quantity has the dimensions of a time called thermal relaxation time, t_1 due to the heat flow associated with the internal variable:

$$t_1 = \frac{\lambda^{(\xi,\xi)}}{a^{(\xi,\xi)}}. \tag{43}$$

Therefore we write the (42) as:

$$t_1 \frac{dJ^{(1)}}{dt} + J^{(1)} = -\frac{\lambda^{(q,\xi)}\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}} \nabla T, \tag{44}$$

the term $\dfrac{\lambda^{(q,\xi)}\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}} \nabla T$ is a heat flux, i.e.:

$$\left[\frac{\lambda^{(q,\xi)}\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}} \nabla T\right] = \left[\frac{\lambda^{(q,\xi)}\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}\right] \times \frac{[Kelvin]}{[meters]} = \frac{[Watt]}{[meters]^2} \Rightarrow \left[\frac{\lambda^{(q,\xi)}\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}\right] = \frac{[Watt]}{[meters]}.$$

Therefore $\lambda^{(q,\xi)}$, $\lambda^{(\xi,\xi)}$, $\lambda^{(\xi,q)}$ have the dimesions of thermal conductivity and $a^{(\xi,\xi)}$ has the dimension of rate thermal conductivity, i.e. $\dfrac{[Watt]}{[meters] \times [Kelvin]}$.

4 General heat equation

In order to obtain an expression of temperature that represents the evolution of an ultrafast process in a viscoanelastic isotropic medium, the conditions that are defined here apriori, are not deductive logical hypotheses but express concrete and experimentally measurable physical aspects maintaining however the mathematical rigor. Before determining the heat equation, produced by a laser source, propagating in a metal film, let us examine the follow conditions [1]:

1. $\dfrac{d\varepsilon_{\alpha\beta}}{dt} = 0$ this condition means that the test material subject constant deformation.
2. Absence of external forces.
3. $\mathbf{v} = 0$, the viscoanelastic medium is not in motion; thus the substantial derivative coincides with the partial derivative, i.e. $\dfrac{d}{dt} = \dfrac{\partial}{\partial t}$.
4. The heat flow vector is given by two component vectors, one due to one diffusion field, $J^{(0)}$ and the other induced by the internal variable $J^{(1)}$.
5. The specific heat at constant volume is: $C_v = \dfrac{du}{dT}$.

Considering the last condition, the equation that expresses the first principle of thermodynamics given by (11) is reduced to:

$$\rho \frac{du}{dt} = -\nabla \cdot \left(J^{(0)} + J^{(1)}\right) \tag{45}$$

where the vector $J^{(0)}$ is proportional to less than the sign at the temperature gradient as shown by the formula (37) while the vector, $J^{(1)}$, always less than the sign, is proportional to the internal variable as shown by the formula (38). Considering the 3 and 5 point conditions and according to the formulas (37) and (38), we have:

$$\rho C_{(v)} \frac{\partial T}{\partial t} = \lambda^{(0)} \triangle T + \frac{\lambda^{(q,\xi)} a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}} \;\nabla \cdot \boldsymbol{\xi}. \tag{46}$$

Multiplying the (46) by t_1 and deriving both members with respect to the time we get the following expression:

$$\rho C_{(v)} t_1 \frac{\partial^2 T}{\partial t^2} = t_1 \lambda^{(0)} \frac{\partial}{\partial t} \triangle T + t_1 \frac{\lambda^{(q,\xi)} a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}} \frac{d}{dt} (\nabla \cdot \boldsymbol{\xi}), \tag{47}$$

by summing member to member (46) with (47), we obtain:

$$\rho C_{(v)} \left(t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} \right) = t_1 \lambda^{(0)} \frac{\partial}{\partial t} \triangle T + \frac{\lambda^{(q,\xi)} a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}} \left(t_1 \nabla \cdot \frac{d\boldsymbol{\xi}}{dt} + \nabla \cdot \boldsymbol{\xi} \right). \tag{48}$$

