



Original Study

Fractional SAQ alcohol model: stability analysis and Türkiye application

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Abstract

Alcohol dependence and alcohol abuse are public health problems. According to the World Health Organization data, approximately two billion people in the world consume alcoholic beverages and approximately 77 million people suffer alcohol use disorder. Epidemiologic studies show that the rate of alcohol dependence varies by region. Fractional derivative models are preferred over integer step models in the control theory of physical, biological, and dynamical systems. Fractional operators are particularly useful in describing the memory and hereditary properties of substances and processes, which are often ignored in integer stepwise derivatives. In this study, we consider a fractional model of alcohol use and analyze its stability. This model consists of three compartments: those who do not use alcohol yet but may use alcohol in the future (S), those who use alcohol (A) and those who quit alcohol for good (Q). The fractional derivative is used in the Caputo sense. Mathematical analysis of the fractional SAQ model for the mathematical model of alcohol use is performed and numerical results are obtained with the help of the Euler method and graphs are drawn.

Keywords: Fractional alcohol model, mathematical modeling, Euler method, Caputo derivative, stability analysis.

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1 Introduction

As per the World Health Organisation, alcohol dependence is a use disorder resulting from repeated or continuous alcohol consumption. Alcohol abuse is a leading risk factor for global health, causing numerous diseases and imposing significant social and economic burdens on societies. Alcohol dependence is frequently linked with various psychological or physical issues. People diagnosed with alcohol dependence are at risk of addiction to different substances, mood disorders, depression, anxiety problems, schizophrenia, personality disorders, and at the same time, physical problems such as enlarged liver, fatty liver, jaundice, cancer and cirrhosis, high blood pressure, enlarged heart and sudden crisis, and vascular blockages [1]. The spread of health risk behaviour within a community can be viewed as a diffusion process with its own incidence rate. In this situation,

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social interaction is considered the key factor in spreading the behaviour, which can thus result. Alcoholism can be considered a treatable contagious disease due to its potential for adverse health effects. Mathematical models are predictive tools that can simulate the spread and control of infectious diseases, and can be applied to alcoholism as well. Alcohol consumption is increasing rapidly in both developed and developing countries. This is a significant problem not only for individual health but also for the public socio-economic situation. The high cost of the Health Public Care System is due to the assistance expenditure of people suffering from diseases related to alcohol consumption. It is important to note that this is an objective evaluation and not a subjective one [2].

Mathematical modelling plays a crucial role in describing alcohol cessation models. These models are analysed using ODE systems, which consider the control of alcohol use. It is widely acknowledged that alcohol use is prevalent in modern society and has a significant impact on social behaviour. Alcohol is a leading cause of violence and reckless behaviour in society. Research has shown that alcohol use often begins at a young age and can spread through society like an infectious disease [3]. The main purpose of mathematical modelling is to express real-life problems mathematically and explain the functioning of processes. Additionally, it is crucial to control the modelled process. Mathematical models aid in system explanation, component analysis, and behaviour prediction. Mathematical modelling is employed not only in epidemic modelling but also in various dynamics modelling [4].

Fractional differential equations play a significant role in the analysis and modelling of various scientific processes, such as damping laws, electrical circuits, fluid mechanics and relaxation processes since the fractional derivative is a nonlocal operator. These problems have attracted the attention of numerous scientists from diverse branches of science [5]. The challenge with fractional differential equations is that they are difficult or impossible to solve analytically. As a result, several numerical methods have been developed to establish a numerical solution in series form, including the reduced differential transform method, the Adomian decomposition method, the homotopy perturbation method, the variational iteration method, the homotopy analysis method, the fractional difference method, the Euler method, the Runge-Kutta method, and the new iterative method [6]. Unlike numerical solutions, there are many analytical solutions for fractional models in literature.

