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Original Study

A novel approach for the numerical solution of nonlinear Fredholm integral equations using the Hosoya polynomial method

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Abstract

In this paper, we study the graph theoretical polynomial known as the Hosoya polynomial obtained from one of the standard classes of graphs called path. Using this polynomial applied for the numerical solution of the nonlinear Fredholm integral equation, which reduces in the algebraic system of equation with collocation points and then solving this system using Newton's iterative with the help of MATLAB, we obtain the required approximate solution. The desired results are in terms of a set of continuous polynomials over a closed interval [0,1]. Illustrative applications show the efficiency, accuracy and validity of the proposed technique.

Keywords: Path of a graph, Hosoya polynomial, collocation points, nonlinear Fredholm integral equations.
AMS 2020 codes: 65R20; 45G10; 45B05; 05C12.

1 Introduction

The real-world problems that arise in mathematical modelling usually result in functional equations, such as partial differential equations, integral and integro-differential equations (IDEs), stochastic equations and others. In particular, integral equations arise in the fields of fluid mechanics, kinetics in chemistry, organic fashion, stable country physics, and many other domains. Unique, crucial equations arise, and these equations are difficult to solve analytically. Then the numerical methods are introduced such as finite difference methods, finite-element methods, boundary element methods, etc. There are advantages and disadvantages to each of these numerical techniques, and the quest for more versatile, user-friendly, precise, and less complicated numerical techniques is a never-ending endeavor. Integral equations can be settled using a variety of techniques, including Picard's method, the Laplace transformation method, the Adomian decomposition technique, successive substitutions [1], and application to soliton theory [2–5]. Some of the recently trending wavelet methods are available in the literature concerning numerical solutions of integral and integro-differential equations, such as the Wavelet Galerkin method [6], Haar wavelet [7–10], Legendre wavelet [11], Hermite wavelet [12], Bernoulli wavelet [13, 14], Wavelet full-approximation scheme [15], Daubechies wavelet new transform method [16], and Biorthogonal Spline Wavelet Full-Approximation Transform method [17].

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In this paper, we consider the nonlinear Fredholm integral equation of the form,

$$y(x) = f(x) + \int_0^1 k(x,t)[y(t)]^m dt,\ 0 \le x,t \le 1. \tag{1}$$

Let $y(x)$ be a unique solution of equation (1) which has to be determined. Where $[y(x)]^m, m > 1$ is the nonlinear term, $f(x)$ and the kernel is assumed to be in $L^2(R)$ over the interval $0 \le x,t \le 1$.

In the field of graph theory, graph theoretical polynomial-based numerical methods have recently been used in the field of numerical analysis. The Hosoya polynomial is one of the most famous in graph theory. A graph polynomial is a graph invariant with polynomial values in algebraic graph theory. There are several graph theoretic polynomials found in literature [18–32]. Among the polynomials utilized in the numerical approximation are the Hosoya polynomials. The first paper has been published [33] for the numerical solution of the linear Fredholm integral equation by using the Hosoya polynomial, whereas [34] is a comparative study of the numerical solution of the Fredholm integral equation by using Haar wavelet and Hosoya polynomial, and also applied for the delay differential equations [35]. In this paper, we applied the Hosoya polynomial method (HPM) for the numerical solution of nonlinear Fredholm integral equations, which reduces to the system of algebraic equations, then using the MATLAB tools, we obtained the required approximate solution. Illustrative applications demonstrate the efficiency of the Hosoya polynomial method, and those results are compared favourably with the corresponding exact solutions, and absolute errors give a more accurate solution than the existing methods shown in tables and figures.

The paper is structured as follows: In Section 2, some basic definitions and properties of the Hosoya polynomial of graphs and Function approximation are included. Section 3 is devoted to the matrix of Hosoya polynomial and Kernal. Section 4 presents the convergence and error analysis. Section 5 represents a description of the Hosoya polynomial method. In Section 6, we solve some illustrative applications and demonstrate the numerical results with high accuracy and efficiency using HPM. Section 7 provides the proposed method's conclusion.

2 Hosoya polynomial and function approximation

In this work, we consider simple graph G and let V be a nonempty finite set having n vertices of graph P_n. Let X be a set with m number of unordered pairs of distinct vertices of set V. Thus, every pair, consider $x = (u,v)$ of points in X is an edge joined by points v and u which are adjacent to each other. Consider $v_1, v_2, \cdots, v_n$ the vertices of the graph a G. Let P_n be a path graph that has n vertices where $v_i, v_{i+1}, i = 1,2,\cdots,n$ are adjacent to each other. The number of edges present in path P_n is the length of P_n. In a connected graph G [18], each pair of points is connected by some path. The distance between such vertices v_i, v_j and in G is the same as the length of the shortest path that joins vertices v_i, v_j. That is given by $d(u_i, v_j)$. Harold Wiener [19] introduced the Wiener index of graph G, as the sum of the distances of these pairs of G. Let $WI(G)$ be a Wiener index of a connected graph G, that is,

$$WI(G) = \sum_{1 \le i < j \le n} d(u_i, v_j).$$

In 1988, the study of the Hosoya polynomial [20] provided necessary content about distance-based graph invariants and also pointed out that the Hosoya polynomial connection with the Wiener index is elementary. Here $H(G,\lambda)$ denotes Hosoya polynomial and is defined as,

$$H(G,\lambda) = \sum_{k \le 0} d(G,k)\lambda^k, \tag{2}$$

where λ is the parameter, and $d(G,k)$ is the number of pairs of vertices of graph G that are at distance k. The relation between the Wiener index $WI(G)$ and the Hosoya polynomial $H(G,\lambda)$, is reported in [20, 21]:

$$WI(G) = H'(G,1),$$

where $H'(G,\lambda)$ indicates the first derivative of $H(G,\lambda)$. Many authors studied these concepts like Hosoya polynomial of tress [22, 23], tori [24], zigzag polyhexnanotorus [25], zig-zag open-ended nanotubes [26], benzenoid graphs [27, 28], cluster graphs [29], Fibonacci and Lucas cubes [30], composite graphs [31], armchair open-ended nanotubes [32], and so on [36–40].