From (41) and (38), we obtain:

$$\rho C_{(v)} \left(t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} \right) = \lambda^{(0)} \triangle T + t_1 \lambda^{(0)} \frac{\partial}{\partial t} \triangle T - \nabla \cdot \left(t_1 \frac{\partial J^{(1)}}{\partial t} + J^{(1)} \right). \tag{49}$$

From (44) and dividing by $\rho C_{(v)}$, we obtain:

$$t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{\lambda^{(q,q)}}{\rho C_{(v)}} \triangle T + \frac{t_1 \lambda^{(0)}}{\rho C_{(v)}} \frac{\partial}{\partial t} \triangle T. \tag{50}$$

Indicating with α' and η', respectively, $\alpha' = \dfrac{\lambda^{(q,q)}}{\rho C_{(v)}}$ and $\eta' = \dfrac{\lambda^{(0)}}{\rho C_{(v)}}$, we obtain:

$$t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha' \triangle T + \alpha' t_1 \frac{\eta'}{\alpha'} \frac{\partial}{\partial t} \triangle T, \tag{51}$$

placing:

$$t_2 = \frac{\eta'}{\alpha'} t_1 \tag{52}$$

i.e.

$$t_2 = \frac{\lambda^{(0)}}{\lambda^{(q,q)}} t_1 \tag{53}$$

$$t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha' \triangle T + \alpha' t_2 \frac{\partial}{\partial t} \triangle T, \tag{54}$$

general heat equation in the absence of external thermal sources.

4.1 Special cases

- If the relaxation time t_1 can be neglected with respect to the heat propagation time t ($t_1 \ll t$ and hence $t_2 \ll t$), the equation (54) becomes

$$\frac{\partial T}{\partial t} = \alpha' \triangle T\,. \tag{55}$$

  i.e. the Fourier equation.

- If the phenomenological coefficients satisfy the following condition:

$$\lambda^{(q,q)} = \frac{\lambda^{(q,\xi)}, \lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}, \tag{56}$$

  from (35) one has: $\lambda^{(0)} = 0$ and therefore from (53), we have:

$$t_2 = 0, \tag{57}$$

and the equation (54) becomes:

$$t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \alpha' \triangle T = 0 . \tag{58}$$

being the Maxwell-Cattaneo-Vernotte.

- If $\lambda^{(0)} = \lambda^{(q,q)}$, i.e. : $t_1 = t_2$, then we have the diffusion and the heat equation admit the Fourier solution.

5 Local physical phenomena and applications to ultra-fast processes

In the case of a constant strain medium, the generalized heat equation allows to characterize the temperature distribution in a metal film under the action of an ultrafast laser pulse. As described in [30], the electrons initially in thermal equilibrium with the crystalline structure of the thin metal film, are heated by the impulsive laser source. In fact, α' is the effective thermal diffusivity in the phonon-electron or pure-phonon system and t_2 and t_1 are two intrinsic time scales characterizing the fast-transient response on microscales. Equation (54) is the generalized energy equation accounting for the effect of microstructural interaction (microscale response in space) in the fast-transient (microscale response in time) process. Table 1 shows the physical significance of the phenomenological and state coefficients with the corresponding units of measurement.
Table 2 shows the parameter estimation, α', t_1 and t_2 values for some metals [31]. After the time t_1 at which the free electrons have heated to a T_e temperature greater than that of the lattice, the electrons couple with phonons, thereby transferring thermal energy. After a time t_2 the lattice rises to the temperature of the free electrons T_e.

Table 1 Physical significance: $\lambda^{(\xi,\xi)}$, $\lambda^{(q,q)}$, $\lambda^{(0)}$, $a^{(\xi,\xi)}$.

$Coefficient$	$Description$	$Measurement\ unit$
$\lambda^{(\xi,\xi)}$, $\lambda^{(q,q)}$, $\lambda^{(0)}$	Thermal conductivity	$W/(meters\ Kelvin)$
$a^{(\xi,\xi)}$	Thermal conductivity rate	$W/(meters\ Kelvin\ seconds)$

Table 2 Parameter estimation values for some metals.

