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Original Study

Odd and even symmetric prime constellations

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Abstract

In this paper, the constellations of primes are discussed. A number of symmetric prime constellations of even and odd order are defined and their existence is computationally verified. Provable minimum extent symmetric constellations are shown to exist for some cases. Conjectures regarding the existence of higher order symmetric constellations are provided and further areas of research are suggested.

Keywords: Prime, prime tuple, quadruple primes, quintuple primes, n-tuple primes, symmetric prime constellations.
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1 Introduction

Number theory is concerned with the properties of integers and the research in number theory has been conducted since antiquity. As an example, one can mention the Erathostenes Sieve which sifts out primes from a number sequence, first mentioned in [1]. Further prime research have been the focus of many researchers following Erathostenes as can be seen from this list of references [2–8]. One of the prominent number theorists of the 20th century was Ramanujan for whom Hardy attributes the following anecdote: "I had ridden in taxi-cab No. 1729, and remarked that to me the number seemed a rather dull one, and that I hoped it was not an unfavourable omen. 'No,' he reflected, it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways. That is, $1729 = 1^3 + 12^3 = 9^3 + 10^3$." [9]. This shows that one of the areas of interest for number theorists is to find various facts about specific integers or combinations of integers. In the past, the establishment of these facts was limited by the extent to which computations could be performed. Today, powerful computers have made it possible to vastly extend the size and complexity of these computations.

One area of number theory research has been to find patterns (constellations) in the sequence of integers and of particular interest has been finding patterns of primes, see for example the article by Granville [10, 11]. Also, the question of whether there is an unbounded number of twin primes has been extensively studied by [12, 13]. Another area of interest has been prime constellations (i.e. particular patterns of primes). Specific examples have been studied by [14–16] and others. For example, the Green-Tao theorem [14] proves that arbitrary long arithmetic progressions of primes exist.

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As noted above, computers have made it possible to perform experiments to validate theoretical results in number theory as well as experimentally suggest conjectures for new theorems [17, 18]. In Zhang et. al. [19], it is stated that "it is known that primes contain unusual patterns". The paper further lists some of these patterns.

The patterns called prime constellations are summarized in [20]. These patterns include twin primes, cousin primes, prime triplets, quadruplets and 5 tuplets.

In this paper, some new patterns called symmetric prime constellations are defined that have certain symmetry properties. These constellations are theoretically and computationally explored and examples of these constellations are exhibited. However, a result like the theorem quoted above [14] is unlikely to be proven for the constellations discussed here for reasons to be discussed hereafter. The theoretical and computational results presented are still a reflection of the fact that primes have certain distributive properties [10, 21]. These results and examples could lead to further conjectures and new results as discussed by Polya [22]. In [23] emerging applications of number theory, where the results in this paper might have applications, are surveyed. These applications include some important areas such as cryptography, information security, DNA modelling, image compression, among others.

2 Prime constellations

By a "constellation", we mean a finite pattern of primes.

This paper is concerned with certain symmetric constellations of n primes embedded in an increasing sequence of m integers $j_1, \cdots, j_m$ where j_1 and j_m are primes and $j_{i+1} - j_i = 1, i = 1, \cdots, m-1$, there are exactly n primes in the sequence and if j_i is a prime of the constellation then so is j_{m-i}. Such symmetric constellations are called $C(m,n)$ constellations. The *extent* of a $C(m,n)$ constellation is $|C(m,n)| = m = j_m - j_1 + 1$.

In this paper, the existence of $C(m,n)$ constellations are demonstrated for various values of n and m. For some values of n, it is shown that there are constellations with the smallest possible extent.

Note that given a sequence of consecutive integers two primes in the sequence are separated by an odd number of integers.

The symmetric constellations $C(m,n)$ can be divided into two classes depending on n being even or odd. Let the sequence of integer of the constellation be $j_1, \cdots, j_m$.

1. If n is odd the center of the constellation is a prime $c = \lfloor j_m / j_1 \rfloor$.
2. If n is even. Then the center of the constellation is $c = (j_m / - j_1)/2 + 1$ where c is now a composite integer. In this case, the *base case* for constellations is given by $\cdots, c-1, c, c+1, \cdots$ where $c-1$ and $c+1$ are are primes and c is an even integer. A *second case* is defined by a central constellation $c-2, c-1, c, c+1, c+2$ where $c-2$ and $c+2$ are primes and c is an odd integer. A *third case* is defined by a central constellation $c-3, c-2, c-1, c, c+1, c+2, c+3$ where $c-3$ and $c+3$ are primes and c is even.

2.1 Definitions

- The "center" c of a symmetric prime n-tuple (constellation) is meant the midpoint between the largest and the smallest primes defining the tuple. It is also called the "n-center".
- The letter E in a table will denote "even".
- The letter P in a table will denote "prime".
- If $C(m,n)$ is a symmetric prime constellation then it will also be called an n-tuple.
- A 2-tuple is a synonym for a double prime constellation.
- A 3-tuple is a synonym for a triple prime constellation.

- A 4-tuple is a synonym for a quadruple prime constellation.
- A 5-tuple is a synonym for a quintuple prime constellation.
- A 6-tuple is s synonym for a sixtuple prime constellation.
- A 7-tuple is a synonym for a seventuple prime constellation.
- An 8-tuple is a synonym for a octuple prime constellation.
- e_i denotes an even i-th element of a constellation.
- c_j denotes a composite j-th element of a constellation.
- p_k denotes a k-th element of constellation that is prime.
- d denotes the center of a tuple.
- m is the "extent" of a $C(m,n)$ constellation.
- "rel. index" will be used to indicate offset relative to a specified number, generally the center of a tuple.

2.2 Preliminaries

The following result is frequently used: given a list of k consecutive positive integers, exactly one of the integers is divisible by k. (For a proof see for example [24].)

The following two results are known and also proven here.

Theorem 1. *The minimum separation between two double prime centers is greater than or equal to 6.*

Proof. Let $p_1,d_1,p_2,c_1,p_3,d_2,p_4$ be two double primes where $d_2-d_1=4$. Then, d_1 and d_2 are both divisible by 3 which is not possible. Hence $d_2-d_1>4$. The sequence $11,12,13,14,15,16,17$ provides an example where $d_2-d_1=6$.

Theorem 2. *The constellation*

$$p_1,d_1,p_2,e_1,c_1,e_2,p_3,d_2,p_4,e_3,c_2,e_4,p_5,d_3,p_6 \tag{1}$$

of three double primes is not possible provided $p_1>5$.

Proof. Consider p_2,e_1,c_1,e_2,p_3. Then, one of the numbers must be divisible by 5. p_2 and p_3 are prime. Then e_1 is not divisible by 5 since it would imply the p_4 was divisible by 5. c_1 is also not divisible by 5 since it would imply that p_6 would be divisible by 5. Similarly e_2 cannot be divisible by 5 since it would imply p_1 be divisible by 5. This proves the theorem since none of the 5 numbers can be divisible by 5 in this constellation.

Note that there is one case of the constellation in Eq. 1 with the list of primes being $(5,7,11,13,17,19)$.

As noted, the divisibility with respect to $2,3,5$ is determined by the two double primes in the sequence $p_1,d_1,p_2,e_1,c_1,e_2,p_3,d_2,p_4$. Carrying these divisibilities to the following sequence

$e_3,c_2,e_4,c_3,e_5,c_4,e_6,c_5,e_7,c_6,e_8,c_7,e_9$

leaves only c_3,c_6,c_7 unconstrained by the divisibilities of $2,3,5$ and leave the possibilities of these entries being composite or primes. Constraining c_6,c_7 to be primes then provides the smallest possible constellation of a pair of double primes at a minimum distance followed by a double prime. The centers of the first pair of double primes are d_1 and d_2 and the center of the double constellation is c_1. From Table 1, it is clear that the third double prime would have a center at c_8. Hence the minimum distance possible between the constellation consisting of the two pairs of double primes and the next double prime must be greater than or equal to $c_8-c_1=15$.

Table 1 Closest double prime to a constellation formed by a double prime pair.

sequence	p_1	d_1	p_2	e_1	c_1	e_2	p_3	d_2	p_4	e_3	c_2
divisors/prime	*prime*	$2,3$	*prime*	2	$3,5$	2	*prime*	$2,3$	*prime*	$2,5$	3
sequence	e_4	c_3	e_5	c_4	e_6	c_5	e_7	c_6	e_8	c_7	e_9
divisors/prime	2		$2,3$	5	2	3	2		$2,3,5$		2

Table 2 The first few examples of computations according to Table 1.

The pair of double primes.	p_1	p_2	p_3	p_4		
The next double primes.					c_6	c_7
sequence 1	11	13	17	19	29	31
sequence 2	18041	18043	18047	18049	18059	18061
sequence 3	97841	97843	97847	97849	97859	97861
sequence 4	165701	165703	165707	165709	165719	165721
sequence 5	392261	392263	392267	392269	392279	392281
sequence 6	663581	663583	663587	663589	663599	663601
sequence 7	1002341	1002343	1002347	1002349	1002359	1002361
sequence 8	1068701	1068703	1068707	1068709	1068719	1068721

In Table 2, it is computationally shown that the minimum distance between the two pairs of double primes is an achievable 15, or equivalently that the minimum extent for such a constellation is $21 = c_7 - p_1 + 1$ where p_1 and c_7 are respectively the smallest and the largest primes.