The paths P_4, P_3, P_2 are shown in Figure 1.

Fig. 1 P_4, P_3, P_2.

The Hosoya polynomial of a path P_n is given as

$$H(P_n,\lambda) = n + (n-1)\lambda + (n-2)\lambda^2 + \cdots + [n-(n-2)]\lambda^{n-2} + [n-(n-2)]\lambda^{n-1}.$$

Particularly,

$$H(P_4,\lambda) = \lambda^3 + 2\lambda^2 + 3\lambda + 4, \;\; H(P_3,\lambda) = \lambda^2 + 2\lambda + 3, \;\; H(P_2,\lambda) = \lambda + 2.$$

2.1 Function approximation

A function $f(x) \in L^2[0,1]$ is expanded as:

$$f(x) = \sum_{i=1}^{n} a_i H(P_i, x) = A^T H_P(x), \tag{3}$$

where $H_P(x)$ and A are $n \times 1$ matrices given by:

$$A = [a_1, a_2, \cdots, a_n]^T, \tag{4}$$

and

$$H_P(x) = [H(P_1,x), H(P_2,x), \cdots, H(P_n,x)]^T. \tag{5}$$

3 Hosoya polynomial and Kernel matrix

We can generalize the Hosoya polynomial of a matrix using the collocation points as follows,

$$H_n(x) = \begin{cases} H_1(x_i) = 1 \\ H_2(x_i) = x_i + 2 \\ H_3(x_i) = x_i^2 + 2x_i + 3 \\ \vdots \\ H_n(x_i) = n + (n-1)x_i + (n-2)x_i^2 + \cdots + [n-(n-2)]x_i^{n-2} + [n-(n-1)]x_i^{n-1} \end{cases}$$

where $x_i = \frac{i-0.5}{n}, i = 1, 2, \cdots, n$. Suppose, for $n = 2$

$$H_2(x) = \begin{cases} H_1(x_i) = 1, \\ H_2(x_i) = x_i + 2. \end{cases}$$

The matrix is of the form where

$$H_{2\times 2} = \begin{bmatrix} 1 & 1 \\ 2.25 & 2.75 \end{bmatrix}$$

Similarly, for $n = 4$

$$H_n(x) = \begin{cases} H_1(x_i) = 1 \\ H_2(x_i) = x_i + 2 \\ H_3(x_i) = x_i^2 + 2x_i + 3 \\ H_4(x_i) = x_i^3 + 2x_i^2 + 3x_i + 4. \end{cases}$$

The matrix is of the form

$$H_{4\times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2.12 & 2.37 & 2.62 & 2.87 \\ 3.26 & 3.89 & 4.64 & 5.51 \\ 4.40 & 5.55 & 6.90 & 8.82 \end{bmatrix}$$

Similarly, for $n = 8$, we get the 8x8 matrix of the form

$$H_{8\times 8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2.0625 & 2.1875 & 2.3125 & 2.4375 & 2.5625 & 2.6875 & 2.8125 & 2.9375 \\ 3.1289 & 3.4101 & 3.7226 & 4.0664 & 4.4414 & 4.8476 & 5.2851 & 5.5739 \\ 4.1955 & 4.6394 & 5.1633 & 5.7790 & 6.4982 & 7.3327 & 8.2941 & 9.3942 \\ 5.2622 & 5.8698 & 6.6135 & 7.5283 & 8.6552 & 10.0412 & 11.7390 & 13.8071 \\ 6.3288 & 7.1006 & 8.0667 & 9.2936 & 10.8686 & 12.9033 & 15.5379 & 18.9442 \\ 7.3955 & 8.3313 & 9.5208 & 11.0659 & 13.1135 & 15.8710 & 19.6245 & 24.7601 \\ 8.4622 & 9.5621 & 10.9752 & 12.8413 & 15.3763 & 18.9113 & 23.9449 & 31.2126 \end{bmatrix}$$