$Metal$	α' $(10^{-4} \times m^2/s)$	t_1 $(picoseconds)$	t_2 $(picoseconds)$
Ag	1.6620	0.7438	89.286
Cu	1.1283	0.4348	70.883
Au	1.2495	0.7438	89.286
Pb	0.2301	0.1670	12.097

6 Case a: 1-D equation of general heat in the case of a laser source with pulse duration proxime to zero

Consideration of a one-dimensional medium (in space), Figure 1, is sufficient to study the time-history of heat propagation. In this case, the equation (54) reduces to:

$$t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha' \frac{\partial^2 T}{\partial x^2} + t_2 \alpha' \frac{\partial}{\partial t}\left(\frac{\partial^2 T}{\partial x^2}\right) . \tag{59}$$

Two initial conditions are needed due to the presence of the wave term in equation (59):
$T = T_0$ and $\dfrac{\partial T}{\partial t} = \Lambda_0$ at $t = 0$.

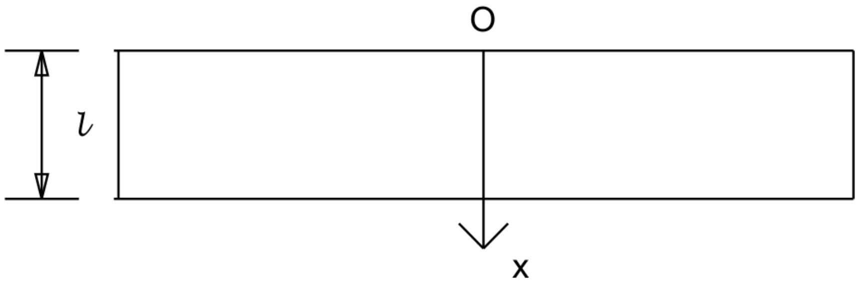

Fig. 1 Thickness of metal sheet: $l = 1.0\mathrm{E}{-6}$ meters.

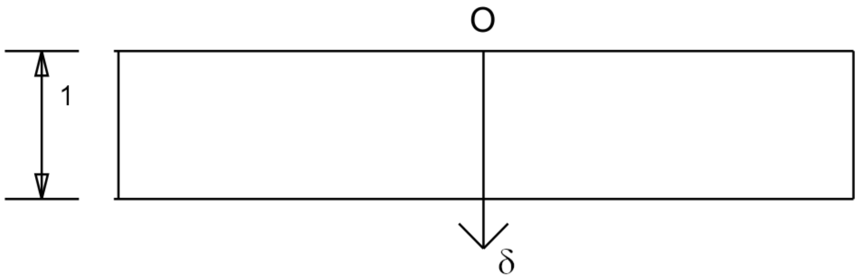

Fig. 2 Geometry of dimensionless problem.

By imposing a suddenly raised temperature at the boundary $T = T_w$ at $x = 0$, we study the way in which heat propagates through the solid medium. At a distance far away from the boundary, the thermal disturbance vanishes: $\dfrac{\partial T}{\partial x} = 0$ at $x \longrightarrow l$.
Introducing the following dimensionless variables:

$$\vartheta = \frac{T - T_0}{T_w - T_0}, \quad \delta = \frac{x}{l}, \quad \beta = \frac{t}{l^2/\alpha'},$$

respectively, temperature, space and time, (see Figure 2) we obtain the dimensionless equation:

$$z_1 \frac{\partial^2 \vartheta}{\partial \beta^2} + \frac{\partial \vartheta}{\partial \beta} = \frac{\partial^2 \vartheta}{\partial \delta^2} + z_2 \frac{\partial}{\partial \beta} \frac{\partial^2 \vartheta}{\partial \delta^2}. \tag{60}$$

with:

$$z_1 = \frac{t_1}{(l^2/\alpha')}, \; z_2 = \frac{t_2}{(l^2/\alpha')}.$$

The initial conditions, and the boundary conditions thus become: $\vartheta = 0$ and $\dfrac{\partial \vartheta}{\partial \beta} = \Gamma_0$ at $\beta = 0$

