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# Criticism of the Analytical Solutions to the Wave-Induced Cyclic Response of a Poro-Elastic Seabed of Finite Thickness

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**Abstract.** This paper deals with the wave-induced cyclic response of a poro-elastic seabed (by means of oscillations of the pore-fluid pressure, soil displacement, effective normal stress and shear stress in a soil skeleton) due to a surface sinusoidal water-wave propagating over a seabed of finite thickness. The main existing analytical solutions to the governing problem, assuming a dual-elastic system of the two-phase (pore-fluid and soil skeleton) seabed medium, are critically discussed, pointing out their limitations, doubtful items, and meaningful errors. The amplitude phenomena is particularly studied as an immanent part of any complex-valued analytical solution of a cyclic nature. A series of calculation analyses, performed for the North Sea wave and soil conditions, has indicated problematic results as far as high values of the shear modulus of soil are concerned. An application of meaningfully different values of the degree of saturation, obtained for one and the same calculation example, has caused many additional doubts as to the quality of the tested analytical solutions.

**Key words:** Wave-induced cyclic response; Analytical solutions; Dual-elasticity of seabed; Partly saturated seabed; Finite thickness of seabed; Sinusoidal progressive wave; Soil permeability; Pore-fluid mobility; Pore-fluid compressibility; Degree of saturation; Shear modulus of soil.

## 1. Introduction

The wave-induced response of poro-elastic seabed sediments loaded by a sinusoidal surface water-wave propagating above a sandy seabed is still an interesting subject in many coastal engineering problems. In order to solve some of them, it is necessary to treat the problem either numerically, especially when a certain coastal engineering structure is involved, or analytically by using one of the existing theories and their particular solutions to the wave-induced cyclic seabed response, especially when a pure case of seabed without any presence of a structure founded on or embedded in seabed sediments is concerned. These analytical solutions can be further used in many scientific and engineering analyses of the wave-induced seabed instability due to the wave-induced momentaneous liquefaction of the seabed or/and the wave-induced residual liquefaction of the upper part of seabed sediments as a result of a continuous build-up of the wave-induced pore-fluid pressure within seabed sediments. Furthermore, it is highly expected that analytically derived solutions are error-free because they are frequently used as a basis for verification and validation of numerical formulations and their solutions.

### 1.1. Review of the Relevant Literature

Focusing on the existing analytical solutions to the wave-induced cyclic seabed response, most interesting from the practical point of view, only a dual-elastic system of the two-phase seabed medium, consisting of the soil skeleton and the pore-fluid, is usually considered, playing one of the dominant role in the whole process of the wave-induced cyclic response of seabed sediments. Dual-elasticity of the seabed means that the above-distinguished phases of the seabed medium are treated simultaneously as compressible components. Therefore, the so-called potential model, where the two phases are assumed to be incompressible and the problem is governed by the Laplace equation, is out of scope of the present paper. Assuming only the pore-fluid to be compressible, Moshagen and Tørum (1975) solved the consolidation problem, governed by the consolidation partial differential equation, and presented analytical solutions to the pore-fluid pressure obtained for both infinite and finite thicknesses of the seabed.

Taking an infinite thickness of the seabed into account, the next research works concerned the case where both phases of the porous seabed are considered to be compressible and the term dual-elasticity or relative compressibility began to take on great importance. This type of analytical solutions is based on Biot's consolidation theory, Hooke's law (the soil has linear, reversible, isotropic, and non-retarded mechanical properties), and Darcy's law for the pore-fluid flow through a porous seabed. All of this leads to the so-called storage problem governed by the coupled equations of static force and moment equilibrium together with the continuity equation in the form of storage partial differential equation proposed by Verruijt (1969). And thus, Yamamoto et al. (1978), Madsen (1978), and Okusa (1985) obtained analytical two-dimensional solutions in terms of six complex-valued wave-induced and cyclically varying parameters: pore-fluid pressure, two soil displacement components, two soil effective normal stress components, and one shear stress component, or at least some of them, assuming the plain strain and partly saturated soil conditions. A detailed analysis of the correctness of these analytical solutions was presented by Magda (2023, 2024).