Next, for $n = 16$ as,

$$H_{16\times 16} = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 2.0 & 2.1 & 2.2 & 2.2 & 2.3 & 2.3 & 2.4 & 2.5 & 2.5 & 2.6 & 2.7 & 2.7 & 2.8 & 2.8 & 2.9 & 3.0 \\ 3.1 & 3.2 & 3.3 & 3.5 & 3.6 & 3.8 & 4.0 & 4.2 & 4.3 & 4.5 & 4.7 & 5.0 & 5.2 & 5.4 & 5.6 & 5.9 \\ 4.1 & 4.3 & 4.5 & 4.8 & 5.0 & 5.3 & 5.6 & 5.9 & 6.3 & 6.7 & 7.1 & 7.6 & 8.0 & 8.6 & 9.1 & 9.7 \\ 5.1 & 5.4 & 5.7 & 6.0 & 6.4 & 6.8 & 7.3 & 7.8 & 8.4 & 9.0 & 9.7 & 10.4 & 11.3 & 12.2 & 13.3 & 14.4 \\ 6.2 & 6.5 & 6.9 & 7.3 & 7.8 & 8.3 & 9.0 & 9.7 & 10.4 & 11.3 & 12.3 & 13.5 & 14.8 & 16.3 & 18.0 & 19.9 \\ 7.2 & 7.6 & 8.1 & 8.6 & 9.2 & 9.9 & 10.6 & 11.5 & 12.5 & 13.7 & 15.1 & 16.7 & 18.6 & 20.8 & 23.3 & 26.3 \\ 8.2 & 8.7 & 9.3 & 9.9 & 10.6 & 11.4 & 12.3 & 13.4 & 14.7 & 16.2 & 17.9 & 20.0 & 22.5 & 25.5 & 29.1 & 33.5 \\ 9.3 & 9.8 & 10.4 & 11.2 & 12.0 & 12.9 & 14.0 & 15.3 & 16.8 & 18.6 & 20.8 & 23.4 & 26.6 & 30.5 & 35.4 & 41.4 \\ 10.3 & 10.9 & 11.6 & 12.4 & 13.4 & 14.4 & 15.7 & 17.2 & 18.9 & 21.0 & 23.6 & 26.8 & 30.8 & 35.8 & 42.1 & 50.2 \\ 11.3 & 12.0 & 12.8 & 13.7 & 14.8 & 16.0 & 17.4 & 19.0 & 21.1 & 23.5 & 26.5 & 30.3 & 35.0 & 41.2 & 49.1 & 59.6 \\ 12.4 & 13.1 & 14.0 & 15.0 & 16.2 & 17.5 & 19.1 & 20.9 & 23.2 & 25.9 & 29.4 & 33.8 & 39.4 & 46.7 & 56.5 & 69.7 \\ 13.4 & 14.2 & 15.2 & 16.3 & 17.5 & 19.0 & 20.7 & 22.8 & 25.3 & 28.4 & 32.3 & 37.3 & 43.8 & 52.4 & 64.2 & 80.5 \\ 14.4 & 15.3 & 16.4 & 17.6 & 18.9 & 20.5 & 22.4 & 24.7 & 27.4 & 30.9 & 35.2 & 40.8 & 48.2 & 58.2 & 72.2 & 92.0 \\ 15.5 & 16.4 & 17.6 & 18.9 & 20.3 & 22.1 & 24.1 & 26.6 & 29.6 & 33.3 & 38.1 & 44.3 & 52.6 & 64.1 & 80.4 & 104.2 \\ 16.5 & 17.5 & 18.7 & 20.1 & 21.7 & 23.6 & 25.8 & 28.5 & 31.7 & 35.8 & 41.0 & 47.8 & 57.1 & 70.1 & 88.9 & 116.9 \end{bmatrix}$$

3.1 Kernel matrix

The kernel matrix is the square matrix obtained by evaluating the kernel function k with Hosoya polynomial $H_P(x)$. As the number of samples n tends to infinity, certain properties of the kernel matrix show a convergent behavior.

The given kernel function is,

$$K = \int_0^1 k(x,t)[y(t)]^m dt,\ 0 \leq x,t \leq 1.$$

Let us put $m = 2, n = 6$ then by applying the collocation points

$$K_i = \int_0^1 k(x_i,t)[y(t)]^2 dt,\ \ x_i = \frac{i-0.5}{6}, i = 1,2,\cdots,6.$$

Then substitute the approximated truncated series $y(x)$ as

$$y(x) = A^T H_P(x),$$

$$K_i = \int_0^1 k(x_i,t)[A^T H_P(t)]^2 dt.$$

Then reduce in the form of a 6×6 matrix as

$$K_i = \left(A^T\right)^2 \left[\int_0^1 k(x_i,t)[H_P(t)]^2 dt\right],$$

$$[K]_{6\times 6} = [A]^2_{1\times 6}\left[(k)_{6\times 6}(H)^2_{6\times 6}\right].$$

In general, we obtain $n \times n$ matrix as,

$$[K]_{n\times n} = [A]^m_{1\times n}\left[(k)_{n\times n}(H)^m_{n\times n}\right],$$

which the given kernel function is reduced in the form of kernel matrix up to $n \times n$ matrices.

4 Convergence and error analysis

In this section, we analyze the convergence and error analysis of the present technique,

Theorem 1. *Let y(x)∈ $L^2(\mathbb{R})$, then the series solution $\sum_{i=1}^{\infty} a_i H(P_i,x)$ converges to y(x).*

Proof. We know that $L^2(\mathbb{R})$ is an infinite dimensional Hilbert space and $\{H(P_1,x), H(P_2,x),\cdots\}$ froms a basis of $L^2(\mathbb{R})$, where $H(P_i,x)$ are Hosoya polynomials of path P_i. Now we shall show that series $\sum_{i=1}^{\infty} a_i H(P_i,x)$ converges to $y(x)$. We have, $a_i = \langle y(x), H(P_i,x)\rangle$, where $\langle\cdot\rangle$ is the inner product on $\mathbb{R}$.

Let $\langle S_n \rangle$ be the sequence of partial sums, given by

$$S_n = \sum_{i=1}^{\infty} a_i H(P_i,x).$$

Consider,

$$\left\langle y(x), S_n \right\rangle = \left\langle y(x), \sum_{i=1}^{\infty} a_i H(P_i,x) \right\rangle.$$

