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Original Study

Unmasking complexity: investigating numbers within Diophantine D(∓2) sets

Özen Özer [1†]

[1]Department of Mathematics, Faculty of Arts and Sciences, Kırklareli University, Kırklareli, 39100, Türkiye

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Abstract

The purpose of this paper is to explore the enigmatic world of numbers in Diophantine D(∓2) sets, revealing fresh insights into their intricate properties and profound connections. Diophantine D(∓2) sets, which are defined by integer-based Diophantine conditions, represent a compelling domain ripe for investigation. Our study delves into these sets, disregarding their cardinalities, aiming to unveil the concealed patterns and unique characteristics they harbor. Through meticulous scrutiny of their structure, our objective is to reveal the presence of prime numbers within these sets. In our investigation, we draw upon the foundational principles of Elementary and Algebraic Number Theory, invoking the Quadratic Reciprocity Law, Diophantine equations, and the enduring contributions of eminent mathematicians such as Gauss, Dirichlet, and Fermat. These tools and insights serve as guides in our exploration, ultimately leading to a deeper comprehension of the numbers within the Diophantine D(∓2) set and their significance within the broader landscape of mathematics.

Keywords: Number theory and Diophantine equations, modular arithmetic and Diophantine sets, prime residues in Diophantine D(m) sets, Legendre symbol and quadratic residues, congruence classes.
AMS 2020 codes: 11D09; 11A07; 11A15.

1 Introduction

Mathematicians have been interested in Diophantine sets for a long time. In the domain of number theory, Diophantine sets have long captivated mathematicians and presented fertile ground for mathematical inquiry. The ancient mathematician Diophantus of Alexandria was the first to investigate the problem of discovering four numbers that, when their pairwise products are increased by one, result in perfect squares. He successfully identified a set of four positive rational numbers possessing this property: {1/16, 33/16, 17/4, 105/16}. However, Fermat was credited with discovering the initial set of four positive integers that fulfilled this condition: {1, 3, 8, 120}. Euler later identified an infinite family of similar sets using a formula involving integers a, b, and r, where $ab+1=r^2$. Several extensions of the initial problem studied by Diophantus and Fermat have been explored. One notable extension involves replacing the number 1 in the definition of Diophantine m-tuples with an arbitrary integer n. Numerous researchers have delved into the existence of Diophantine quadruples with the property D(n), achieving partial solutions. These discoveries led to the formulation of definitions for Diophantine m-tuples, which involve sets of positive integers or non-zero rationals that satisfy specific mathematical conditions. In this work, the Diophantine D(∓2) sets, characterized by integer values, stand as an intriguing