The non-integer order derivative is considered in the sense of Caputo, instead of Riemann-Liouville, which is consistent with the initial biological conditions. The Caputo fractional derivative is a method for defining the fractional derivative. It involves differentiation from 0 to t , encompassing the entire interval and resulting in a non-local derivative. The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations, contain the limit values of integer-order derivatives of unknown functions at the terminal $t = \alpha$. As a result, the derivative is dependent on all points within the interval, providing an advantage over the local and limited classical derivative, which only considers the initial and final points [4–30].

In population models, the future state of a population is dependent on its past state, which is known as the memory effect. The analysis of a population's memory effect can be achieved by introducing a delay term or by using a fractional differentiation in the model [4–30]. Mathematical epidemic models have been used to create models for phenomena such as smoking, alcohol use, and internet use. The R_0 ratio of a mathematical model provides information about the increase or decrease of the epidemic or social contagion. R_0 consists of the parameters of parameters of the model [7].

This paper is divided into four parts. The first part discusses the significance of fractional mathematical modelling and alcohol consumption. It is important to note that technical term abbreviations are explained when first used. The second part covers the development of the fractional SAQ model, including the mathematical

analysis of its existence, uniqueness, and non-negativity, as well as the Generalised Euler Method and stability analysis. The third section presents a novel application of the fractional SAQ alcohol model, providing numerical results and accompanying graphs. The fourth section offers conclusions.

2 Fractional derivative and fractional SAQ alcohol model

The most commonly used definitions of the fractional derivative are Riemann-Liouville, Caputo, Atangana-Baleanu and the Conformable derivative. In this study, because the classical initial conditions are easily applicable and provide ease of calculation, the Caputo derivative operator was preferred and modeling was created. The definition of the Caputo fractional derivative is given below.

Definition 1. Let $f(t)$ be a function that can be continuously differentiable n times [4]. The value of the function $f(t)$ for the value of α that satisfies the condition $n - 1 < \alpha < n$. The Caputo fractional derivative of α -th order $f(t)$ is defined by $D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{(n-\alpha-1)} f^n(x) dx$.

Definition 2. The Riemann-Liouville (RL) fractional-order integral of a function $A(t) \in C_n$ ($n \geq -1$) is given by [4]

$$J^\gamma A(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{(\gamma-1)} A(s) ds, J^0 A(t) = A(t). \quad (1)$$

Definition 3. The series expansion of two-parametrized form of Mittag-Leffler function for $a, b > 0$ is given by [4]

$$E_{a,b}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(ai+b)}. \quad (2)$$

2.1 Fractional SAQ alcohol model

The fractional order SAQ alcohol model consists of three compartments. The first refers to those who have not yet used alcohol, but will be able to use alcohol in the future, the second refers to those who use alcohol and the third refers to those who have definitively stopped using alcohol. The expression of the SAQ alcohol model as a system of fractional differential equations is as follows.

$$\begin{aligned} \frac{d^\alpha S}{dt^\alpha} &= \mu N - \mu S - \frac{\beta AS}{N} \\ \frac{d^\alpha A}{dt^\alpha} &= \frac{\beta AS}{N} - \mu A - mA - wA \\ \frac{d^\alpha Q}{dt^\alpha} &= mA - \mu Q \end{aligned} \quad (3)$$

where $\frac{d^\alpha}{dt^\alpha}$ is the Caputo fractional derivative with respect to time t and $0 < \alpha \leq 1$. Initial values are given as,

$$S(0) = S_0, A(0) = A_0, Q(0) = Q_0$$

Here $S + A + Q = N$ and it is easy to see

$$\frac{d^\alpha N}{dt^\alpha} = \frac{d^\alpha S}{dt^\alpha} + \frac{d^\alpha A}{dt^\alpha} + \frac{d^\alpha Q}{dt^\alpha}.$$

Fractional order models possess memory features for time-dependent events and thus generate results that are more precise and realistic when compared to integer order models. Table 1 and Table 2 display the parameters and compartments of the model, respectively.

Table 1 Variables used in the systems and their meanings.

Variables used in the systems	Meaning
$S(t)$	The number of people who have not yet drunk alcohol at time t
$A(t)$	The number of individuals who consume alcohol at time t
$Q(t)$	The number of individuals who have stopped drinking at time t
$N(t)$	Total population

Table 2 Parameters and their meanings.