By properties of inner product space, we obtain

$$\left\langle y(x), S_n \right\rangle = \sum_{i=1}^{n} |a_i|^2.$$

Consider,

$$\|S_n - S_m\|^2 = \left\| \sum_{i=m+1}^{n} |a_i|^2 \right\|^2.$$

By norm properties, we have

$$\|S_n - S_m\|^2 = \sum_{i=m+1}^{n} |a_i|^2, \forall n > m.$$

From Bessel's inequality, we determine that

$$\sum_{i=m+1}^{n} |a_i|^2 \leq \|y(x)\|^2.$$

Hence $\left\langle S_n \right\rangle$ is bounded.
Therefore, $\left\langle S_n \right\rangle$ is a Cauchy sequence and converges to say t. Now we shall show that, $y(x) = t$
Consider,

$$\begin{aligned}
\left\langle t - y(x), H(P_i, x) \right\rangle &= \left\langle tH(P_i, x) \right\rangle - \left\langle y(x)H(P_i, x) \right\rangle \\
&= \left\langle \lim_{n\to\infty} S_n H(P_i, x) \right\rangle - a_i \\
&= \lim_{n\to\infty} \left\langle S_n H(P_i, x) \right\rangle - a_i \\
\left\langle t - y(x), H(P_i, x) \right\rangle &= 0.
\end{aligned}$$

This implies, $t = y(x)$ and $\sum_{i=1}^{n} H(P_i, x) \to y(x)$ as $n \to \infty$. Hence the proof.
Theorem 2 gives the error estimation due to Hosoya polynomial expansion.

Theorem 2. *Let $y(x) \in H_P^n[0,1]$ and $A^T H_P(x)$ is an approximate solution by making use of the Hosoya polynomial. Then bound for error is given by*

$$\|E(x)\| \leq \left\| \frac{1}{n!2^{2n-1}} \max_{x\in[0,1]} \left| y^{(n)}(x) \right| \right\|.$$

Proof.

$$\begin{aligned}
\|E(x)\|^2 &= \int_0^1 \left(y(x) - A^T H_P(x) \right)^2 dx \\
&\leq \int_0^1 \left(y(x) - P_n(x) \right)^2 dx,
\end{aligned}$$

in which $P_n(x)$ represents the interpolating polynomial of degree n that approximates $y(x)$ over $[0,1]$. By making use of the maximum error estimate for the polynomial on $[0,1]$, we have

$$\begin{aligned}
\|E(x)\|^2 &\leq \int_0^1 \left(\frac{2}{n!4^n} \max_{x\in[0,1]} \left| y^{(n)}(x) \right| \right) dx \\
&= \left\| \frac{1}{n!2^{2n-1}} \max_{x\in[0,1]} \left| y^{(n)}(x) \right| \right\|^2
\end{aligned}$$

where maximum error bound for the interpolation has been used.

5 Description of the Hosoya polynomial method

Here, we consider the nonlinear Fredholm integral equation,

$$y(x) = f(x) + \int_0^1 k(x,t)[y(t)]^m dt,\ 0 \le x,t \le 1. \tag{6}$$

The steps to solve equation (6), are as follows:
Step 1 Using the above defined equation (6), approximate the truncated series as. That is,

$$y(x) = A^T H_P(x), \tag{7}$$

where $H_P(x)$ and A are defined in equation (5) and equation (4).
Step 2 Substitute equation (7) in equation (6), which results as,

$$A^T H_P(x) = f(x) + \int_0^1 k(x,t)[A^T H_P(t)]^m dt. \tag{8}$$

Step 3 Next, we substitute the collocation point $x_i = \frac{i-0.5}{n}, i = 1,2,\cdots,n$ in equation (8), to obtain,

$$A^T H_P(x_i) = f(x_i) + \int_0^1 k(x_i,t)[A^T H_P(t)]^m dt, \tag{9}$$

$$A^T H_P(x_i) = f(x_i) + \left(A^T\right)^m \left[\int_0^1 k(x_i,t)[H_P(t)]^m dt\right].$$

Step 4 Thus, the nonlinear integral equation converts into the nonlinear system of algebraic equations with unknown coefficients as,

$$[A]_{1\times n}[H]_{n\times n} = [f]_{1\times n} + [A]^m_{1\times n}[(k)_{n\times n}(H)^m_{n\times n}]. \tag{10}$$

Step 5 On solving this system of nonlinear algebraic equations, we obtain the Hosoya coefficients A using the Newton's iterative method and then substitute these coefficients in equation (7). Hence, finally we claim the desired approximate solution of equation (6).

6 Numerical applications

To demonstrate the capability of this method, we consider a few illustrative applications from the literature and verify the accuracy and efficiency of the results:

$$Errorfunction = ||y_e(x_i) - y_A(x_i)||_\infty = \sqrt{\sum_{i=1}^{n} [y_e(x_i) - y_A(x_i)]^2},$$

where, y_e is an exact solution and y_A is the approximate solution. In this case, we study numerical results that are compared with the exact solution and existing method.
Application 1 Consider the nonlinear Fredholm integral equation [40],

$$y(x) = \frac{5}{6}x^2 - \frac{8}{105}x - 1 + \int_0^1 (x^2 t + x t^2) y^2(t) dt, 0 \le x \le 1, \tag{11}$$

which has an exact solution $y(x) = x^2 - 1$. Here we considered $n = 3$, by the proposed technique to solve equation (11) is reduced into a system of algebraic equations. Solving this system, we get the three unknown Hosoya coefficients as,

$$a_1 = 0, a_2 = -2, a_3 = 1,$$

substituting these coefficients in $y(x) = A^T H_P(x)$ as,

$$y(x) = a_1(1) + a_2(x+2) + a_3(x^2 + 2x + 3),$$

we obtain $y(x) = x^2 - 1$, which coincides with the exact solution which is the required solution of equation (11). This demonstrates the current technique has stability, accuracy and fast convergence.