†Corresponding author.
Email address: ozenozer39@gmail.com

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realm awaiting exploration. Our investigation immerses into these sets, overlooking their cardinalities, with the intention of uncovering the hidden patterns and distinct traits they possess. By meticulously examining their structure, our aim is to expose the existence of prime numbers within these sets. Due to these reasons, we considered literature about this issue as follows: The book [1] serves as an entry point to the fascinating world of analytic number theory. It covers fundamental concepts, such as prime number theory, Dirichlet series, and zeta functions, making it an essential read for those delving into the intricate connections between number theory and analysis. It is a valuable resource for students and mathematicians interested in the deeper connections between number theory and analysis. It introduces key techniques and theories that form the basis for advanced studies in the field. Cox [2] explores the fascinating connections between Fermat's last theorem, class field theory, and complex multiplication. It provides a detailed account of the theory behind quadratic forms and their applications in number theory. The book is a valuable resource for those interested in understanding the intricate relationships between algebraic number theory, elliptic curves, and the theory of quadratic forms. A masterpiece by Carl Friedrich Gauss, it [3] presents a comprehensive exploration of number theory concepts, including congruences, quadratic forms, and the theory of integers. Gauss's pioneering work laid the foundation for numerous mathematical discoveries. Gauss insights and mathematical rigor continue to influence and inspire mathematicians, providing a rich source of knowledge in the realm of pure mathematics. Focused specifically on Pell's equation, the book [4] offers a comprehensive collection of solutions and properties related to this particular type of Diophantine equation. Pell's equation has intrigued mathematicians for centuries, and this book delves into its various aspects, providing a wealth of solutions and their properties. It's a resource that offers a deep dive into a specific area of Diophantine equations, making it a must-read for mathematicians interested in this field. "Topics from the Theory of Numbers" as the title book [5] covers a wide range of topics within the realm of number theory and an excellent primer for individuals seeking an introduction to various facets of number theory. Foundational books [6–8] in number theory provide a gentle introduction to key concepts, including divisibility, prime numbers, and Diophantine equations. Bridging classical principles with modern advancements, these books offer a comprehensive approach to the study of number theory. They link historical insights with contemporary developments, catering to a broad audience of mathematicians seeking a well-rounded understanding of the subject. Focusing on additive number theory, the book [9] explores the intricate relationship between number theory and geometry. It delves into sumsets, inverse problems, and their geometric interpretations, offering a fresh perspective on this branch of mathematics. The author of the [10–13], examined a selection of Diophantine D(400) triples, quadruples and D($\mp$2) pairs to triples utilizing methods concerning Diophantine equations. The outcomes derived in these studies found out significance in demonstrating the application of techniques and unveiling new insights in Diophantine theory within the literature. Covering arithmetic from a broad perspective, the book [14] offers a comprehensive course, including number theory, algebra, and more. It aims to provide a well-rounded understanding of arithmetic, making it suitable for students and mathematicians at various levels. Focused on algebraic geometry, the book [15] concentrates on the study of varieties in projective space. It provides a foundational understanding of algebraic geometry and its connections to various mathematical concepts. The book [16] "Elliptic Curves: Number Theory and Cryptography" explores the intricate connections between elliptic curves, number theory, and their applications in cryptography. It delves into the underlying mathematical principles that make elliptic curves a vital component in modern cryptographic systems. It's a valuable resource that not only covers the mathematical intricacies but also sheds light on practical applications in secure communication systems. Wright in [17] investigated the properties of quadratic residues and non-residues within arithmetic progressions. By studying the distribution and behavior of these residues, Wright likely contributes to our understanding of fundamental questions in number theory, such as the distribution of primes and the existence of certain types of Diophantine equations. The article likely presents new results and techniques for analyzing quadratic residues and non-residues, providing valuable insights for further research in the field. The provided references [18, 19] explore the applications of AG-Groupoids and Linear Diophantine Fuzzy Sets in decision-making and information theory contexts, too. This paper embarks on an investigative journey into the enigmatic world of Diophantine D($\mp$2) sets' numbers, shedding new light on their intricate properties and pro-

found connections. The Diophantine D(∓2) sets, characterized by integer values, stand as an intriguing domain ripe for exploration. Our study delves into these sets (irrespective of their cardinality) seeking to unravel the hidden patterns and unique characteristics they hold. By scrutinizing their composition, we aim to unveil the presence of prime numbers within these sets.

The rest of the paper is organized as follows. Preliminaries and some important lemmas are derived in Section 2. Theorems and properties are described in Section 3. Main results are presented in Section 4. Finally, main contributions of this paper are reported in Section 5.

2 Preliminaries

In order to ensure clarity and precision within the paper, it is essential to articulate definitions in a comprehensive manner. Definitions serve as the foundation upon which the entire discourse is built, providing readers with a clear understanding of the key concepts and terms used throughout the text.

2.1. Diophantine *s*-Tuples with property D(k)

A set of distinct positive integers, denoted as Æ={æ$_1$, æ$_2$, æ$_3$, $\cdots$, æ$_s$} is referred to as a Diophantine s-Tuples with property D(k) if (æ$_i$ * æ$_j$+k) is a perfect square integer for every pair of indices i and j where $1 \leq i \neq j \leq s$.