Parameters	Meaning
β	Annual rate of alcohol initiation
μ	Annual birth and death rate
m	Annual drop-off rate
w	Annual alcohol-related death rate

All individuals are born into a vulnerable class. When individuals reach the age of adolescence, they start to recognise and consume alcohol, which is one of the bad habits, under the influence of the social environment they live in. Thus, they become acquainted with alcohol. Natural birth and death rates are accepted as equal in the model. All births are considered to be in the sensitive class. The parameters defined in the model do not change over time. There is no mortality rate associated with some diseases caused by alcohol use [5–20]. The population N was considered dimensionless and constructed as follows with the help of new variables.

$s = \frac{S}{N}, a = \frac{A}{N}, q = \frac{Q}{N}$ where $s + a + q = 1$. The new form of the fractional SAQ alcohol model is written as follows.

$$\begin{aligned}
 D^\alpha s(t) &= \mu - \mu s(t) - \beta a(t)s(t) \\
 D^\alpha a(t) &= \beta a(t)s(t) - \mu a(t) - ma(t) - wa(t) \\
 D^\alpha q(t) &= ma(t) - \mu q(t).
 \end{aligned} \tag{4}$$

2.2 Existence, uniqueness and non-negativity of the system

We investigate the existence and uniqueness of the solutions of the fractional-order system (3) in the region $B \times [t_0, T]$ where

$$B = \{(S, A, Q) \in \mathbb{R}_+^3 : \max\{|S|, |A|, |Q|\} \leq \Psi, \min\{|S|, |A|, |Q|\} \geq \Psi_0\} \tag{5}$$

and $T < +\infty$.

Theorem 1. For each $H_0 = (S_0, A_0, Q_0) \in B$ there is a unique solution $H(t) \in B$ of the fractional-order system (3) with initial condition H_0 , which is defined for all $t \geq 0$.

Proof: We denote $H = (S, A, Q)$ and $\bar{H} = (\bar{S}, \bar{A}, \bar{Q})$. Consider a mapping $M(H) = (M_1(H), M_2(H), M_3(H))$ and

$$\begin{aligned}
 M_1(H) &= \mu N - \mu S - \frac{\beta AS}{N} \\
 M_2(H) &= \frac{\beta AS}{N} - \mu A - mA - wA \\
 M_3(H) &= mA - \mu Q.
 \end{aligned} \tag{6}$$

For any $H, \bar{H} \in B$ it follows from equation (6) that

$$\|M(H) - M(\bar{H})\| = \|M_1(H) - M_1(\bar{H})\| + \|M_2(H) - M_2(\bar{H})\| + \|M_3(H) - M_3(\bar{H})\|$$

$$\|M_1(H) - M_1(\bar{H})\| = \left| \mu N - \mu S - \frac{\beta SA}{N} - \mu N + \mu \bar{S} + \frac{\beta \bar{S} \bar{A}}{N} \right|$$

$$= \left| -\mu(S - \bar{S}) - \frac{\beta}{N}(SA - \bar{S}\bar{A}) \right|$$

$$\leq \mu |S - \bar{S}| + \frac{\beta}{N} \Psi |S - \bar{S}| + \frac{\beta}{N} \Psi |A - \bar{A}|$$

$$\|M_2(H) - M_2(\bar{H})\| = \left| \frac{\beta SA}{N} - \mu A - mA - wA - \frac{\beta \bar{S} \bar{A}}{N} + \mu \bar{A} + m\bar{A} + w\bar{A} \right|$$

$$= \left| \frac{\beta}{N}(SA - \bar{S}\bar{A}) - \mu(A - \bar{A}) - m(A - \bar{A}) - w(A - \bar{A}) \right|$$

$$\leq \frac{\beta}{N} \Psi |S - \bar{S}| + \frac{\beta}{N} \Psi |A - \bar{A}| + \mu |A - \bar{A}| + m |A - \bar{A}| + w |A - \bar{A}|$$