$\vartheta = 1$ at $\delta = 0$ and $\dfrac{\partial \vartheta}{\partial \delta} = 0$ at $\delta = 1$
with:

$$\Gamma_0 = \frac{\Lambda_0 \, (l^2/\alpha')}{T_w - T_0}.$$

The parameters z_1 and z_2 weigh the relative importance of the lagging times (in both the temperature gradient and the heat flux vector) to the diffusion time. They are the physical parameters governing the transition of the physical mechanisms. In the simulation we consider: $\Lambda_0 = 0$.

6.1 Results-case a

Using numerical FEM with Comsol and assuming the values of the geometrical and physical parameters in Table 2, we obtain the following results: From Figures 3 and 4 left panels, it is observed that the trend of the Fourier solution, that is, in the case in which heat transmission is considered without taking into account the microscopic effects, coincides with that of diffusion. This is not surprising since the diffusion equation, characterized by two equal relaxation times, allows Fourier's solution. The presence of two relaxation times makes it possible to discriminate the material according to the heat propagation rate. In fact, in the right panel of Figure 5, it is observed that silver, gold and copper are brought more quickly to the temperature of hot electrons than lead. This behaviour cannot be shown by using both the Fourier and diffusion models.

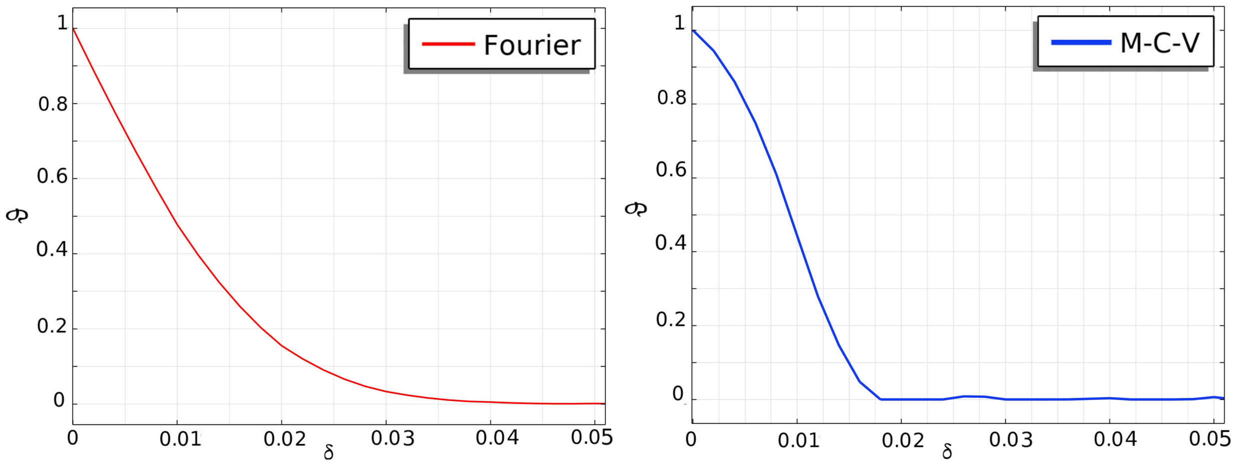

Fig. 3 Left panel: Fourier's parabolic heat equation. Right panel: MCV's hyperbolic heat equation. $\beta = 1 \times 10^{-4}$.