2.2. Quadratic or non-quadratic residue

A quadratic residue r modulo n is an integer r such that there exists an integer x satisfying the congruence $x^2 \equiv r \pmod{n}$, where $n > 1$. In other words, r is a quadratic residue if there exists an integer x such that when x^2 is divided by n, the remainder is r. Conversely, a quadratic non-residue is an integer that does not have a solution to the congruence $x^2 \equiv r \pmod{n}$ for a given modulus $n > 1$. Quadratic residues and non-residues (given as follows) have various applications in number theory, cryptography, and other mathematical fields. They are essential in algorithms like the Quadratic Residue Cryptosystem.

2.3. Legendre symbol

The Legendre symbol $\left(\frac{a}{p}\right)$ is defined as:

1. It equals 0 if a is divisible by p.
2. It equals 1 if a is a quadratic residue modulo p.
3. It equals -1 if a is a quadratic non-residue modulo p.

The Legendre symbol is a mathematical concept used in number theory to determine whether a given integer is a quadratic residue or non-residue modulo a prime number. It's denoted as $\left(\frac{a}{p}\right)$, where a is an integer and p is an odd prime.

2.4. Law of quadratic reciprocity

The law of quadratic reciprocity states that for distinct odd prime numbers p and q,

$$\left(\frac{q}{p}\right).\left(\frac{p}{q}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

Here, $\left(\frac{a}{p}\right)$ is the Legendre symbol, denoting whether a is a quadratic or non-residue modulo p. It establishes relationships between the quadratic residues and non-residues of two distinct prime numbers. One of the most well-known forms of this theorem is the Law of Quadratic Reciprocity, attributed to both Gauss and Euler

2.5. Diophantine equation

A Diophantine equation is typically expressed as $f(x_1,x_2,\cdots,x_m)=0$ where $x_1,x_2,\cdots,x_m$ are variables and f represents a polynomial or a set of polynomial equations with integer coefficients and solutions. Diophantine equations (like Fermat's last theorem) can involve linear, quadratic, cubic, or higher degrees. Each degree might require different solutions.

Specifically, the Pell equation, named after the mathematician John Pell, is a type of Diophantine equation of the form $x^2-Ny^2=1$ as classically. Here, N is a nonsquare positive integer, and the task is to find integer solutions for x and y. Pell equations are a special case of a broader class of equations known as generalized Pell equations. Some solution techniques for the Pell equation can be mentioned as continued fractions, recurrence relations, algebraic and number theoretic methods, the Samasa method, Brahmagupta's technique and so on. They are trivially known by mathematicians from numerous books related with mathematics especially in number theory.

3 Theorems and properties

The theorems and properties stated below contain the details of the part mentioned above and summarized from the literature. Each of these expressions will be used in transitions in the proofs of the theory in our article.

3.1. Dirichlet's theorem

If two positive integers, a and b, are coprime (meaning they share no common factors except 1), then there are infinitely many primes of the form $an+b$, where n is a non-negative integer. In other words, for any two coprime integers a and b, there are infinitely many primes in the sequence $b,b+a,b+2a,b+3a,\cdots$.
Dirichlet's theorem has implications and applications in various other number theoretic investigations and problems. It is a key tool for exploring the distribution of primes in certain mathematical progressions, which has implications in many areas within number theory.

3.2. Supplement to the quadratic reciprocity laws

1. If $a\equiv b$ (mod p) where a and b are integers and p is an odd prime number, then $\left(\frac{a}{p}\right)=\left(\frac{b}{p}\right)$.
2. $\left(\frac{-1}{p}\right)$=(-1)$^{\frac{(p-1)}{2}}$. It equals 1 if $p\equiv 1$ (mod 4).
3. $\left(\frac{2}{p}\right)$ equals 1 if $p\equiv\mp 1$ (mod 8).

3.3. Wilson's theorem

For a prime number $(p-1)!\equiv -1$ (mod p).