For sequences $2,3,5,6$ and 7 the entry c_3 is a composite number and for sequences $1,3$ and 8 c_3 is prime. Note also the regularity of the last digits of the primes. This is due to c_1 having a factor of 5. There are 35 such constellations up to $10 < 7$ consisting of two double primes at minimum distance between them followed by a third double prime at smallest distance from the first two double primes.

The question is now, can Table 1 be expanded so that symmetric constellations can be defined? A possible expansion is shown in the table in Figure 1 where an added entry called *rel. index* has been added for each entry. The rel. index is a number indicating the relative integer distance to the central entry e_8 in the table. In the new table, the original divisor constraints with respect to $2,3,5$ of Table 1 have been expanded by the divisor 7 tested for the range of rel. indices -19 to -13 with the original divisors $2,3,5$ resulting in $c_3, c_6, c_7, c_{10}, c_{12}, c_{13}, c_{17}, c_{19}$ being unspecified. Then the red arrows show that all the entries at the 7 rel. indices -19 to -13 cannot be a multiple of 7 hence a contradiction to the assumption that a symmetric table with the primes $p_1, p_2, p_3, p_4, c_3, c_6, c_7, c_{10}, c_{12}, c_{13}, c_{17}, c_{19}$, i.e. a 12-tuple, is impossible in this form.

The possible symmetric constellations $C(m,n)$ which are now considered are restricted by the following theorems.

Theorem 3. *Consider a symmetric $C(m,n)$ constellation where n is odd. Then the center of the constellation is a prime P and the constellation can only have primes at offsets $k*6, k = 1, \cdots$ from P.*

Proof. The only possibility for the symmetric distribution of the primes constrains the center of the constellation to be a prime P. Let the sequence of integers around P be denoted by e_i, where $|i| = 1,2,\cdots$. Consider e_{-1}, P, e_1. Then, this is a sequence of three consecutive integers hence one and only one element of the sequence is divisible by 3. Suppose this is e_1. Then, the symmetry requirement means that e_2 is also divisible by 3. Hence e_{-1} and e_1 are both divisible by 3, a contradiction to the original claim that only one of the numbers in the sequence e_{-1}, P, e_1 is divisible by 3. Now consider the sequence $e_{-5}, c_{-4}, e_{-3}, c_{-2}, e_{-1}, P, e_1, c_2, e_3, c_4, e_5$. The same argument as above implies that c_2 and c_{-2} must be divisible by 3. Since e_1, e_3, e_5 and e_{-1}, e_{-3}, e_{-5} are even the first possible

Range for testing divisibility by 7

rel. index	−19	−18	−17	−16	−15	−14	−13	−12	−11
sequence	p_1	d_1	p_2	e_1	c_1	e_2	p_3	d_2	p_4
divisors/prime	*prime*	2, 3	*prime*	2	3, 5	2	*prime*	2, 3	*prime*
rel. index	−10	−9	−8	−7	−6	−5	−4	−3	−2
sequence	e_3	c_2	e_4	c_3	e_5	c_4	e_6	c_5	e_7
divisors/prime	2, 5	3	2		2, 3	5	2	3	2
rel. index	−1	0	1						
sequence	c_6	e_8	c_7						
divisors/prime		2, 3, 5							
rel. index	2	3	4	5	6	7	8	9	10
sequence	e_9	c_8	e_{10}	c_9	e_{11}	c_{10}	e_{12}	c_{11}	e_{13}
divisors/prime	2	3	2	5	2, 3		2	3	2, 5
rel. index	11	12	13	14	15	16	17	18	19
sequence	c_{12}	e_{14}	c_{13}	e_{15}	c_{14}	e_{16}	c [illegible]	e_{17}	[illegible]
divisors/prime		2, 3		2	3, 5	2		2, 3	

Fig. 1 Expanded Table 1.

prime would be e_6. In the tables in Figures 2 and 3, the rel. index denotes the relative offsets of the integers from the center prime P which is at rel. index 0. By extending the above results to a table it follows that the only possible indices that can be prime are at locations offset from P by $6*k, |k| = 1, 2, \cdots$ as shown in the tables in Figure 2 and 3 depending on whether rel. index -1 or rel. index +1 is divisible by 3. The possible symmetric constellation are the same in both cases.

rel. index	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18	-17
divisibility	2		2,3		2	3	2		2,3		2
rel. index	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6
divisibility	3	2		2,3		2	3	2		2,3	
rel. index	-5	-4	-3	-2	-1	0	1	2	3	4	5
divisibility	2	3	2		2,3	P	2	3	2		2,3
rel. index	6	7	8	9	10	11	12	13	14	15	16
divisibility		2	3	2		2,3		2	3	2	
rel. index	17	18	19	20	21	22	23	24	25	26	27
divisibility	2,3		2	3	2		2,3		2	3	2

Fig. 2 Divisibility relative to center prime - odd case - rel. index -1 divisible by 3.

Theorem 4. *Consider a symmetric $C(m,n)$ constellation where n is even. Then the center of the constellation is a composite number c and the constellation base case can only have primes at locations given in table in Figure 4. The even second case can only have primes at locations given in table in Figure 5.*

Proof. Due to the symmetry requirement the center of the constellation would have to be a composite number

c with index 0 as used in Figures 4 and 5 or not an index item. In the first case, the extent of the constellation is m and the smallest resp. largest primes are p_1 and p_m. Then $m = 2w + 1$ where w is the number of items the left, resp. right of c. For the second case $m = 2w$ which contradicts the fact that there is an odd number of times between two primes.

The base case of the 3 entries around the prime center for the even case is therefore p_{i-1}, c_i, p_{i+1}. Clearly c_i is divisible by 2 and 3. The possible base cases of even symmetric constellations would therefore have to conform to the divisibilities given in Figure 4.

The second case of even symmetric n-tuple has the 5 entries around the prime center as $p_{i-2}, c_{i-1}, c_i, c_{i+1}, p_{i+2}$ where c_{-1} and $c+1$ are even. Consider c_i, c_{i+1}, p_{i+1}. Suppose c_{i+1} is divisible by 3. This would imply that p_{i-2} is divisible by 3, hence a contradiction. So, c_i is divisible by 3. The resulting divisibilities are given in Figure 5.

rel. index	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18	-17
divisibility	2	3	2		2,3		2	3	2		2,3
rel. index	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6
divisibility		2	3	2		2,3		2	3	2	
rel. index	-5	-4	-3	-2	-1	0	1	2	3	4	5
divisibility	2,3		2	3	2	P	2,3		2	3	2
rel. index	6	7	8	9	10	11	12	13	14	15	16
divisibility		2,3		2	3	2		2,3		2	3
rel. index	17	18	19	20	21	22	23	24	25	26	27
divisibility	2		2,3		2	3	2		2,3		2

Fig. 3 Divisibility relative to center prime - odd case - rel. index 1 divisible by 3.

Note that in Figure 2, there are empty non-colored entries at rel. indices $-26, -14, -2, 10, 22$ as well as $-20, -8, 4, 16$ and for Figure 3 the empty entries are $-22, -10, 2, 14, 26$ and $-16, -4, 8, 20$. As the value of n is increased some of these entries may be primes which will not have corresponding symmetric primes hence the constellation might be non-symmetric.

Some simple examples illustrating these definitions are provided when discussing the double (two-tuple) and quadruple (four-tuple) prime constellations.

In this paper, symmetric prime constellations are considered for various values of m and n. Of particular

rel. index	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18	-17
divisibility	3	2		2,3		2	3	2		2,3	
rel. index	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6
divisibility	2	3	2		2,3		2	3	2		2,3
rel. index	-5	-4	-3	-2	-1	0	1	2	3	4	5
divisibility		2	3	2	P	2,3	P	2	3	2	
rel. index	6	7	8	9	10	11	12	13	14	15	16
divisibility	2,3		2	3	2		2,3		2	3	2
rel. index	17	18	19	20	21	22	23	24	25	26	27
divisibility		2,3		2	3	2		2,3		2	3

Fig. 4 Divisibility relative to composite center - even base case.

rel. index	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18	-17
divisibility	2,3		2	3	2		2,3		2	3	
rel. index	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6
divisibility		2,3		2	3	2		2,3		2	3
rel. index	-5	-4	-3	-2	-1	0	1	2	3	4	5
divisibility	2		2,3	*P*	2	3	2	*P*	2,3		2
rel. index	6	7	8	9	10	11	12	13	14	15	16
divisibility	3	2		2,3		2	3	2		2,3	
rel. index	17	18	19	20	21	22	23	24	25	26	27
divisibility	2	3	2		2,3		2	3	2		2,3

Fig. 5 Divisibility relative to composite center - even second case.

interest are symmetric constellations having the smallest possible extent for a fixed value of n.

Since a double prime is a pair of primes p_1 and p_2 separated by a composite even number e_1 it is an even symmetric $C(3,2)$ constellation which clearly has a minimal extent 3.

As noted, a great deal of research has also focused on whether there is an infinite number of double primes or not, see for example [12, 25].

3 Symmetric triple (three-tuple) prime constellations

One might ask if there are non-symmetric and symmetric prime constellations where $n = 3$ of the form p_1, e_1, p_2, e_2, p_3, i.e. $m = 5$. Indeed, there is at least one example: $3,4,5,6,7$ where $3,5,7$ are primes and $4,6$ are both even and composite.