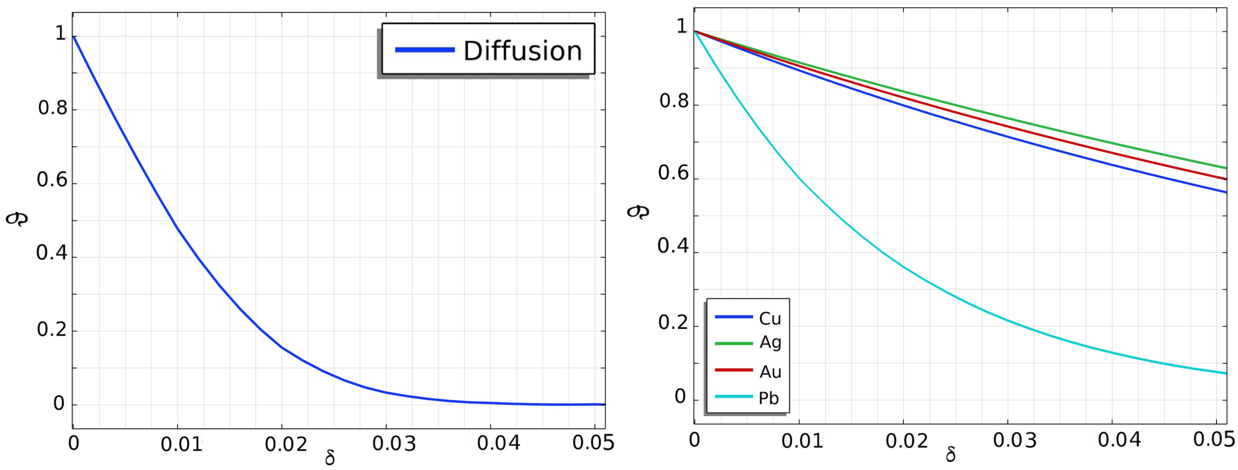

Fig. 4 Left panel: Diffusion $z_1 = z_2 \neq 0$. Right panel: Generalised heat equation. $\beta = 1 \times 10^{-4}$.

7 Case b: 1-D equation of dimensionless heat in the case of a laser source with pulse duration comparable to thermal phase-lag

If the duration of the laser pulse is comparable to the thermal phase lag t_1, the equation (45) representing the first principle of thermodynamics must include a term $\mathscr{S}(t,x)$ which takes into account the rate of energy absorbed due to the laser beam. Hence:

$$\rho \frac{du}{dt} = -\nabla \cdot \left(J^{(0)} + J^{(1)}\right) + \mathscr{S}(t,x). \tag{61}$$

Applying the same procedure as in Section 6, one obtains

$$t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha' \triangle T + \alpha' t_2 \frac{\partial}{\partial t} \triangle T + \frac{\alpha'}{K}\left(S(t,x) + t_1 \frac{\partial S(t,x)}{\partial t}\right), \tag{62}$$

where K is the thermal conductivity of metal film, $S(t,x)$ is the approximate intensity function of the laser pulse and the time derivative of $S(t,x)$ in (62) is the apparent heat source resulting from the fast-transient effect of thermal inertial described by t_1. From an experimental and computational point of view the function $S(t,x)$ is analytically expressible by the following equation [39]:

$$S(t,x) = S_0\, Exp\left(-\frac{x}{\phi}\right) I(t), \tag{63}$$

where $I(t)$ is the light intensity of the laser beam; ϕ denoting the optical depth of penetration; S_0 the intensity of the laser absorption rate, measured in $\frac{[Watt]}{[meters]^3}$ that depends on the laser fluence $J\left(\frac{[Joule]}{[meters]^2}\right)$, the radiative reflectivity of the sample to the laser beam (R), and the full width at half-maximum pulse duration (t_p).