3.4. First supplement of Legendre's theorem

This supplement extends the original Law of quadratic reciprocity for the case where p and q are odd primes,

with $p \equiv q \equiv 3 \pmod 4$. It states that if p and q are distinct odd primes, then:

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right).(-1)^{\frac{(p-1)(q-1)}{4}}.$$

Here, $\left(\frac{p}{q}\right)$ is the Legendre symbol, indicating whether the congruence $x^2 \equiv p \pmod q$ has a solution or not.

3.5. Second supplement of Legendre's theorem

The second supplement provides a condition for when a prime p is a quadratic residue modulo $4q$. For an odd prime q, it states that if$p$ is an odd prime not dividing q, then:

$$\left(\frac{p}{4q}\right) = \left(\frac{p}{q}\right).$$

This supplement gives the relationship between the Legendre symbol for p modulo q and modulo $4q$. It helps to deduce whether a prime p is a quadratic residue modulo $4q$.

4 Main results

Remark 1. Note that prime numbers such as $p = 5,11,13,19,29,37,43,53,59,61,67,83,101,107,109,\cdots$ **are not** in the Diophantine sets showcasing the D(+2) property. (i.e. $p = 17,41,97,113,137,193,233,257,313,337,353,401,409,457,521,569,577,593,601,617,641,673,761,769,809,857$ and $p = 7,23,31,47,71,79,103,127,151,167,$ 191, 199, 223, 239, 263, 271, 283, 311, 359, 367, 383, 431, 439, 463, 479, 487, 503,$\cdots$ belong to the Diophantine sets with the D(+2) property.) While not comprehensive, this compilation comprises prime numbers conforming to the given criteria.

The classifications of the primes in [11] was not determined. So, the following can be given to determine the primes falling inside the Diophantine sets with the D(+2) property. This completes the classification.

Theorem 1. *Assuming p is an odd prime number. If the prime p adheres to any of the subsequent scenarios, it belongs to the Diophantine sets characterized by property D(+2). Conversely, primes situated within the Diophantine sets with property D(+2) are of the following categories:*

(i) *$p \equiv 1 \pmod 8$*
(ii) *$p \equiv 7 \pmod 8$*

Proof. Only the proof for (i) will be demonstrated in this section, leaving the remaining case for the reader to explore.

(i) Let $p \equiv 1 \pmod 8$ be a prime number and $\delta \in \mathbb{Z}^+$ be in the Diophantine sets with property D(+2). From the definition of the Diophantine sets with property D(+2), we get the following equation:

$$\delta.p + 2 = \Sigma^2$$

and

$$\Sigma^2 \equiv 2 \pmod p.$$

Considering following definitions and theorems/lemmas from the preliminaries part

$$\left(\frac{2}{p}\right) = (-1)^{[(p^2-1)/8]},$$

and

$$\left(\frac{2}{p}\right) = +1.$$

It implies that $p \equiv 1 \pmod 8$ belongs to the Diophantine sets characterized by property D(+2) and vice versa is also satisfied.

Remark 2. Observe that the prime numbers such as $p = 3, 11, 19, 43, 59, 67, 83, 107, 131, 139, 163, 179, 211, 227, 251, \cdots, 643, 659, 691, 739, 787, \cdots$ belong to the Diophantine sets exhibiting the D(-2) property. This is not an exhaustive list, but it contains prime numbers that meet the specified conditions.

Hence, we can now present our other main theorem.

Theorem 2. *Let p represent a prime number greater than 3. If p conforms to the subsequent scenery, it belongs to the Diophantine sets with the property D(-2). Conversely, primes falling inside the Diophantine sets with the D(-2) property adhere to the subsequent category.*

$$p = a^2 + 2b^2$$

where $a,\ b \in \mathbb{Z}$*, and also p is prime.*

Proof. Examining Euler's approach to demonstrate Fermat's result, this theorem can be readily ascertained. In order for the prime number p to be in the Diophantine D(-2) set, (-2) must be a quadratic residue with respect to p prime number. So $\left(\frac{-2}{p}\right)$ the Legendre symbol value should be +1. If we remember that $\left(\frac{-1}{p}\right)$=$(-1)^{\frac{p-1}{2}}$ if $p \equiv 1 \pmod 4$ and $\left(\frac{2}{p}\right)$ equals 1 if $p \equiv \pm 1 \pmod 8$ from the definitions and lemmas mentioned in the Preliminaries section, we get that the prime value p is written as the type $p = a^2 + 2b^2$, $(a, b \in \mathbb{Z})$. Conversely, it is trivial. (Evidence for the remaining case is deferred to the readers.)