$$\|M_3(H) - M_3(\bar{H})\| = |mA - \mu Q - m\bar{A} + \mu \bar{Q}|$$

$$= |m(A - \bar{A}) - \mu(Q - \bar{Q})|$$

$$\leq m |A - \bar{A}| + \mu |Q - \bar{Q}|.$$

Then equation (6) becomes,

$$\begin{aligned} \|M(H) - M(\bar{H})\| &\leq \mu |S - \bar{S}| + \frac{\beta}{N} \Psi |S - \bar{S}| + \frac{\beta}{N} \Psi |A - \bar{A}| \\ &+ \frac{\beta}{N} \Psi |S - \bar{S}| + \frac{\beta}{N} \Psi |A - \bar{A}| + \mu |A - \bar{A}| + m |A - \bar{A}| + w |A - \bar{A}| + m |A - \bar{A}| + \mu |Q - \bar{Q}| \\ &\leq \left(\mu + \frac{2\beta\Psi}{N} \right) |S - \bar{S}| + \left(\mu + \frac{2\beta\Psi}{N} + 2m + w \right) |A - \bar{A}| + \mu |Q - \bar{Q}|, \\ \|M(H) - M(\bar{H})\| &\leq L \|H - \bar{H}\| \end{aligned}$$

where $L = \max(\mu + \frac{2\beta\Psi}{N}, \mu + \frac{2\beta\Psi}{N} + 2m + w, \mu)$. Therefore $M(H)$ obeys Lipschitz condition which implies the existence and uniqueness of solution of the fractional-order system (3).

Theorem 2. $\forall t \geq 0, S(0) = S_0 \geq 0, A(0) = A_0 \geq 0, Q(0) = Q_0 \geq 0$, the solutions of the system in (3) with initial conditions $(S(t), A(t), Q(t)) \in R_+^3$ are not negative.

Proof: (Generalized Mean Value Theorem) Let $f(x) \in C[a, b]$ and $D^\alpha f(x) \in C[a, b]$ for $0 < \alpha \leq 1$. Then we have

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\varepsilon)(x-a)^\alpha \quad (7)$$

with $0 \leq \varepsilon \leq x, \forall \varepsilon \in (a, b]$.

The existence and uniqueness of the solution (3) in $(0, \infty)$ can be obtained via Generalized Mean Value Theorem. We need to show that the domain R_+^3 is positively invariant. Since

$$\begin{aligned} D^\alpha S &= \mu N - \mu S - \frac{\beta AS}{N} \geq 0 \\ D^\alpha A &= \frac{\beta AS}{N} - \mu A - mA - wA \geq 0 \\ D^\alpha Q &= mA - \mu Q \geq 0 \end{aligned}$$

on each hyperplane bounding the non negative orthant, the vector field points into R_+^3 .

2.3 Stability analysis of the fractional SAQ alcohol model

In order to find the equilibrium point without alcohol (4) in the system, $D^\alpha s = 0$, $D^\alpha a = 0$, $D^\alpha q = 0$ is taken.

$$\begin{aligned} D^\alpha s(t) &= \mu - \mu s(t) - \beta a(t)s(t) \\ D^\alpha a(t) &= \beta a(t)s(t) - \mu a(t) - ma(t) - wa(t) \\ D^\alpha q(t) &= ma(t) - \mu q(t). \end{aligned}$$

To determine the non-alcoholic equilibrium point in system (4), $a(t) = 0$ is taken.

$$E_0 = (s_0, a_0, q_0) = (1, 0, 0) \quad (8)$$

alcohol-free equilibrium point is obtained. The Jacobian matrix of the system at the alcohol-free equilibrium point

$$J(E_0) = \begin{bmatrix} -\mu & -\beta & 0 \\ 0 & \beta - \mu - m - w & 0 \\ 0 & m & -\mu \end{bmatrix} \quad (9)$$

is obtained. The eigenvalues obtained from the Jacobian matrix (9) are given below.

$$\begin{aligned} \lambda_1 &= -\mu \\ \lambda_2 &= -\mu \\ \lambda_3 &= \beta - \mu - m - w \end{aligned}$$

eigenvalues are obtained. The parameters β, μ, m, w are positive definite real numbers. It is clear that $\lambda_1 < 0$ and $\lambda_2 < 0$. If $\lambda_3 < 0$ the non-alcoholic equilibrium point is locally asymptotically stable. If $\lambda_3 > 0$, the non-alcoholic equilibrium point is unstable.