$$S_0 = 0.94 J\left(\frac{1-R}{t_p\,\phi}\right). \tag{64}$$

In this case we consider a femto-laser stimulator, i.e. $t_p = 100$ (femto-seconds). The analytical expression of $I(t)$ which best approximates the real trend on the basis of experimental data is:

$$I(t) = Exp\left(-a\,\frac{|t-2t_p|}{t_p}\right). \tag{65}$$

For gold film, whose actual behaviour deviates from the model (59), if phase delays are used of the Table 2, it has been found experimentally that suitable values are $t_1 = 8.5$ ps and $t_2 = 90$ ps. The reason for this variability of the estimate of t_1 is due to the temperature dependence of the coupling factor between electron and phonon. The estimate is improved if an average of t_1 is measured and simulated over the whole initial and final temperature range of free electrons:

$$\bar{t}_1 = \frac{\int_{T_0}^{T_F} t_1\, dT}{(T_F - T_0)}, \tag{66}$$

where T_0 and T_F are the initial and final temperatures in the range of t_1. For $T_0 = 300K$ and $T_F = 5\times 10^4 K$ the average value obtained, $\bar{t}_1$ is equal to 6.5 ps. Hence while the t_2 value remains almost independent of temperature, it is observed that the value of t_1 varies by about one order of magnitude. Applying the same procedure of dimensionless as in Section 5, we obtain:

$$z_1 \frac{\partial^2 \vartheta}{\partial \beta^2} + \frac{\partial \vartheta}{\partial t} = \frac{\partial^2 \vartheta}{\partial \delta^2} + z_2 \frac{\partial}{\partial \beta}\left(\frac{\partial^2 \vartheta}{\partial \delta^2}\right) + \left(Q(\beta,\delta) + z_1 \frac{\partial Q(\beta,\delta)}{\partial \beta}\right), \tag{67}$$

where

$$Q(\beta,\delta) = Q_0 \, Exp\left(-\frac{x}{\phi}\right) I(\beta), \tag{68}$$

with:

$$Q_0 = \frac{l^2 \, S_0}{K \, (T_w - T_0)}, \tag{69}$$

and

$$I(\beta) = Exp\left(-a \, \frac{|\beta - 2 z_p|}{z_p}\right). \tag{70}$$

7.1 Results-case b

Using numerical FEM with Comsol and assuming the values for Au metal, of the geometrical and physical parameters: $t_p = 10^{-}15$ (s), $t_1 = 8.5$ (ps), $t_2 = 90$ (ps), $l = 0.1 \times 10^{-6}$ (m), $J = 13.4$ (Joule/m^2), $R = 0.93$ $\phi = 15.3 \times 10^{-}9$ (m), $\alpha' = 1.2 \times 10^{-4}$ (m^2/s), and $K = 315$ (W/m Kelvin), we obtain the following results, shown in Figures 5 and 6.

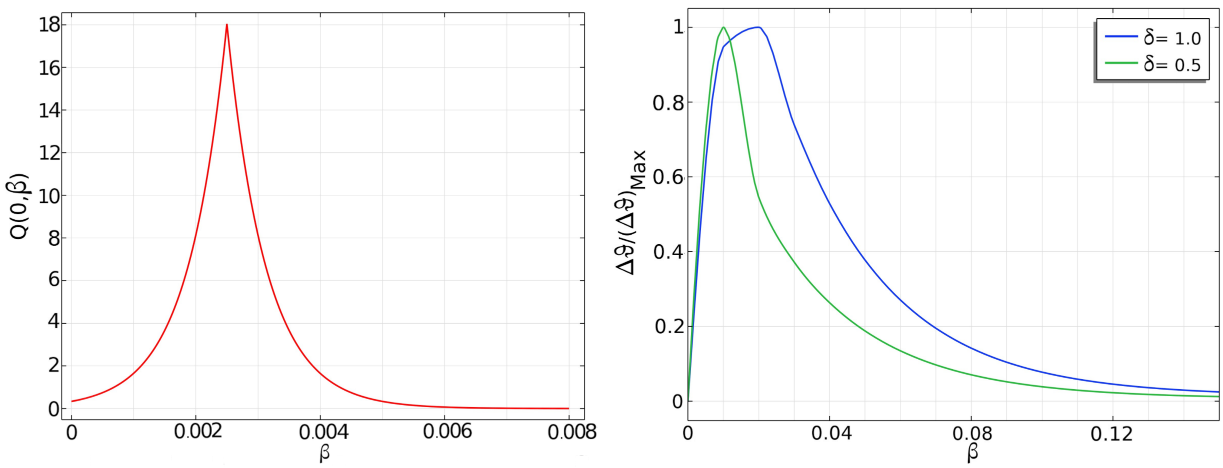

Fig. 5 Left panel: Trend of the dimensionless of heat absorption respect to time dimensionless, due at stimulation of laser. Right panel: Trend rate temperature normalized respect to maximum rate temperature dimensionless.