Application 2 Consider the nonlinear Fredholm integral equations [7],

$$y(x) = f(x) + \int_0^1 xt[y(t)]^2 dt, 0 \le x \le 1, \tag{12}$$

where

$$f(x) = f_1(x) - \left(\frac{9}{128} - \frac{9}{32e} + \frac{7}{16e^2} + \frac{1}{16e^4}\right)x,$$

and

$$f_1(x) = \begin{cases} e^{2x-2}, & 0 \le x < 1/2 \\ -x^2 + \frac{1}{e} + \frac{1}{4}, & 1/2 \le x \le 1 \end{cases}$$

and the exact solution is $y(x) = f_1(x)$. Solving equation (12), transform into system of equations using the described method. We obtain the required numerical solution of equation (12), compared with the exact solution and existing method [7] as shown in Table 1 and Figure 2 at $n = 6$. The error analysis is represented graphically as shown in Figure 3 compared with existing method [7]. This shows it converges fast and the efficiency, stability and accuracy of the present method.

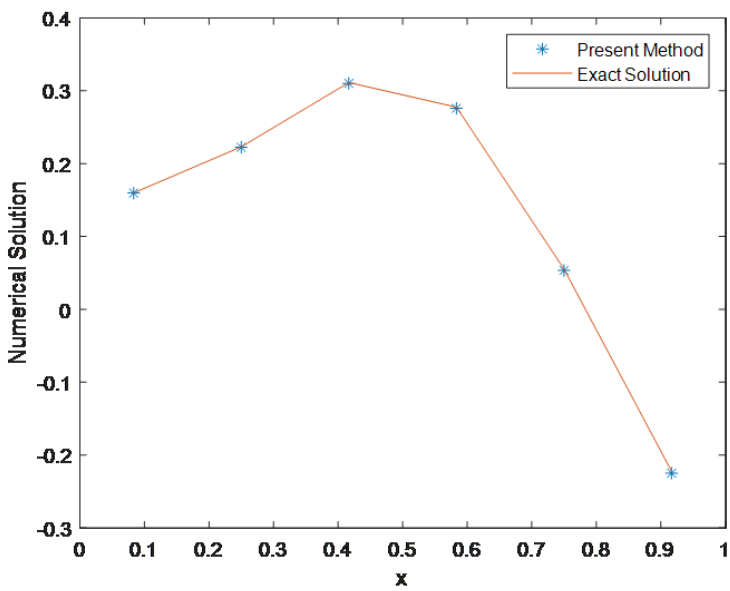

Fig. 2 Numerical solution of present method (HPM) with exact solution at $n = 6$ of application 2.

Table 1 Comparison of exact, method [7] and HPM with Abs. Error of application 2.

x	Exact solution	Method [7]	Abs. Error of Method [7]	HPM at $n=6$	Abs. Error of HPM at $n=6$
0.1	0.1652988882	0.1625177090	2.78E-03	0.1608554812	4.44E-03
0.2	0.2018965180	0.1921647474	9.73E-03	0.1942241951	7.67E-03
0.3	0.2465969639	0.2290236855	1.76E-02	0.2531871269	6.59E-03
0.4	0.3011942119	0.2744773506	2.67E-02	0.3047983984	3.60E-03
0.5	0.3678794412	0.3240321944	4.38E-02	0.3658308083	2.04E-03
0.6	0.2578794412	0.2134699868	4.44E-02	0.2619872746	4.11E-03
0.7	0.1278794412	0.0853296542	4.25E-02	0.1371122768	9.23E-03
0.8	-0.0221205588	-0.0603813505	3.83E-02	-0.0375967015	1.55E-02
0.9	-0.1921205588	-0.2236779332	3.16E-02	-0.2043777302	1.23E-02

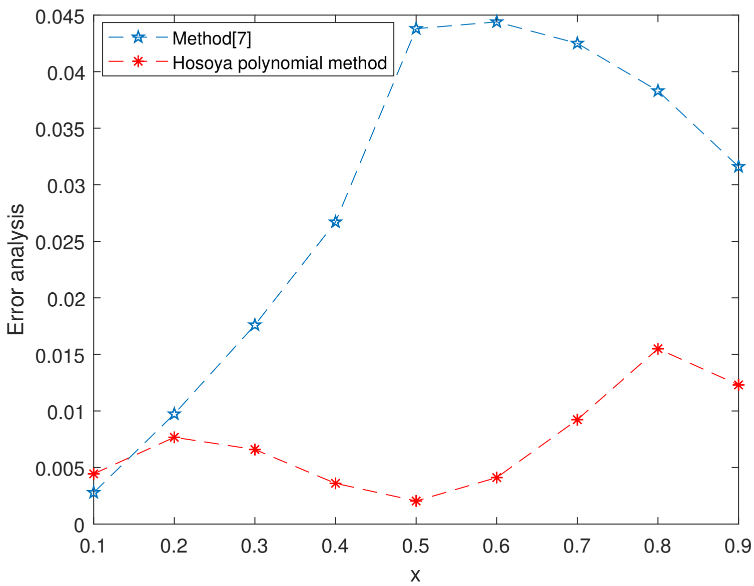

Fig. 3 Error analysis of HPM at $n=6$ with existing method [7] of application 2.

Application 3 Consider the nonlinear Fredholm integral equation [6, 37],

$$y(x) = exp(x) - \frac{(1+2exp(3))x}{9} + \int_0^1 xt[y(t)]^3 dt, 0 \le x < 1, \tag{13}$$

and the exact solution is $y(x) = exp(x)$. Solving equation (13), convert into system of equations using the proposed method. We obtain the approximate solution of equation (13), compared with the exact solution and existing methods [6, 37] as shown in Table 2 and Table 3. Graphically, numerical results and exact solutions are shown in Figure 4. The error analysis is represented graphyically as shown in Figure 5 compared with existing methods [6, 37]. This shows it converges fast, efficiency and the stability and validity of the present method.