The following conclusion can be given, equivalent to what is stated in Theorem 2.

Corollary 1. Let $p > 3$ be a prime number. If p conforms to following equivalents, it belongs to the Diophantine sets with the property D(-2). Conversely, primes are in the Diophantine sets with the D(-2) property adhere to the following statement

$$p \equiv 1 \pmod 8 \text{ or } p \equiv 3 \pmod 8$$

where p is prime.

Proof. Previously, some approximations were given in [10] by the author. Using the definition of the Diophantine sets with property D(-2), remainders modulo p, the Legendre symbol and reference [10], the Corollary's proof is readily attained.

The following theorem gives the primes that belong to Diophantine sets with the properties both D(-2) and D(+2).

Theorem 3. *Let* $p > 3$ *represents a prime number.*
$p \equiv 1$ *(mod 24)* $\Longleftrightarrow$ *p simultaneously belongs to the Diophantine sets with properties D(-2) and D(+2).*

Proof. Using Theorem 1 and Theorem 2 with their proofs, we get what follows

$$\left(\frac{2}{p}\right) = 1 \text{ if and only if } \ p \equiv 1 \pmod 8 \text{ or } \ p \equiv 7 \pmod 8,$$

and

$$\left(\frac{-2}{p}\right) = 1 \text{ if and only if } \ p \equiv 1 \pmod 8 \text{ or } \ p \equiv 3 \pmod 8.$$

From the above results, we obtain the system of congruences: If we solve them, it is obtained that primes $p \equiv 1 \pmod{24}$ are in the D(±2).

5 Conclusions

In this paper, the exploration into the enigmatic realm of Diophantine D(∓2) sets has unveiled compelling insights into their intricate properties and profound interconnections. The investigation has highlighted the captivating nature of these sets within the landscape of number theory, showcasing their potential for rich mathematical exploration.

This study delved into the mysterious realm of Diophantine D(∓2) sets' numbers, illuminating their complex properties and deep-seated connections. These sets, governed by integers, presented an alluring landscape ready for investigation. This research delved into these sets, regardless of their size, with the aim of unveiling concealed patterns and distinctive attributes. Through meticulous examination of their structure, we endeavoured to uncover the prevalence of prime numbers nestled within these sets. Throughout this study, the focus on Diophantine D(∓2) sets, defined by integer values, has provided a fascinating platform for investigation. Regardless of their cardinality, this research aimed to uncover hidden patterns and distinctive attributes embedded within these sets. The meticulous scrutiny of their composition has offered a deep insight of the presence of prime numbers, shedding light on the fundamental characteristics these sets embody. Presenting the world of Diophantine D(∓2) sets has not only expanded our understanding of their properties but has also emphasized their significance in the broader context of number theory. The discoveries made throughout this exploration pave the way for further research and analysis, fostering a deeper appreciation for the intricate relationships and complexities inherent in these mathematical constructs.

6 Declarations

6.1 Conflict of interest:

The corresponding author states that there is no conflict of interest.

6.2 Funding:

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6.3 Author's contribution:

Ö.Ö.-Conceptualization, Methodology, Software, Writing-Review Editing, Formal Analysis, Validation, Writing-Original Draft. The author read and approved the final submitted version of this manuscript.

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6.5 Data availability statement:

All data that supports the findings of this research can be created by the use of the constructed analog of this paper.

6.6 Using of AI tools:

The author declares that she has not used Artificial Intelligence (AI) tools in the creation of this article.

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