8 Conclusions

In this paper, the theory of thermodynamics of ultra-fast non-linear irreversible processes was applied with the use of internal variables, and in particular, the generalized equation of heat which gives it was derived in the case of medium isotropic visco-anelastic. The physical significance of the phenomenological coefficients was given by highlighting the thermal properties of materials in the case of metals and specifically of a gold film. The approach followed demonstrates the validity of the results obtained and compared in the literature reporting experimental data. The advantage of this approach is that it allows complex physical processes to be studied in a simple and immediate manner, independently of the system or material subject to the process is based on a combination of physical properties, such as mechanical and thermal. The study of the anisotropic materials as in the case of biological tissues can be extended to reveal substantial abnormalities characterizing specific pathologies compared with a healthy tissue.

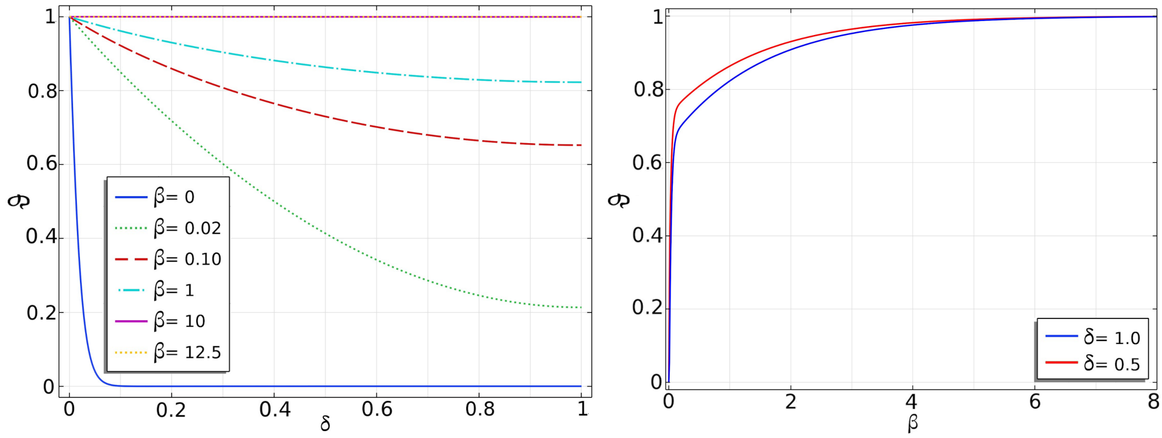

Fig. 6 Left panel: Trend of the dimensionless temperature of gold metal film when the dimensionless depth changes for various values of the dimensionless time. Right panel: Trend of the dimensionless temperature of gold metal film when the dimensionless time changes for various values of the dimensionless depth.

9 Declarations

9.1 Conflict of interests:

Not applicable.

9.2 Funding:

Not applicable.

9.3 Author's contributions:

A.C.-Methodology, Simulations, Resources, Writing-Original Draft, Validation, Conceptualization and Formal Analysis, Data curation, Investigation, Visualization. B.F.F.F.-Methodology, Simulations, Resources, Writing-Original Draft, Validation, Conceptualization and Formal Analysis, Data curation, Investigation, Visualization. All authors read and approved the final submitted version of this manuscript.

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9.5 Data availability statement:

All data that support the findings of this study are included within the article.

9.6 Using of AI tools:

The author affirms that Artificial Intelligence (AI) techniques were not used in creating this content.

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