Table 2 Comparison with exact, method [37] and HPM with Abs. Error of application 3.

x	Exact solution	Method [37] at k=8	Abs. Error of Method [37]	HPM at $n = 6$	Abs. Error of HPM at $n = 6$
0.1	1.105171	1.0658	3.94E-02	1.105171	5.60E-07
0.2	1.221403	1.2091	1.23E-02	1.221403	6.57E-07
0.3	1.349859	1.3712	2.13E-02	1.349859	1.32E-07
0.4	1.491825	1.5547	6.29E-02	1.491825	4.13E-07
0.5	1.648721	1.7225	7.38E-02	1.648722	7.63E-07
0.6	1.822119	1.7625	5.96E-02	1.822119	6.47E-07
0.7	2.013753	1.9978	1.60E-02	2.013753	5.91E-07
0.8	2.225541	2.2641	3.86E-02	2.225542	1.40E-06
0.9	2.459603	2.5258	6.62E-02	2.459605	1.56E-06

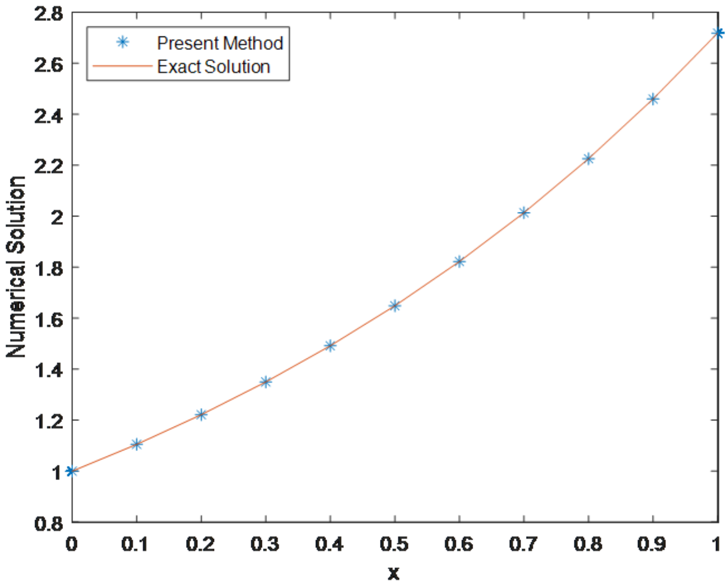

Fig. 4 Numerical solution of present method (HPM) with exact solution at $n = 6$ of application 3.

Table 3 Comparison with exact, method [6] and HPM with Abs. Error of application 3.

x	Exact solution	Method [6] at $M = 4, k = 2$	Abs. Error of Method [6]	HPM at $n = 6$	Abs. Error of HPM at $n = 6$
0	1	0.999956	4.40E-05	1.000000	7.47E-06
0.2	1.221403	1.221391	1.18E-05	1.221403	6.57E-07
0.4	1.491825	1.491845	2.00E-05	1.491825	4.13E-07
0.6	1.822119	1.822157	3.80E-05	1.822119	6.47E-07
0.8	2.225541	2.225517	2.39E-05	2.225542	1.40E-06
1	2.718282	2.718217	6.48E-05	2.718274	7.44E-06

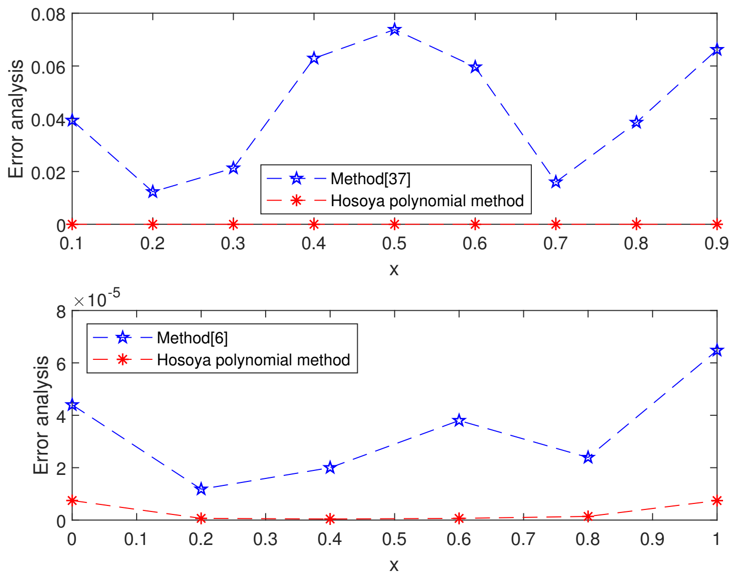

Fig. 5 Error analysis of HPM at $n = 6$ with existing method [6, 37] of application 3.

Application 4 Consider the nonlinear Fredholm integral equations [36],

$$y(x) = sin(\pi x) + \frac{1}{5}\int_0^1 [cos(\pi x)sin(\pi t)]\, y^3(t)dt, 0 \le x \le 1, \tag{14}$$

which has an exact solution $y(x) = sin(\pi x) + \frac{1}{3}\left(20 - \sqrt{391}\right) cos(\pi x)$. On simplifying, equation (14), convert into system of equations using the present technique. We obtain the required numerical solution of equation (14), compared with the exact solution and existing method [36] as shown in Table 4 and Figure 6 at $n = 6$. The error analysis is represented graphically as shown in Figure 7 compared with existing method [36]. This shows it converges fast and the stability and validity of the present method.