If $\beta - \mu - m - w < 0$, $\beta < \mu + m + w$ is. $R_0 = \frac{\beta}{\mu + m + w} < 1$ is the basic reproduction rate. If $R_0 < 1$, alcohol consumption will decrease over time. If $R_0 > 1$, alcohol consumption will increase over time.

Theorem 3. When $R_0 > 1$, alcohol-free equilibrium point is globally asymptotically stable [4–30].

2.4 Generalized Euler method

In this paper, we used the Generalized Euler method to solve the initial value problem with the Caputo fractional derivative. Many of the mathematical models consist of nonlinear systems and finding solutions to these systems can be quite difficult. In most cases, analytical solutions cannot be found and a numerical approach should be considered for this. One of these approaches is the Generalized Euler method [15]. $D^\alpha y(t) = f(t, y(t))$, $y(0) = y_0$, $0 < \alpha \leq 1$, $0 < t < \alpha$ for the initial value problem, $h = \frac{a}{n}$ impending $[t_j, t_{j+1}]$ is divided into n sub with $j = 0, 1, \dots, n-1$. Suppose that $y(t)$, $D^\alpha y(t)$ and $D^{2\alpha} y(t)$ are continuous in range $[0, a]$ and using the generalized Taylor's formula, the following equation is obtained [15].

$$y(t_1) = y(t_0) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_0, y(t_0)).$$

This process will be repeated to create an array. Let $t_j = t_{j+1} + h$ such that

$$y(t_{j+1}) = y(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_j, y(t_j))$$

$j = 0, 1, \dots, n-1$ the generalized formula in the form is obtained. For every $k = 0, 1, \dots, n-1$

$$\begin{aligned} S(k+1) &= S(k) + \frac{h^\alpha}{\Gamma(\alpha+1)} (\mu N - \mu S(k) - \frac{\beta S(k)A(k)}{N}) \\ A(k+1) &= A(k) + \frac{h^\alpha}{\Gamma(\alpha+1)} (\frac{\beta S(k)A(k)}{N} - \mu A(k) - mA(k) - wA(k)) \\ Q(k+1) &= Q(k) + \frac{h^\alpha}{\Gamma(\alpha+1)} (mA(k) - \mu Q(k)). \end{aligned} \quad (10)$$

3 Numerical simulation of fractional SAQ alcohol model for Türkiye

The numerical simulation of the fractional SAQ alcohol model using the Generalized Euler method will now be obtained. Let us consider the following parameters according to the data in [18]. $S = 47236000$, $A = 13324075$, $Q = 111914$, $\beta = 0.0003$, $m = 0.086$, $\mu = 0.022$, $w = 0.003$ and let's take size of step $h = 1$. Hence we get the following results and tables. The Table 3, Table 4 and Table 5 are obtained using the Euler method. If we calculate the basic reproduction rate R_0 , $R_0 = \frac{\beta}{\mu + m + w}$ then according to the parameters we have, $R_0 = 0,0027027027$ is obtained. $R_0 < 1$, alcohol consumption will decrease over time.