To show that there are no further constellations of this kind the simple result used in Theorem 3 is needed. That is, given a sequence of n consecutive positive integers one and only one of the integers is divisible by n. Therefore considering that p_1, e_1, p_2 is a sequence of three numbers at least one has to be divisible by 3. Since p_1 and p_2 are primes it means that e_1 is divisible by 3. In the same manner p_2 or p_3 has to be divisible by 3. This means that one of the other sequences e_1, p_2, e_2 and e_1, p_2, e_2 has two numbers divisible by 3, which is not possible. The only resolution is that one of p_1, p_2 or p_3 is composite hence the $C(5,3)$ constellation p_1, e_1, p_2, e_2, p_3 is not possible. The next triple constellations possible are therefore either case_1: $p_1, e_1, p_2, e_2, c_1, e_3, p_3$ where c_1 is a composite integer or case_2: $p_1, e_1, c_1, e_2, p_2, e_3, p_3$ where c_1 is a composite. In both cases, the constellations are non-symmetric.

The first few examples of case_1 constellations are the triple sets of primes: $(5,7,11)$, $(11,13,17)$, $(17,19,23)$, $(41,43,47)$, $(101,103,107)$, $(107,109,113)$ where the first four constellations overlap and the last two constellations overlap. The first few examples of case_2 constellations are the triple sets of primes: $(7,11,13)$, $(13,17,19)$, $(37,41,43)$, $(67,71,73)$, $(97,101,103)$, $(103,107,109)$ where the first two constellations and the last two constellations overlap. In each case parentheses are used to clearly distinguish the constellations. These constellations are not symmetric and hence do not fall into the general theme of this paper. Checking if they can form a basis for 6-tuples or 7-tuples results in one case that will be considered in the section on 6-tuples. There are no possible 7-tuples on the basis of these two cases due to Theorem 3.

The question now is whether symmetric triple constellations exist and if therefore there are triple minimal constellations. The first potential triple symmetric constellation would be the $C(9,3)$ constellation p_1, e_1, c_1, e_2, p_2, e_3, c_2, e_4, p_3 where c_1, c_2 and c_3 are composite. Divisibility properties for this constellation are given in Table 3. The sequence is read from left to right and top to bottom.

Then, both from the divisibility by 3 for the sequence e_2, p_2, e_3 and from Theorem 3 it follows that this is

Table 3 Potential triple prime, $C(9,3)$.

Parameter	value	value	value	value	value	value	value	value	value
rel. index	-4	-3	-2	-1	0	1	2	3	4
constellation	p_1	e_1	c_1	e_2	p_2	e_3	c_2	e_4	p_3
divisors/prime	P	2	C	2	P	2	C	2	P

not a possible symmetric constellation. The next potential symmetric constellation with 3 primes is the $C(13,3)$ constellation p_1, e_1, c_1, e_2, c_2, e_3, p_2, e_4, c_3, e_5, c_4, e_6, p_3. The properties of this constellation are given in Table 4.

Table 4 Potential triple prime, $C(13,3)$.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val	val
rel. index	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
constellation	p_1	e_1	c_1	e_2	c_2	e_3	p_2	e_4	c_3	e_5	c_4	e_6	p_3
divisors/prime	P	2	C	2	C	2	P	2	C	2	C	2	P

Since this does not contradict Theorem 3, it is a possible symmetric triple constellation. The first example of such a constellation is given by Table 5.

Table 5 Triple prime $(47,53,59)$ with $m = 13$.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val	val
constellation	47	48	49	50	51	52	53	54	55	56	57	58	59
factors	47	$2^4 * 3$	7^7	$2 * 5^2$	$3 * 17$	$2^2 * 13$	53	$2 * 3^3$	$5 * 11$	$2^3 * 7$	$3 * 19$	$2 * 29$	59

Further examples for the centers p_2 of triple primes of this kind are $157, 167, 257, 263, 273$. There are 758163 such triple primes up to 10^9 showing that they are relatively common. Having eliminated the possibilities of smaller symmetric constellations of triple primes it follows that this version of $C(13,3)$ is a minimal symmetric triple constellation.

4 Symmetric quadruple (four-tuple) prime constellations

Quadruple prime constellations were discussed in the context of a hierarchy of primes in [26, 27].

Quadruple primes might be formed from two double primes each having 2-centers d_1 and d_2, with $d_1 < d_2$. This would define an even base case for a quadruple prime if they are separated by 3 non-primes. It is the smallest distance possible between two double primes if $d_1 \geq 6$ (assuming that the 2-centers are greater than 6) (see also Theorem 4). A symmetric quadruple prime constellation sequence would therefore be $p_1, e_1, p_2, e_2, c, e_3, p_3, e_4, p_4$ where $e_1 = d_1$, $e_4 = d_2$ and the centers of such quadruple primes are given by $c = (d_2 + d_2)/2$.

Let's consider e_2, c, e_3. One of the integers has to be divisible by 3. It cannot be e_2 since this would imply p_2 would be divisible by 3, a contradiction since p_4 is a prime. For e_3 being divisible by 5 it would imply p_1 being composite, a contradiction. Hence c is divisible by 3. Then consider p_2, e_2, c, e_3, p_3 and assume one of the numbers is divisible by 5. It cannot be e_3 since this would imply that p_1 is composite, assume $p_1 > 5$. Similarly for e_3. Since p_2 and p_3 are primes, it shows that c must have a factor of 5. Note also that both one of e_1,e_2,c,e_3 and e_2,c,e_3,e_4 must be divisible by 7. In Table 6, the divisibilities for these quadruple primes are shown assuming p_1 is a prime greater than 5.

Table 6 Divisibility of symmetric 4-tuple prime $C(3,9)$ by 2, 3 and 5.

Parameter	value	value	value	value	value	value	value	value	value
rel. index	-4	-3	-2	-1	0	1	2	3	4
constellation	p_1	e_1	p_2	e_2	c	e_3	p_3	e_4	p_4
divisors/prime	P	$2,3$	P	2	$3,5$	2	P	$2,3$	P

Computational examples of symmetric quadruple prime constellations are provided in Table 7.

Table 7 Some 4-tuple symmetric primes, $C(9,4)$.

constellations	p_1	p_2	p_3	p_4	q
quadruple prime, special case 1	5	7	11	13	9
quadruple prime 2	11	13	17	19	15
quadruple prime 3	101	103	107	109	105
quadruple prime 4	191	193	197	199	195
quadruple prime 5	821	823	827	829	825
quadruple prime 6	1481	1483	1487	1489	1485
quadruple prime 7	1871	1873	1877	1879	1875

There are 4768 symmetric quadruple $C(9,3)$ primes up to 10^8.

It is also interesting that the sequences of the lowest digit are $1,3,7,9$ starting with the second quadruple prime. Since no counterexample to this was found up to 10^7, it is conjectured that this sequence of lowest digit holds for all quadruple primes starting from the second one. The conjecture was easily verified by considering the trailing 5 for the center and then counting forwards and backwards from the center. I.e. p_1 is four steps backwards from q hence counting down from 5 results in 1 and so on.

These quadruple constellations are also minimal symmetric quadruple prime constellations from the above discussion.

5 Symmetric quintuple (five-tuple) prime constellations

The most obvious choice for a quintuple symmetric prime constellation would be to simply change the center of the quadruple prime constellation in Table 6 to a prime. This would result in a sequence of 5 primes only separated by even integers which can be easily shown to be impossible. It is also not possible from consulting the tables in Figures 2 and 3.

For the next choices for a symmetric quintuple prime Theorem 3 is consulted. Hence the first possible constellation with 5 primes will be with primes at the first two possible locations shown in Figure 2 or 3. Consider therefore the sequence of consecutive integers starting with p_1

$$p_1,e_1,c_1,e_2,c_2,e_3,p_2,e_4,c_3,e_5,c_4,e_6,p_3,e_7,c_5,e_8,c_6,e_9,p_4,e_{10},c_7,e_{11},c_8,e_{12},p_5 \tag{2}$$

where $p_1,\cdots,p_5$ are the primes of the 5-tuple and p_3 the center. Divisibilities of this sequence are shown in Table 8.

Table 8 Possible symmetric quintuple prime $C(25,5)$.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val
rel. index	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
constellation	p_1	e_1	c_1	e_2	c_2	e_3	p_2	e_4	c_3	e_5	c_4	e_6
divisors/prime	*prime*	2		2		2	*prime*	2		2		2
rel. index	0											
center	p_3											
prime	*prime*											
rel. index	1	2	3	4	5	6	7	8	9	10	11	12
constellation	e_7	c_5	e_8	c_6	e_9	p_4	e_{10}	c_7	e_{11}	c_8	e_{12}	p_5
divisors/prime	2		2		2	*prime*	2		2		2	*prime*

Consider then the sub-sequence c_4, e_6, p_3, e_7, c_5. One of the numbers in the sequence must be divisible by 5. Assume it is c_4. This would imply p_1 is composite which is not possible since it is a prime. p_3 is prime by assumption. c_5 is not possible since it would imply p_5 was composite. In a similar manner e_6 and e_7 cannot be divisible by 5. Hence there is a contradiction since one of the 5 consecutive numbers must be divisible by 5.