Table 4 Comparison with exact, method [36] and HPM with Abs. Error of application 4.

x	Exact solution	Method [36]	Abs. Error of Method [36]	HPM at $n = 6$	Abs. Error of HPM at $n = 6$
0	0.07542668890494	0.02559478612236	4.98E-02	0.07975849286936	4.33E-03
0.1	0.38075203836056	0.33335908249980	4.74E-02	0.38049498956402	2.57E-04
0.2	0.64880672544600	0.60849186923286	4.03E-02	0.64856295681349	2.44E-04
0.3	0.85335168974252	0.82406123219325	2.93E-02	0.85347811951489	1.26E-04
0.4	0.97436464499621	0.95896574017436	1.54E-02	0.97439793249943	3.33E-05
0.5	1.00000000000000	1.00000000000000	0.00E+00	0.99990360585242	9.64E-05
0.6	0.92774838759410	0.94314729241595	1.54E-02	0.92778213023346	3.37E-05
0.7	0.76468229900737	0.79397275655664	2.93E-02	0.76480830219652	1.26E-04
0.8	0.52676377913895	0.56707863535209	4.03E-02	0.52652674951005	2.37E-04
0.9	0.23728195038934	0.28467490625009	4.74E-02	0.23703395647723	2.48E-04
1	-0.07542668890494	-0.02559478612236	4.98E-02	-0.07123971074411	4.19E-03

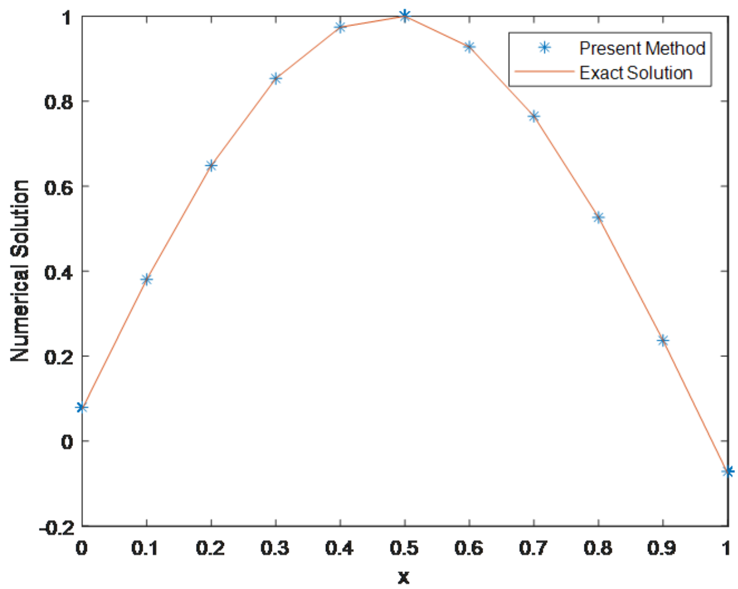

Fig. 6 Numerical solution of present method (HPM) with exact solution at $n = 6$ of application 4.

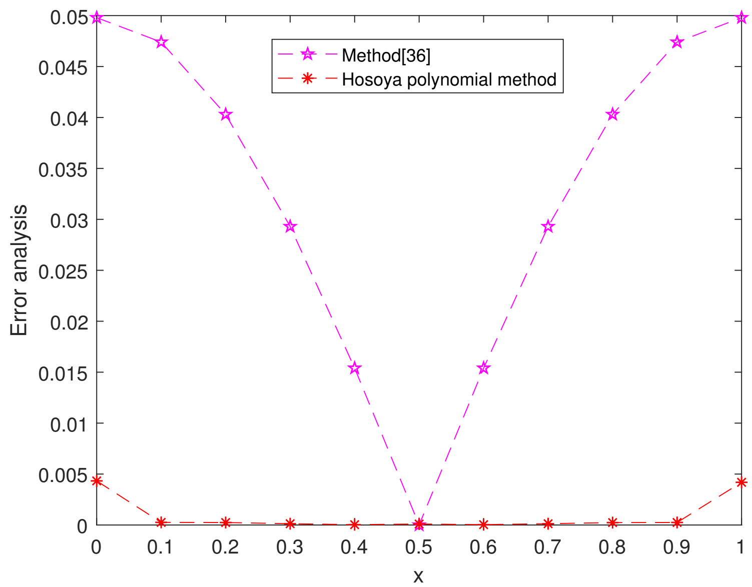

Fig. 7 Error analysis of HPM at $n = 6$ with existing method [36] of application 4.

Application 5 Consider the nonlinear Fredholm integral equations [7, 39],

$$y(x) = exp(x+1) - \int_0^1 exp(x-2t)y^3(t)dt, 0 \le x \le 1, \tag{15}$$

which has an exact solution $y(x) = exp(x)$. On simplifying, equation (15), transform into system of equations using the present technique. We obtain the required approximate solution of equation (15) and exact solution are shown in Table 5 and compared with the existing methods [7, 39] as shown in Table 6. Graphically, numerical results and exact solutions are shown in Figure 8, and Figure 9 shows error analysis for $n = 3$ and $n = 6$ compared with existing methods [7, 39]. This demonstrates the fast convergence and its stability.

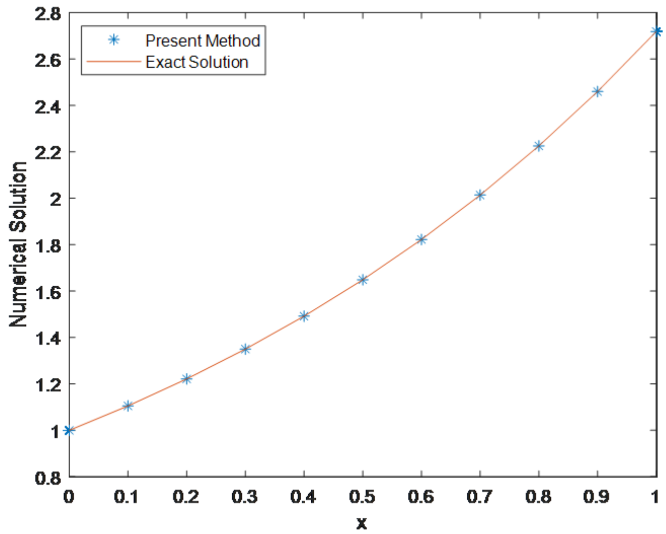

Fig. 8 Numerical solution of present method (HPM) with exact solution at $n = 6$ of application 5.