Table 3 The values of S , A and Q at the moment t and $\alpha = 1$.

t	$S(t)$	$A(t)$	$Q(t)$
0	47236000,00	13324075,00	111914,00
1	48020534,09	11847377,52	1255322,34
2	48788026,96	10534374,94	2246579,71
3	49538833,66	9366916,98	3103111,20
4	50273303,06	8328866,40	3840397,62
5	50991777,89	7405875,67	4472191,38
6	51694594,77	6585188,43	5010708,48
7	52382084,20	5855462,94	5466799,10
8	53054570,66	5206615,18	5850099,33
9	53712372,62	4629679,34	6169166,05
10	54355802,55	4116683,74	6431596,82
11	54985167,05	3660540,64	6644136,49
12	55600766,82	3254948,13	6812771,99
13	56202896,76	2894303,02	6942816,54
14	56791846,02	2573623,34	7038984,64

Table 4 The values of S , A and Q at the moment t and $\alpha = 0.9$.

t	$S(t)$	$A(t)$	$Q(t)$
0	47236000,00	13324075,00	111914,00
1	48051719,30	11788678,83	1300772,80
2	48849015,86	10430249,76	2325144,23
3	49628288,72	9228385,98	3204614,99
4	50389929,72	8165038,40	3956499,44
5	51134323,51	7224239,19	4596101,97
6	51861847,62	6391861,78	5136949,05
7	52572872,46	5655408,44	5590994,60
8	53267761,39	5003822,41	5968801,52
9	53946870,75	4427321,72	6279702,46
10	54610549,93	3917252,16	6531941,87
11	55259141,43	3465957,24	6732801,78
12	55892980,89	3066663,10	6888713,00
13	56512397,21	2713376,78	7005353,54
14	57117712,58	2400796,14	7087735,66

Table 5 The values of S , A and Q at the moment t and $\alpha = 0.8$.

t	$S(t)$	$A(t)$	$Q(t)$
0	47236000,00	13324075,00	111914,00
1	48078330,74	11738589,18	1339557,26
2	48901017,27	10341805,13	2391805,87
3	49704498,93	9111258,54	3290226,66
4	50489206,87	8027160,15	4053802,80
5	51255564,10	7072077,09	4699241,80
6	52003985,50	6230652,22	5241246,87
7	52734877,91	5489356,88	5692755,89
8	53448640,12	4836273,14	6065151,96
9	54145663,01	4260901,95	6368448,88
10	54826329,51	3753994,17	6611454,43
11	55491014,77	3307401,73	6801914,37
12	56140086,16	2913946,51	6946639,11
13	56773903,38	2567304,87	7051615,42
14	57392818,59	2261905,86	7122104,73

Table 3, Table 4, and Table 5 show variations of S , A , and Q for different states of α .

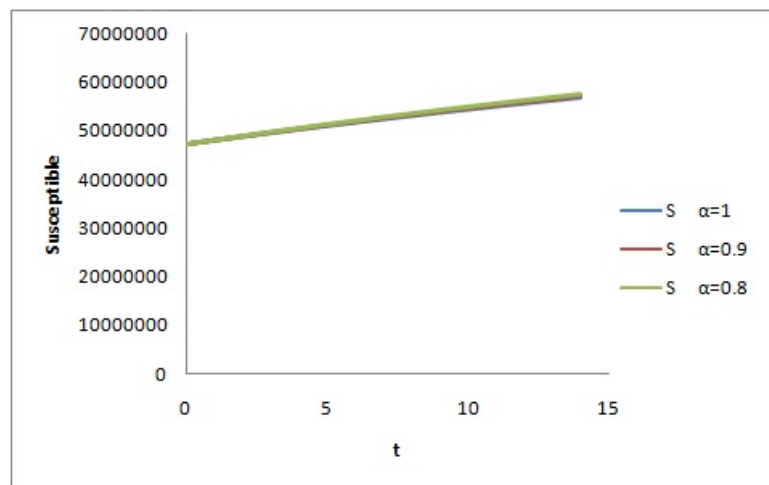


Fig. 1 The graph of change of the S compartment model.