For the next possible constellations expanded the constellations by adding $p_1 - 6$ and $p_5 + 6$ to be prime at rel. indices $-18, 18$ (Case_1) and then choosing either the entries at rel. location $-6, 6$ to be prime or rel. entries locations $-12, 12$ to be prime (Case_2).

For case 1, it results in Table 9 where the constellation entries have been renamed.

Table 9 Case_1: possible symmetric quintuple prime.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val
rel. index	-18	-17	16	-15	-14	-13	-12	-11	-10	-9	-8	-7
constellation	p_1	e_1	c_1	e_2	c_2	e_3	c_3	e_4	c_4	e_5	c_5	e_6
divisors/prime	*prime*	2		2		2		2		2		2
rel. index	-6	-5	-4	-3	-2	-1						
constellation	p_2	e_7	c_6	e_8	c_7	e_9						
divisors/prime	*prime*	2		2		2						
rel. index	0											
center	p_3											
prime	*prime*											
rel. index	1	2	3	4	5	6						
constellation	e_{10}	c_8	e_{11}	c_9	e_{12}	p_4						
divisors/prime	2		2		2	*prime*						
rel. index	7	8	9	10	11	12	13	14	15	16	17	18
constellation	e_{13}	c_{10}	e_{14}	c_{11}	e_{15}	c_{12}	e_{16}	c_{13}	e_{17}	c_{14}	e_{18}	p_5
divisors/prime	2		2		2		2		2		2	*prime*

However, consider the sequence e_9, p_3, e_{10}. One of these integers must be divisible by 3. Suppose this is e_9. Then, this would mean that p_2 is composite hence a contradiction. A similar argument holds for e_{10}. Hence, the Case_1 constellation is not possible.

The second choice is to allow the original p_2 and p_4 to be composite and require the entries at rel. index -6 and $+6$ to be prime. The resulting table with renamed entries and divisibilities is Table 10 and where p_3 is still the center prime of the constellation.

Table 10 Case_2: possible symmetric quintuple primes.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val
rel. index	-18	-17	16	-15	-14	-13	-12	-11	-10	-9	-8	-7
constellation	p_1	e_1	c_1	e_2	c_2	e_3	p_2	e_4	c_3	e_5	c_4	e_6
divisors/prime	*prime*	2		2		2	*prime*	2		2		2
rel. index	-6	-5	-4	-3	-2	-1						
constellation	c_5	e_7	c_6	e_8	c_7	e_9						
divisors/prime		2		2		2						
rel. index	0											
center	p_3											
prime	*prime*											
rel. index	1	2	3	4	5	6						
constellation	e_{10}	c_8	e_{11}	c_9	e_{12}	c_{10}						
divisors/prime	2		2		2							
rel. index	7	8	9	10	11	12	13	14	15	16	17	18
constellation	e_{13}	c_{11}	e_{14}	c_{12}	e_{15}	p_4	e_{16}	c_{13}	e_{17}	c_{14}	e_{18}	p_5
divisors/prime	2		2		2	*prime*	2		2		2	*prime*

Case_2 was tested numerically and the following results were obtained as shown in Table 11.

Table 11 Some Case_2 $C(37,5)$ 5-tuple symmetric primes.

constellation	p_1	p_2	p_3	p_4	p_5
quintuple 1	18713	18719	18731	18743	18749
quintuple 2	25603	25609	25621	25633	25639
quintuple 3	28051	28057	28069	28081	28087
quintuple 4	31033	31039	31051	31063	31069
quintuple 5	97423	97429	97441	97453	97459
quintuple 6	103651	103657	103669	103681	103687

There are 5162 instances of these quintuple primes up to 10^9. It is interesting to note that there are two types of these primes depending on the sequence of low order digits. These types are distinguished by the divisibility by 3 of the even integers that are just before or just after the center prime.

6 Symmetric six-tuple prime constellations

In the same manner, as in the previous section an obvious choice for symmetric six-tuple constellations would be to start with the symmetric quadruple constellations and extend those with a prime at the first extension from the extremes of the quadruple constellation obtaining a new constellation

$p_1, e_1, p_2, e_2, p_3, e_3, c_1, e_4, p_4, e_5, p_5, e_6, p_6$.

This is clearly not a possible constellation since it would include sequences of three consecutive primes (i.e. p_1, e_1, p_2, e_2, p_3) only separated by even integers which has been shown to be impossible, except for the trivial case of $3,4,5$.

Hence, either p_1 and p_2 (and symmetrically p_5 and p_6) or p_2 and p_3 (and symmetrically p_4 and p_5) have to be separated by an even-odd-even sequence.

In the first case, the resulting sequence would be

$$p_1, e_1, c_1, e_2, p_2, e_3, p_3, e_4, c_2, e_5, p_4, e_6, p_5, e_7, c_3, e_8, p_6. \tag{3}$$

Let's consider e_4, c_2, e_5. One of the integers in this sequence must be divisible by 3. Suppose this is e_4. This would imply p_4 being divisible by 3 which is not possible since it is a prime. A similar argument holds for e_5. Hence c_2 is divisible by 3.

Now consider p_3, e_4, c_2, e_5, p_4. In this case, one of the integers in the sequence must be divisible by 5. Let's consider e_4. It is is divisible by 5 then this implies p_5 is divisible by 5 which is not possible since p_5 is a prime. By symmetry e_5 is not divisible by 5. Since p_3 and p_4 are primes it leaves c_2 which is therefore divisible by 5.

For the sequence $p_3, e_4, c_2, e_5, p_4, e_6, p_5$ consider again e_4. If e_4 is divisible by 7 then so would be p_1 as well, which is not possible since p_1 is a prime. Similarly e_5 is excluded from being divisible by 7. For e_6 being divisible by 7 would imply p_2 begin divisible by 7, yet another contradiction. The remaining integers are prime, except for c_2 which therefore is divisible by 7.

The resulting divisibility properties of the sequence in Eq. 3 are provided in Table 12. This is an example of an even constellation that conforms to the divisibility properties found in Figure 5.

Table 12 Divisibility of symmetric six-tuple prime constellations - $C(17,6)$. Center configuration is even second case p_3, e_4, c_2, e_5, p_4.

Parameter	value	value	value	value	value	value
rel. index	-8	-7	-6	-5	-4	-3
constellation	p_1	e_1	c_1	e_2	p_2	e_3
divisibility/prime	*prime*	2,3,7		2,5	*prime*	2
center	p_3	e_4	c_2	e_5	p_4	
rel. index	-2	-1	0	1	2	
constellation	*prime*	2,3	3,5,7	2	*prime*	
rel. index	3	4	5	6	7	8
constellation	e_6	p_5	e_7	c_3	e_8	p_6
divisibility/prime	2	*prime*	2,3,7		2	*prime*

Table 13 The first few symmetric sixtuple constellations with $C(17,6)$ are given here.

constellation	p_1	p_2	p_3	p_4	p_5	p_6
sixtuple prime 1	7	11	13	17	19	23
sixtuple prime 2	97	101	103	107	109	113
sixtuple prime 3	16057	16061	16063	16067	16069	16073
sixtuple prime 4	19417	19421	19423	19427	19429	19433
sixtuple prime 5	43777	43781	43783	43787	43789	43793
sixtuple prime 6	1091257	1091261	1091263	1091267	1091269	1091273

There are two items of interest in this case. First of all, the low order digits of the primes in the constellations form a sequence $7,11,13,17,19,23$ by subtracting the first prime less 7 from the sequence of primes for a given sixtuple. That is, for example for the second sixtuple the same sequence as is given for the first sixtuple prime is repeated if $97-7$ is subtracted from the primes. This is explained by considering that c_2 in Table 12 is divisible by 5 and not divisible by 2, hence the low digit of c_2 is 5. The rest follows by counting up and down from c_2.

There is also a large jump from the first 5 sixtuples to the following ones as shown in Table 13.

There are 18 sixtuples of this kind up to 10^7.

One can ask if it is possible to combine two triple prime constellations symmetrically around an even center to obtain sixtulpe primes and if so, do they have a smaller width. The divisibility of such a constellation would be as in Table 14. That is, the sequence p_3, e_7, p_4 constants of three consecutive integers, one of which has to be divisible by 3. Since e_7 is the only possible choice the entries divisible by 3 can be entered into the table together with the divisibility by 2. Then consider the sequence c_4, e_6, p_3, e_7, p_4 of consecutive integers. One of these integers must be divisible by 5. It cannot be c_4 since this would imply that p_1 is a prime. Similarly e_6 is not a candidate since this would imply that p_2 would be a prime. Since p_3 and p_4 are primes it leaves e_7 which is therefore divisible by 5.

Table 14 Divisibility of symmetric sixtuple prime constellations with $C(27,6)$, even base case.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val
rel. index	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2
constellation	p_1	e_1	c_1	e_2	c_2	e_3	p_2	e_4	c_3	e_5	c_4	e_6
divisibility	*prime*	2,3		2,5	3	2	*prime*	2,3	5	2	3	2
rel. index	-1	0	1									
constellation	p_3	e_7	p_4									
divisibility	*prime*	2,3,5	*prime*									
rel. index	2	3	4	5	6	7	8	9	10	11	12	13
constellation	e_8	c_5	e_9	c_6	e_{10}	p_5	e_{11}	c_7	e_{12}	c_8	e_{13}	p_6
divisibility	2	3	2	5	2,3	*prime*	2	3	2,5		2,3	*prime*

The first few symmetric sixtuples of this kind are given in Table 15.