Table 5 Numerical solution of present method(HPM) with Abs. Error of application 5.

x	Exact solution	HPM at $n = 3$	Abs. Error at $n = 3$	HPM at $n = 6$	Abs. Error at $n = 6$
0.1	1.105170918	1.110184967	5.01E-03	1.105195722	2.48E-05
0.2	1.221402758	1.219896572	1.51E-03	1.221455388	5.26E-05
0.3	1.349858808	1.346250257	3.61E-03	1.349857068	1.74E-06
0.4	1.491824698	1.489246022	2.58E-03	1.491797307	2.74E-05
0.5	1.648721271	1.648883866	1.63E-04	1.648712515	8.76E-06
0.6	1.822118800	1.825163790	3.04E-03	1.822138610	1.98E-05
0.7	2.013752707	2.018085793	4.33E-03	2.013770656	1.79E-05
0.8	2.225540928	2.227649877	2.11E-03	2.225522500	1.84E-05
0.9	2.459603111	2.453856039	5.75E-03	2.459586419	1.67E-05

Table 6 Comparison with exact, HPM and existing method with error analysis of application 5.

x	Exact solution	Method [7]	Abs. Error of Method [7]	Abs. Error of Method [39]	HPM at $n=6$	Abs. Error of HPM at $n=6$
0.1	1.105170918	1.107217811	2.05E-03	2.00E-04	1.105195722	2.48E-05
0.2	1.221402757	1.218102916	3.30E-03	9.00E-03	1.221455388	5.26E-05
0.3	1.349858806	1.341165462	8.69E-03	1.00E-03	1.349857068	1.74E-06
0.4	1.491824696	1.474918603	1.69E-02	1.00E-03	1.491797307	2.74E-05
0.5	1.648721268	1.667402633	1.87E-02	1.00E-03	1.648712515	8.76E-06
0.6	1.822118797	1.833861053	1.17E-02	1.00E-03	1.822138610	1.98E-05
0.7	2.013752703	2.016679830	2.93E-03	1.00E-03	2.013770656	1.79E-05
0.8	2.225540923	2.217456630	8.08E-03	1.00E-03	2.225522500	1.84E-05
0.9	2.459603104	2.437978177	2.16E-02	1.00E-03	2.459586419	1.67E-05

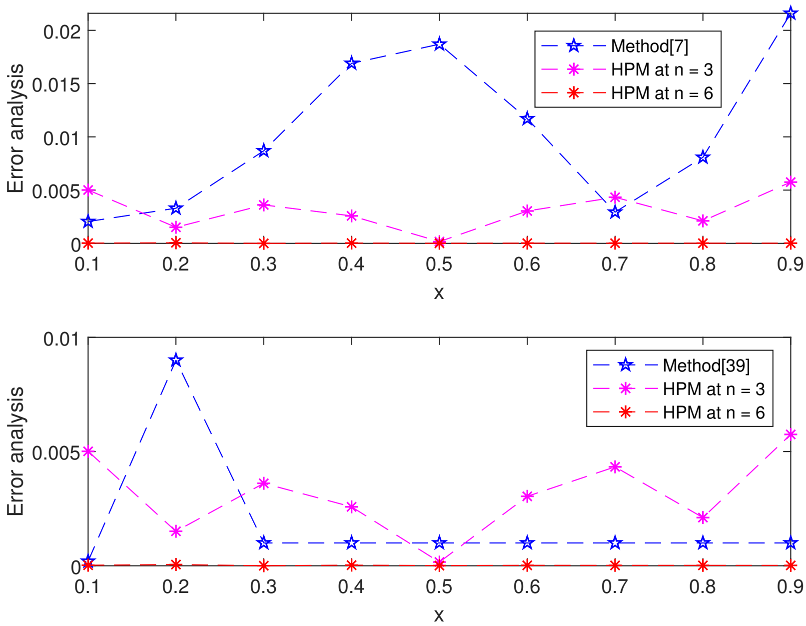

Fig. 9 Error analysis of HPM at $n=3$ and $n=6$ with existing method [7, 39] of application 5.

7 Conclusions

This paper explores the solution of nonlinear Fredholm integral equations using the Hosoya polynomial method. Through illustrative applications, we transform integral equations into algebraic systems and represent them in matrix form using Matlab. By simplifying the nonlinear system with the help of Newton's iterative method, we obtain the Hosoya coefficients. Substituting these coefficients into the function approximation, we obtain the required approximate solution. The resulting numerical solutions are compared with exact solutions and existing methods presented in tables and figures. The error analysis is demonstrated through the graphical representations. The proposed technique represents its efficiency, validity, accuracy, stability, and convergence as compared to the existing method. In the future, we can apply this technique to solve real-world problems in diverse fields, including physical models, fluid dynamics, and engineering problems.

8 Declarations

8.1 Conflict of interest:

The authors have no competing interests to declare that are relevant to the content of this article.

8.2 Funding:

No funding was received to assist with the preparation of this manuscript.

8.3 Author's contribution:

R.A.M.-Conceptualization, Methodology, Software, Writing-Review Editing. R.B.J.-Formal Analysis, Validation, Writing-Original Draft. All authors read and approved the final submitted version of this manuscript.

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8.5 Data availability statement:

All data that support the findings of this study are included within the article.

8.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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