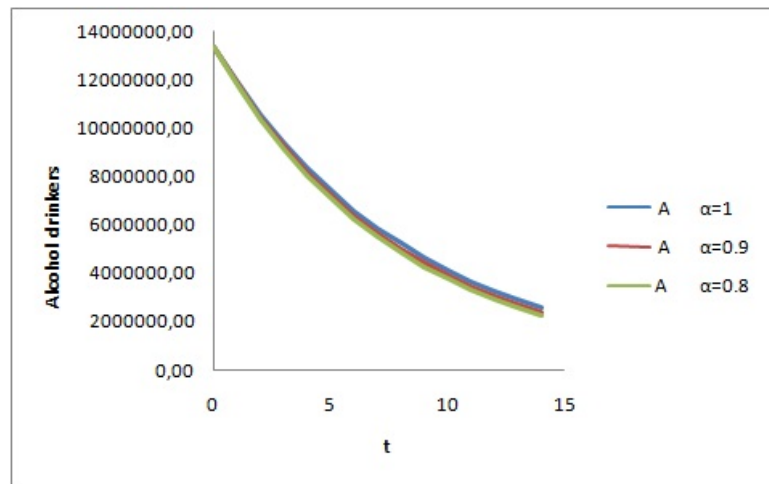


Fig. 2 The graph of change of the A compartment model.

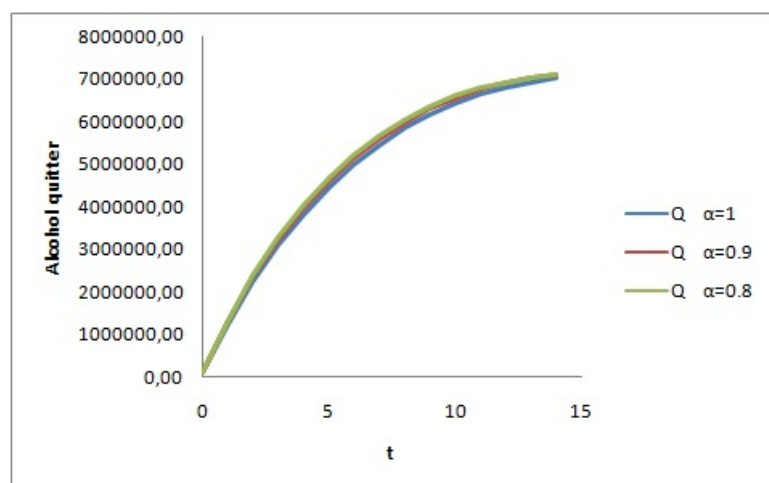


Fig. 3 The graph of change of the Q compartment model.

The above graphs demonstrate that the following observations can be made.

1. It is observed that the number of people who do not yet drink alcohol, but may drink alcohol in the future, increases slowly and steadily over time (Figure 1).
2. It is observed that individuals who use alcohol decrease over time (Figure 2).
3. It is observed that individuals who stop using alcohol increase over time (Figure 3).

4 Conclusions and comments

This study applies the fractional SAQ model to the year 2019 data on alcohol consumption in Türkiye [18]. Numerical results were used to create graphs. The mathematical analysis examined the existence, uniqueness,

and non-negativity of the system. The stability analysis of the fractional SAQ model was performed by obtaining the equilibrium point without alcohol, and the basic reproduction rate R_0 was determined. The graphs show a gradual increase in the number of individuals who do not currently use alcohol but may do so in the future. Additionally, there is a decrease in the number of individuals who currently use alcohol and an increase in the number of individuals who have stopped using alcohol over time. As it is common with mathematical models, the model presented in this paper may be approached with caution due to the assumptions made and the difficulty in estimating the model parameters.

5 Declarations

5.1 Conflict of interest:

Not applicable.

5.2 Funding:

Not applicable.

5.3 Author's contribution:

Z.Ö.-Data Curation, Conceptualization, Design, Formal analysis, Project administration. H.B.-Data Curation, Conceptualization, Design. S.S.-Writing - Original Draft, Investigation, Writing-Original Draft, Resources. All authors reviewed the results and approved the final version of the manuscript.

5.4 Acknowledgement:

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5.5 Data availability statement:

All data that support the findings of this study are included within the article.

5.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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