Table 15 The first few symmetric sixtuple constellations $C(27,6)$, even base case.

constellation	p_1	p_2	p_3	p_4	p_5	p_6
sixtuple prime 1	587	593	599	601	607	613
sixtuple prime 2	19457	19463	19469	19471	19477	19483
sixtuple prime 3	101267	101273	101279	101281	101287	101293
sixtuple prime 4	179807	179813	179819	179821	179827	179833
sixtuple prime 5	193367	193373	193379	193381	193387	193393

There are 66 such sixtuples up to 10^7.

There is also another similar case as shown in Table 16. In the previous case there was one composite center with a prime on each side. In the new case there are two primes at the center of the sixtuple separated by three composite numbers. Otherwise the two cases are similar. Here c_5 is clearly divisible by 3. Then consider the sequence p_3, e_7, c_5, e_8, p_4. One of these integers must be divisible by 5. Since e_7 cannot be divisible by 5 since it would imply p_6 being composite it follows that c_5 is the only choice due to the symmetry and the fact that p_3 and p_4 are prime.

There are 34 sixtuples of this kind up to 10^7 and the first 7 examples are shown in Table 17. Note that the last digits of the primes are the same for each sixtuple due to the center being odd and divisible by 5.

Table 16 Divisibility of symmetric sixtuple prime constellations - with $C(29,6)$.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val
constellation	p_1	e_1	c_1	e_2	c_2	e_3	p_2	e_4	c_3	e_5	c_4	e_6
divisibility	*prime*	2	3	2	5	2,3	*prime*	2	3	2		2,3
constellation	p_3	e_7	c_5	e_8	p_4							
divisibility	*prime*	2	3,5	2								
constellation	e_9	c_6	e_{10}	c_7	e_{11}	p_5	e_{12}	c_8	e_{13}	c_9	e_{14}	p_6
divisibility	2,3	2	2,5	3	2	*prime*	2,3	5	2	3	2	*prime*

Table 17 The first few symmetric sixtuple constellations, third case, are given with $C(29,6)$.

constellation	p_1	p_2	p_3	p_4	p_5	p_6
sixtuple prime 1	151	157	163	167	173	179
sixtuple prime 2	20101	20107	20113	20117	20123	20129
sixtuple prime 3	128461	128467	128473	128477	128483	128489
sixtuple prime 4	297601	297607	297613	297617	297623	297629
sixtuple prime 5	350431	350437	350443	350447	350453	350459
sixtuple prime 6	354301	354307	354313	354317	354323	354329
sixtuple prime 7	531331	531337	531343	531347	531353	531359

Considering the table in Figure 5 the question is whether the minimal possible configuration
$p_1, e_1, p_2, e_2, c_1, e_3, p_3, e_4, p_4, e_5, c_2, e_6, p_5, e_7, p_6$
in Table 18 will produce sixtuple primes.

Table 18 Divisibility of possible symmetric six-tuple prime constellation - $C(15,6)$. Even base case.

Parameter	value	value	value	value	value	value
constellation	p_1	e_1	p_2	e_2	c_1	e_3
divisibility/prime	*prime*	2,3	*prime*	2		2
center	p_3	e_4	p_4			
constellation	*prime*	2,3	*prime*			
constellation	e_5	c_2	e_6	p_5	e_7	p_6
divisibility/prime	2		2	*prime*	2	*prime*

A computation up to 10^9 produced one example: $5, 7, 11, 13, 17, 19$.

Proof for the impossibility of such constellations for $p_1 > 5$ was given in Theorem 2.

From the table in Figure 5 it can be seen that another possible minimal sixtuple constellation in this case would be $p_1, e_1, c_1, e_2, p_2, e_3, p_3, e_4, c_2, e_5, p_4, e_6, p_5, e_7, c_3, e_8, p_6$. Will this $C(17,6)$ constellation produce symmetric sixtuple prime constellations?

Again a computation produces one example $5, 9, 11, 15, 17, 21$. A proof similar to the proof in Theorem 2 shows that there are no further examples when $p_1 > 5$.

7 Symmetric seventuple prime constellations

From Theorem 3, it is clear that the only possible symmetric seventuple prime constellations have to involve primes which are offset $k*6, k = 1\cdots$ steps away from the prime center of the constellation.

In the following the rel. index will denote the locations of primes symmetrically relative to a prime center c. As an example, the locations of the primes in the case of $-24,-12,-6,0,6,12,24$ are as shown in Eq. 4.

$$\cdots,-24,\cdots,-12,\cdots,-6,\cdots,c,\cdots,6,\cdots,12,\cdots,24,\cdots \tag{4}$$

Also, from the discussion above it turns out that the first offset with the first prime greater than 5 is not a possible choice as noted when discussing the quintuple constellations. Hence the prime locations of Eq. 4 are not possible.

Numerically experimenting with the choice $1-24,-18,-12,0,12,18,24$ provided no seventuples for values up to 10^9. The possible proof that there are no seventuples of this form is missing currently.

The next possible choice was therefore offsets $-30,-18,-12,0,12,18,30$ with the center prime p_4. In Table 19, the sequence of primes is shown interleaved between composite even and composite odd numbers. The divisibility with respect to 2,3 in this case is then shown in Table 20 for the case when the even number to the left of the center prime p_4 is divisible by 3. The divisibility of the case where the number to the right of the center is divisible by 3 is symmetric with respect to the center to the first case and not displayed here.

Table 19 With rhe rel. indices $(-30,-18,-12,0,12,18,30)$ for possible symmetric seven-tuple prime constellations.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val
rel. index	-30	-29	-28	-27	-26	-25	-24	-23	-22	-21	-20	19
constellation	p_1	e_1	c_1	e_2	c_2	e_3	c_3	e_4	c_4	e_5	c_5	e_6
rel. index	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	7
constellation	p_2	e_7	c_6	e_8	c_7	e_9	p_3	e_{10}	c_8	e_{11}	c_9	e_{12}
rel. index	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
constellation	c_{10}	e_{13}	c_{11}	e_{14}	c_{12}	e_{15}	p_4	e_{16}	c_{13}	e_{17}	c_{14}	e_{18}
rel. index	6	7	8	9	10	11	12	13	14	15	16	17
constellation	c_{15}	e_{19}	c_{16}	e_{20}	c_{17}	e_{21}	p_5	e_{22}	c_{18}	e_{23}	c_{19}	e_{24}
rel. index	18	19	20	21	22	23	24	25	26	27	28	29
constellation	p_6	e_{25}	c_{20}	e_{26}	c_{21}	e_{27}	c_{22}	e_{28}	c_{23}	e_{29}	c_{24}	e_{30}
rel. index	30											
constellation	p_7											

The first few seventuples of the type shown in Tables 19 and 20 are given in Table 21 and there are 73 such seventuples up to 10^9.

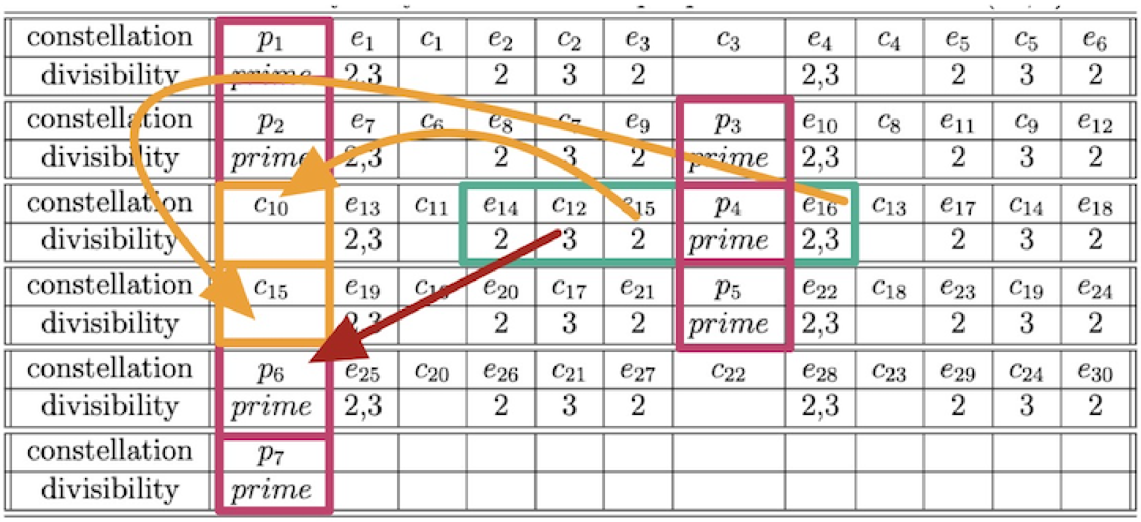
Note the number of blank entries for the divisibilities in Table 20. These entries can be composite or prime. If one or more of these entries are prime then the constellation might not satisfy the symmetry condition and it would have more than 7 primes. However, if the two entries c_{10} and c_{15} are the added primes then the resulting constellation is symmetric and might possibly be a 9-tuple. Referring to Figure 6, this possibility is now contradicted and shown as follows. The red boxes are the original 7-tuple primes and yellow boxes the two added possible primes. The green rectangle have 5 consecutive entries and hence one should be a multiple of 5. Then the red arrows point to p_5 and p_6 which are a multiple of 5 steps from e_{14} and c_{12} respectively. The yellow arrows point to c_{10} and c_{15} which are 5 steps each from e_{15} and e_{16}. The last item is p_4 which is a prime. Since none of the entries in the green box can be a multiple of 5 it contradicts the original assumption and hence the constellation is not possible.

Table 20 Divisibility of symmetric seventuple prime constellations $C(61,7)$.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val
rel. index	-30	-29	-28	-27	-26	-25	-24	-23	-22	-21	-20	19
constellation	p_1	e_1	c_1	e_2	c_2	e_3	c_3	e_4	c_4	e_5	c_5	e_6
divisibility	*prime*	2,3		2	3	2		2,3		2	3	2
rel. index	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	7
constellation	p_2	e_7	c_6	e_8	c_7	e_9	p_3	e_{10}	c_8	e_{11}	c_9	e_{12}
divisibility	*prime*	2,3		2	3	2	*prime*	2,3		2	3	2
rel. index	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
constellation	c_{10}	e_{13}	c_{11}	e_{14}	c_{12}	e_{15}	p_4	e_{16}	c_{13}	e_{17}	c_{14}	e_{18}
divisibility		2,3		2	3	2	*prime*	2,3		2	3	2
rel. index	6	7	8	9	10	11	12	13	14	15	16	17
constellation	c_{15}	e_{19}	c_{16}	e_{20}	c_{17}	e_{21}	p_5	e_{22}	c_{18}	e_{23}	c_{19}	e_{24}
divisibility		2,3		2	3	2	*prime*	2,3		2	3	2
rel. index	18	19	20	21	22	23	24	25	26	27	28	29
constellation	p_6	e_{25}	c_{20}	e_{26}	c_{21}	e_{27}	c_{22}	e_{28}	c_{23}	e_{29}	c_{24}	e_{30}
divisibility	*prime*	2,3		2	3	2		2,3		2	3	2
rel. index	30											
constellation	p_7											
divisibility	*prime*											

Table 21 The first few symmetric seventuple constellations $C(61,7)$.

constellation	p_1	p_2	p_3	p_4	p_5	p_6	p_7
seventuple 1	12003179	12003185	12003191	12003197	12003209	12003227	12003221
seventuple 2	14907619	14907625	14907631	14907637	14907649	14907667	14907661
seventuple 3	19755271	19755277	19755283	19755289	19755301	19755319	19755313



constellation	p_1	e_1	c_1	e_2	c_2	e_3	c_3	e_4	c_4	e_5	c_5	e_6
divisibility	*prime*	2,3		2	3	2		2,3		2	3	2
constellation	p_2	e_7	c_6	e_8	c_7	e_9	p_3	e_{10}	c_8	e_{11}	c_9	e_{12}
divisibility	*prime*	2,3		2	3	2	*prime*	2,3		2	3	2
constellation	c_{10}	e_{13}	c_{11}	e_{14}	c_{12}	e_{15}	p_4	e_{16}	c_{13}	e_{17}	c_{14}	e_{18}
divisibility		2,3		2	3	2	*prime*	2,3		2	3	2
constellation	c_{15}	e_{19}	c_{16}	e_{20}	c_{17}	e_{21}	p_5	e_{22}	c_{18}	e_{23}	c_{19}	e_{24}
divisibility		2,3		2	3	2	*prime*	2,3		2	3	2
constellation	p_6	e_{25}	c_{20}	e_{26}	c_{21}	e_{27}	c_{22}	e_{28}	c_{23}	e_{29}	c_{24}	e_{30}
divisibility	*prime*	2,3		2	3	2		2,3		2	3	2
constellation	p_7											
divisibility	*prime*											

Fig. 6 Proof of impossibility of suggestion for Table 20.

In general, as the tuples increase in number these added non-specified entries cause greater and greater problems when searching for pure n-tuple constellations, especially for the odd numbered tuples, where by pure is meant that there are no other primes in the constellation.

8 Symmetric octuple prime constellations

In [26, 27], general octuple prime constellations with centers $n+19$ were discussed.

Table 22 Divisibility of symmetric octuple prime constellations.

Parameter	val	val	val	val	val	val	val	val	val	val
rel. index	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10
constellation	p_1	e_1	p_2	e_2	c_1	e_3	p_3	e_4	p_4	e_5
divis	*prime*	2,3	*prime*	2	3,5	2,7	*prime*	2,3	*prime*	2,5
rel. index	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
constellation	c_2	e_6	c_3	e_7	c_4	e_8	c_5	e_9	c_6	e_{10}
divis	3	2	7	2,3	5	2	3	2	?	2,3,5,7
rel. index	1	2	3	4	5	6	7	8	9	10
constellation	c_7	e_{11}	c_8	e_{12}	c_9	e_{13}	c_{10}	e_{14}	c_{11}	e_{15}
divis	?	2	3	2	5	2,3	7	2	3	2,5
rel. index	11	12	13	14	15	16	17	18	19	20
constellation	p_5	e_{16}	p_6	e_{17}	c_{12}	e_{18}	p_7	e_{19}	p_8	e_{20}
divis	*prime*	2,3	*prime*	2,7	3,5	2	*prime*	2,3	*prime*	2,5

These octuple constellations were generated on the bases of two quadruple constellations based on one 4-tuple with entries p_1, e_1, p_2, e_2, c_1, e_3, p_3, e_4, p_4 and the first possible following 4-tuple with entries p_5, e_{16}, p_6, e_{17}, c_{12}, e_{18}, p_7, e_{19}, p_8, e_{20}, as shown in Table 22. In the general case the entries at c_6 and c_7 were allowed to be primes or composite numbers. Only the cases where both of those entries are composite numbers confirm to the strict definition used here for symmetric prime constellations. The widths of these octal constellations are $m = 39$.

Note that the number of undetermined entries is 2 in this case whereas in the 7-tuple case there were 14 such entries (Table 20).

Examples of these symmetric prime constellations were computed up to $210 * 10^9$ finding 439 such constellations.

Another possibility is to consider Table 12 and then asking that c_1 and c_3 be primes. This would result in two double primes that are too close, hence disproving the existence of such constellations.

A further approach to symmetric octuple constellations is to start with the sequence

prime-composite-prime and then check to see if this sequence can be the central core of an octal constellation.

Computing with this $C(39,8)$ constellation resulted in 29 cases of 8-tuple up to 10^9. As examples, the primes of the first and last case are given in Table 24.

Yet another possibility is to start with a 6-tuple constellation as shown in Table 14. Noting that in this case the entries c_1 and c_8 are required to be composite for the 6-tuple case it seems reasonable to ask what happens if these entries are required to be prime. Testing this up to 10^{10} results in 102 cases of eigthtuple primes where the first few are given in Table 25.

A further example of octal primes is provided by the sequence of rel. indices

Table 23 Divisibility of symmetric octuple prime constellation - second case.

Parameter	val	val	val	val	val	val	val	val	val	val	val	val
rel. index	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8
constellation	p_1	e_1	c_1	e_2	c_2	e_3	p_2	e_4	c_3	e_5	c_4	e_6
divis	*prime*	2,3	?	2	3,5	2	*prime*	2,3	?	2,5	3	2
rel. index	-7	-6	-5	-4	-3	-2						
constellation	p_3	e_7	c_5	e_8	c_6	e_9						
divis	*prime*	2	5	2	3	2						
rel. index	-1	0	1									
constellation	p_4	e_{10}	p_5									
divis	prime	2,3,5	prime									
rel. index	2	3	4	5	6	7						
constellation	e_{11}	c_7	e_{12}	c_8	e_{13}	p_6						
divis	2	3	2	5	2,3	*prime*						
rel. index	8	9	10	11	12	13	14	15	16	1	18	19
constellation	e_{14}	c_9	e_{15}	c_{10}	e_{16}	p_7	e_{17}	c_{11}	e_{18}	c_{12}	e_{19}	p_8
divis	2	3	2,5	?	2,3	*prime*	2	3,5	2	?	2,3	*prime*

Table 24 First and last computed cases for constellation in Table 23.

Parameter	value	value	value	value	value	value	value	value
first case	p_1 344231	p_2 344237	p_3 344243	p_4 344251	p_5 344253	p_6 344259	p_7 344263	p_8 344267
last case	p_1 944554301	p_2 944554307	p_3 944554313	p_4 944554321	p_5 944554323	p_6 944554329	p_7 944554333	p_8 944554337

Table 25 The first few symmetric eight-tuple constellations $C(27,8)$ with rel. indices at $(-13,-11,-7,-1,1,7,11,13)$.

constellation	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
eight-tuple 1	17	19	23	29	31	37	41	43
eight-tuple 2	1277	1279	1283	1289	1291	1297	1301	1303
eight-tuple 3	113147	113149	113153	113159	113161	113167	113171	113173
eight-tuple 4	2580647	2580649	2580653	2580659	2580661	2580667	2580671	2580673
eight-tuple 5	20737877	20737879	20737883	20737889	20737891	20737897	20737901	20737903

$(-17,-13,-11,-1,1,11,13,17)$

This results in 160 case where the first few have centers: $(30, 103980, 113160, 1322160, 2668230)$. Of these cases 28 have a prime at rel. index -7 or 7 or both.

An example of octal primes with a larger span is given by the case $C(47,8)$ with rel. indices $(-23,\ -19,\ -17,\ -1,\ 1,\ 17,\ 19,\ 23)$. When testing for examples 66 cases were found up to 10^{10} with 24 of the cases having one or more of integers with rel. indices $(-13,\ -11,\ -7,\ 7,\ 11,\ 13)$ being prime.

A second example with $C(47,8)$ is provided by rel. indices $(-23,-19,-13,-1,13,19,23)$. Up to $30*10^{10}$ 199 examples were found. 89 of the examples had one or more of the integers at re. indices $(-17,\ -11,\ -7,\ 7,\ 11,\ 17)$ being prime.

A third example with $C(47,8)$ with rel. indices of the primes being $(-23,-19,-11,-1,1,11,19,23)$ provided 90 examples of which 40 had one or more primes at the rel. indices $(-17,\ -13,\ -7,\ 7,\ 13,\ 17)$.

A further example $C(47,8)$ constellations with rel. indices of the primes being $(-23,\ -13,\ -7,\ -1,\ 1,\ 7,\ 13,\ 23)$ provided 115 examples of which 60 had one or more primes at the rel. indices $(-19,-17,-11,11,17,19)$.

9 Symmetric 9-tuple prime constellations

As noted when the number of the primes increases for the odd symmetry case the number of possible constellations are reduced.

This lead to a conjecture that there were no symmetric 9-tuple constellations possible. The conjecture was disproved by selecting the sequence of primes at rel. indices $(-60,-42,-30,0,18,30,42,60)$.

Checking for 9-tuple primes up to $2*10^{10}$ one case was found as shown in Table 26.

Table 26 9-tuple prime, $C(120,9)$.

Parameter	value	value	value	value
constellation location divisors/prime	p_1 $c-60$ 12383210011	p_2 $c-42$ 12383210029	p_3 $c-30$ 12383210041	p_4 $c-18$ 12383210053
center location prime	p_5 c 12383210071			
constellation location divisors/prime	p_6 $c+18$ 12383210089	p_7 $c+30$ 12383210101	p_8 $c+42$ 12383210113	p_9 $c+60$ 12383210131

A further larger number of examples with one or more added primes were also found which did not satisfy the strict symmetry requirements, but some of these could possibly be 12-tuples.

10 10-tuple prime constellations

For some 10-tuple prime constellations consider Table 23 and the sequence of rel. indices
$(-19,-17,-13,-11,-7,-1,1,7,11,13,17,19)$
in the table. As noted, it was shown that for this case all of the rel. indices could not be prime. Removing a pair of rel. indices might, however, provide examples for 10-tuples.

For the first case of changing the entries at rel. indices $-19, 19$ to be composite two examples of 10-tuples with rel. indices $(-17,-13,-11,-7,-1,1,7,11,13,17)$ were found when testing up to 10^{10} as shown in Table

27.

Table 27 The two 10-tuples found up to $30*10^{10}$ with rel. indices $(-17,-13,-11,-1,1,11,13,17)$ when entries with indices $(-19,19)$ in the table in Figure 1 are composite.

10-tuple center	factors of center	entry index -1	entry index +1
30	$(2,3,5)$	prime	composite
113160	$(2(3),3,5,23,41)$	composite	composite

For the second case of rel. indices the three $(-19,-17,-13,-11,-1,1,11,13,17,19)$ allowing the entries at rel. indices (-7,7) to be composite the three cases listed in Table 28.

Table 28 The three 10-tuples found with rel. indices $(-19,-17,-13,-11,-1,1,11,13,17,19)$ when entries with indices $(-7,7)$ in the table in Figure 1 are composite.

10-tuple center	factors of center	entry index -7	entry index +7
39713433690	$(2,3,5,7,23,8222243)$	composite	composite
66419473050	(2, (3,2), (5,2), (7,2), 3012221)	composite	composite
71525244630	(2, 3, 5, (7,3), 6950947)	composite	composite

Table 29 The four 10-tuples found when rel. indices $(-19,-17,-13,-11,-7,7,11,13,17,19)$ are specified with $(-1,1)$ in the table in Figure 1 being composite.

10-tuple center	factors of center	entry index -1	entry index +1
30	$(2,3,5)$	composite	composite
1864508550	$(2,3,5,5,241,151577)$	composite	composite
4763132670	$(2,3,5,7193,22073)$	composite	composite
5302859550	$2,3,5,5,167,211691$	composite	composite

For the third case of rel. indices $(-19,-17,-13,-11,-7,7,11,13,17,19)$ allowing the entries at rel. indices $-1,1$ to be composite the four cases listed in Table 29 were found.

In Table 22 the entries c_6 and c_7 were not specified (see also Figure 5 from [26]). If these additional entries were primes then this would be examples of symmetric $C(39,10)$ of even case one prime constellations.

Numerically testing for this case resulted in three examples of such constellations up to $210*10^9$. These constellations had centers $39713433690, 66419473050, 71525244630$. In Table 30 the center of the first 10-tuple prime is $t = 39713433690 = 2*3*5*7*23*8222243$.

In Figure 7 the complete list of integers and their divisors are given for the first prime listed in Table 30.

Some examples pf 10-tuple symmetric constellations $C(47,10)$ are now provided.

The first example is provided by the sequence of primes at rel/index

$(-23,-17,-13,-7,-1,1,7,13,17,23)$

Testing up to $30*10^{10}$ resulted in one example with the rel. index center 30.

For the sequence of primes at rel/index

$(-23,-19,-17,-13,-1,1,13,17,19,23)$

no results were obtained when testing for centers up to $30*10^{10}$.

Another example of $C(47,10)$ is provided by the sequence of primes at rel. index

Table 30 The first 10-tuple $C(39,10)$ prime constellation, center $c = 39713433690$.

Parameter	val	val	val	val	val
rel. index	-19	-17	-13	-11	-1
prime	p_1	p_2	p_3	p_4	p_5
value	39713433671	39713433673	39713433677	39713433679	39713433689
rel. index	1	11	13	17	19
prime	p_6	p_7	p_8	p_9	p_{10}
value	39713433691	39713433701	39713433703	39713433707	39713433709

```
39 713 433 671   {{39 713 433 671, 1}}   PRIME
39 713 433 672   {{2, 3}, {3, 1}, {11, 1}, {67, 1}, {151, 1}, {14 869, 1}}
39 713 433 673   {{39 713 433 673, 1}}   PRIME
39 713 433 674   {{2, 1}, {97, 1}, {204 708 421, 1}}
39 713 433 675   {{3, 1}, {5, 2}, {43, 1}, {12 314 243, 1}}
39 713 433 676   {{2, 2}, {7, 1}, {5563, 1}, {254 959, 1}}
39 713 433 677   {{39 713 433 677, 1}}   PRIME
39 713 433 678   {{2, 1}, {3, 3}, {17, 1}, {47, 1}, {920 443, 1}}
39 713 433 679   {{39 713 433 679, 1}}   PRIME
39 713 433 680   {{2, 4}, {5, 1}, {19, 1}, {2633, 1}, {9923, 1}}
39 713 433 681   {{3, 1}, {12 823, 1}, {1 032 349, 1}}
39 713 433 682   {{2, 1}, {13, 1}, {179, 1}, {8 533 183, 1}}
39 713 433 683   {{7, 1}, {11, 1}, {515 758 879, 1}}
39 713 433 684   {{2, 2}, {3, 1}, {89, 1}, {139, 1}, {267 517, 1}}
39 713 433 685   {{5, 1}, {353, 1}, {887, 1}, {25 367, 1}}
39 713 433 686   {{2, 1}, {31, 1}, {137, 1}, {397, 1}, {11 777, 1}}
39 713 433 687   {{3, 2}, {263, 1}, {491, 1}, {34 171, 1}}
39 713 433 688   {{2, 3}, {149, 1}, {33 316 639, 1}}
39 713 433 689   {{39 713 433 689, 1}}   PRIME
39 713 433 690   {{2, 1}, {3, 1}, {5, 1}, {7, 1}, {23, 1}, {8 222 243, 1}}

39 713 433 690   {{2, 1}, {3, 1}, {5, 1}, {7, 1}, {23, 1}, {8 222 243, 1}}
39 713 433 691   {{39 713 433 691, 1}}   PRIME
39 713 433 692   {{2, 2}, {29, 1}, {342 357 187, 1}}
39 713 433 693   {{3, 1}, {13 237 811 231, 1}}
39 713 433 694   {{2, 1}, {11, 1}, {41, 1}, {44 028 197, 1}}
39 713 433 695   {{5, 1}, {13, 1}, {17, 1}, {619, 1}, {58 061, 1}}
39 713 433 696   {{2, 5}, {3, 2}, {541, 1}, {254 887, 1}}
39 713 433 697   {{7, 1}, {367, 1}, {461, 1}, {33 533, 1}}
39 713 433 698   {{2, 1}, {197, 1}, {4691, 1}, {21 487, 1}}
39 713 433 699   {{3, 1}, {19, 1}, {157, 1}, {4 437 751, 1}}
39 713 433 700   {{2, 2}, {5, 2}, {397 134 337, 1}}
39 713 433 701   {{39 713 433 701, 1}}   PRIME
39 713 433 702   {{2, 1}, {3, 1}, {37, 1}, {107, 1}, {359, 1}, {4657, 1}}
39 713 433 703   {{39 713 433 703, 1}}   PRIME
39 713 433 704   {{2, 3}, {7, 1}, {709 168 459, 1}}
39 713 433 705   {{3, 4}, {5, 1}, {11, 1}, {1021, 1}, {8731, 1}}
39 713 433 706   {{2, 1}, {19 856 716 853, 1}}
39 713 433 707   {{39 713 433 707, 1}}   PRIME
39 713 433 708   {{2, 2}, {3, 1}, {13, 2}, {113, 1}, {173 297, 1}}
39 713 433 709   {{39 713 433 709, 1}}   PRIME
```

Fig. 7 Complete table for the first 10-tuple.

$(-23,-17,-13,-11,-1,1,11,13,17,23)$.

In this case there were 3 resulting 10-tuples with centers $30, 807213960, 6296082240$. The result with the center 30 has three extra primes at the integers $11, 23, 37$ whereas the other two results had no extra primes in the ranges of the constellations..

Another example $C(47,10)$ is provided by the sequence of primes at rel. index
$(-23,-19,-17,-13,-11,11,13,17,19,23)$
where one result was found with the central prime being $6697423110 = (2,3,5,7,167,353,541)$.

A further example $C(47,10)$ is provided by the sequence of primes at rel. index
$(-23,-17,-13,-7,-1,1,7,13,17,23)$
One case with center at 30 when testing for examples up to $30*10^{10}$.

An example where the rel. indices of the 10-tuple are
$(-23,-19,-17,-11,-7,7,11,17,19,23)$
results in 5 cases when testing up to $30*10^{10}$ one of which has an added prime at rel. index 3. The centers of the 5 case are listed in Table 31.

Table 31 Centers of the 5 10-tuple constellations $C(47,10)$.

90	1011208680	2233694520	4143953640	6486125010

For a last example the rel. indices are $(-23,-19,-13,-7,-1,1,7,13,19,23)$. Of the three examples found with centers listed in Table 32, the first constellation has primes at re. indices $(-17,11)$ and the second constellation has a prime at rel index (11). The last case has no primes at the rel. indices $(-17,-11,11,17)$.

Table 32 Centers of the last 3 10-tuple constellations $C(47,10)$.

60	967352040	4407582630

11 The possible minimal symmetric constellations

Table 33 lists the minimal even symmetric constellations found so far.

Table 33 Summarizing the even symmetric constellations, smallest extent so far.

configurations	value of m	number of cases	m is minimal ?	comment
2-tuples (double primes)	3	no limit	YES obvious	extensively studied
4-tuples (quadruple primes)	9	166 cases up to 10^7	YES as shown in section 4.	
6-tuples (sixtuple primes)	17	18 cases up to 10^7	YES as shown in section 6	The cases with $m < 17$ were shown to be not possible using divisibility by 2,3,5,7.
8-tuples (octuple primes)	27	28 cases up to 10^{10}	Not verified	The cases with $m = 27$ have the smallest extent found so far
10-tuples	35	2 cases up to 10^{10}	Not verified	The cases with $m = 35$ were the minimal extent found so far

Table 34 lists the moinimal odd symmetric constellations found so far.

Table 34 Summarizing the odd symmetric constellations, smallest extent so far.

constellations	value of m	number of cases	is minimal ?	comment
3-tuples (triple primes)	13	758163 up to 10^9	YES by construction	$m = 7, 11$ not possible by symmetry, $m = 5, 7$ not possible by construction
5-tuples (qunintuple primes)	37	124 up to 10^7	YES as shown in section 5	The cases with $m < 37$ were shown to be not possible using Theorem 3
7-tuples (qunintuple primes)	73	124 up to 10^9	NO	
9-tuples	121	124 up to $2 * 10^{10}$	NO	121 is the smallest extent found so far

12 Conclusion

Symmetric prime constellations have been defined and a number of cases of such constellations have been found numerically.

There are a number of possible further questions that can be explored.

1. Is there any relationship between and n-tuple and the minima extent of the n-tuple?
2. What is the maximal n for an odd n-tuple and for an even n-tuple?
3. What are the conditions for two n-tuple forming a $2n$-tuple for even n and a $2n+1$-tuple for odd n (assuming the center of the $2n+1$-tuple is a prime).
4. Can a symmetric 12-tuple constellation be found?
5. For each value of n can all the possible constellations up to a specified limit be found?
6. Does a 16-tuple exist as based on the octuples from the hierarchy presented in [26]?

If this would be the case, such a 16-tuple would have to conform to the divisibilities shown in Figure 8.

13 Declarations

13.1 Conflict of interest:

There is no conflict of interest.

13.2 Author's contributions:

J.R.-Conceptualization, Methodology, Formal Analysis, Writing-Review and Editing, Validation, Writing-Original Draft, Resources. All authors read and approved the final submitted version of this manuscript.

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13.5 Data availability statement:

All data that support the findings of this study are included within the article.

13.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

o-19	o-18	o-17	o-16	o-15	o-14	o-13	o-12	o-11	o-10
P	2,3	P	2	3,5	2,7	P	2,3	P	2,5
o-9	o-8	o-7	o-6	o-5	o-4	o-3	o-2	o-1	o
3	2	7	2,3	5	2	3	2	?	2,3,5,7
o+1	o+2	o+3	o+4	o+5	o+6	o+7	o+8	o+9	o+10
?	2	3	2	5	2,3	7	2	3	2,5
o+11	o+12	o+13	o+14	o+15	o+16	o+17	o+18	o+19	o+20
P	2,3	P	2,7	3,5	2	P	2,3	P	2,5
o+21	o+22	o+23	o+24	o+25	o+26	o+27	o+28	o+29	o+30
3,7	2		2,3	5	2	3	2,7		2,3,5
o+31	o+32	o+33	o+34	o+35	o+36	o+37	o+38	o+39	o-40
	22	3	2	5,7	2,3		2	3	2,5
o+41	o+42	o+43	o+44	o+45	o+46	o+47	o+48	o+49	o+50
	2,3,7		2	3,5	2		2,3	7	2,5
o+51	o+52	o+53	o+54	o+55	o+56	o+57	o+58	o+59	o+60
3	2		2,3	5	2,7	3	2		2,3,5
o+61	o+62	o+63	o+64	o+65	o+66	o+67	o+68	o+69	o+70
	2	3,7	2	5	2,3		2	3	2,5,7
o+71	o+72	o+73	o+74	o+75	o+76	o+77	o+78	o+79	o+80
	2,3		2	3,5	2	7	2,3		2,5
o+81	o+82	o+83	o+84	o+85	o+86	o+87	o+88	o+89	o-90
3	2		2,3,7	5	2	3	2		2,3,5
o+91	o+92	o+93	o+94	o+95	o+96	o+97	o+98	o+99	o+100
7	2	3	2	5	2,3		2,7	3	2,5
o+101	o+102	o+103	o+104	o+105	o+106	o+107	o+108	o+109	o+110
	2,3		2	3,5,7	2		2,3		2,5
o+111	o+112	o+113	o+114	o+115	o+116	o+117	o+118	o+119	o+120
3	2,7		2,3	5	2	3	2	7	2,3,5
o+121	o+122	o+123	o+124	o+125	o+126	o+127	o+128	o+129	o+130
	2	3	2	5	2,3,7		2	3	2,5
o+131	o+132	o+133	o+134	o+135	o+136	o+137	o+138	o+139	o-140
	2,3	7	2	3,5	2		2,3		2,5,7
o+141	o+142	o+143	o+144	o+145	o+146	o+147	o+148	o+149	o+150
3	2		2,3	5	2	3,7	2		2,3,5
o+151	o+152	o+153	o+154	o+1155	o+56	o+157	o+158	o+159	o+160
	2	3	2,7	5	2,3		2	3	2,5
o+161	o+162	o+163	o+164	o+165	o+166	o+167	o+168	o+169	o+170
7	2,3		2	3,5	2		2,3,7		2,5
o+71	o+172	o+173	o+174	o+175	o+176	o+177	o+178	o+179	o+180
3	2		2,3	5,7	2	3	2		2,3,5
o+181	o+182	o+183	o+184	o+185	o+186	o+1187	o+188	o+189	o+190
	2,7	3	2	5	2,3		2	3,7	2,5
o+191	o+192	o+193	o+194	o+195	o+196	o+197	o+198	o+199	o+200
P	2,3	P	2	3,5	2,7	P	2,3	P	2,5
o+201	o+202	o+203	o+204	o+205	o+206	o+207	o+208	o+209	o+210
3	2	7	2,3	5	2	3	2		2,3,5,7
o+211	o+212	o+213	o+214	o+215	o+216	o+217	o+218	o+219	o+220
	2	3	2	5	2,3	7	2	3	2,5
o+221	o+222	o+223	o+224	o+225	o+226	o+227	o+228	o+229	o+230
P	2,3	P	2,7	3,5	2	P	2,3	P	2,5

Fig. 8 Table for a possible 16-tuple based on octuples.

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