A direct analytical solution to the wave-induced cyclic response of the seabed, with its two component phases compressible, is mathematically very demanding and complicated particularly in the case of limited thickness of the seabed layer. Therefore, Mei and Foda (1981) proposed a boundary-layer approximation where the porous and elastically deformable seabed layer is divided into the inner region close to the seabed surface, where a full solution is required, and the outer region, where a simplified solution is good enough. Richwien and Magda (1994), reprinted later by Magda (1995, 1998), presented an exact analytical two-dimensional solution, although obtained only to the wave-induced pore-fluid pressure. This solution is correct for the most commonly assumed condition of completely rough surface of a rigid and impermeable base laying under the seabed layer. However, it must be admitted that an additional particular variant of the novel solution, assuming a totally smooth surface of the base underlying the seabed layer, is burdened with two errors. Hsu and Jeng (1994) extended the seabed cyclic response theory to more complicated three-dimensional short-crested wave configurations (for both progressive and standing waves), presenting particular two-dimensional solutions for both infinite and finite thicknesses of the seabed. Their analytical solution was obtained in terms of the wave-induced pore-fluid pressure, soil displacement, and stresses in the soil matrix. Jeng and Hsu (1996) obtained a three-dimensional 'finite thickness' analytical solution for nearly saturated seabed sediments loaded by either progressive or standing surface water-waves. Although the title says otherwise, the final form of the solution is limited only to fully saturated soil conditions. Moreover, the solution contains mistakes in the wave-induced pore-water pressure equations. It also seems that a combination of two different sign conventions for strains and stresses brings additional confusion. The original equation (49) in Jeng and Hsu (1996, p. 432) for the effective vertical normal stress shows evidently that the compressive stress, e.g. under the water-wave crest, obtains negative values which stays in contradiction of the following sentence written by Jeng and Hsu (1996, p. 430) with respect to the stress-displacement relationships: 'A positive sign is used in the present paper, as in equations (7)–(12), i.e. compressive stresses are defined as positive'. A closer analysis reveals that these equations, with the positive-signed right-hand sides, are derived under the assumption of solid mechanics sign convention for strains and stresses where compressive stresses are defined as negative, as it was formerly adopted by Yamamoto et al. (1978). Zhang and Li (2010) presented an analytical solution to the Biot two-dimensional consolidation problem. Unfortunately, their solution is not a well-posted problem due to a false assumption for the wave-induced pore-fluid pressure in infinity for the seabed of infinite thickness. Additionally, the authors incorrectly assumed the excess pore-water pressure vanishing at a rigid and impermeable base underlaying a porous and permeable seabed layer of finite thickness. These two incorrect assumptions disqualify the solution at the beginning. Jeng (2013, 2018) published a comprehensive study on the wave-induced poro-elastic seabed response, mainly gathering information on results of comparison analyses given in his previous works and studies of other researchers. It is worth noting that Hsu and Jeng (1994), Jeng and Hsu (1996), and Jeng (2013, 2018) used very often the results obtained by Mei and Foda (1981) as a reference point in their computational analyses.

### 1.2. Objective of the Paper

Based on a long scientific and engineering practice with different analytical solutions to the wave-induced poro-elastic cyclic response of the seabed of finite thickness, the author of the present paper has noticed that the above-mentioned two-dimensional 'finite thickness' analytical solutions, giving the pore-fluid pressure, soil displacements, and stresses in the soil matrix as an output, are not free of limitations and even errors. Therefore, an extended critical review of the most promoted analytical solutions obtained by Hsu and Jeng (1994) and Jeng (2013, 2018) will be presented in detail, generally pointing out their weaknesses.

## 2. Critical Review of the Main Theories

A basic definition sketch of the governing two-dimensional plain strain problem is illustrated in Fig. 1. A porous and elastically deformable seabed is loaded by a progressive sinusoidal surface water-wave passaging above it. This causes wave-induced cyclic variations of the main six (in a two-dimensional case) parameters: pore-fluid pressure, two soil displacement components, two effective normal stress components, and one shear stress component. Usually, it is assumed that the direction of surface water-wave propagation, and — as a consequence — the direction of propagation of the wave-induced hydrodynamic pressure at the seabed surface is consistent with the positive horizontal $x$-axis, whereas the vertical $z$-axis is traditionally taken to be negative when directed downwards.

Different analytical solutions to the governing problem were obtained analytically and numerically by many researchers — a comprehensive list of the past work from the period from 1949 to 2007, regarding stresses and pore-fluid pressure in seabed sediments loaded by surface water-waves, can be found in Sumer (2014). However, this paper deals only with some widely advertised and promoted two-dimensional analytical solutions to the wave-induced cyclic seabed response published by Hsu and Jeng (1994) and Jeng (2013, 2018).

### 2.1. Theory by Hsu and Jeng

Hsu and Jeng (1994) published an analytical two-dimensional solution to the wave--induced cyclic response of a poro-elastic seabed of finite thickness. Besides the wave-induced pore-fluid pressure, they also presented the solutions to the soil displacement components and effective normal stress and shear stress components. Two particular analytical solutions were derived for both progressive and standing water-wave types of loading. As stated by Sumer (2014, p. 159), and admitted by Jeng (2013, p. 45), there were two errors (they called them typos) in coefficients $C_{11}$ and

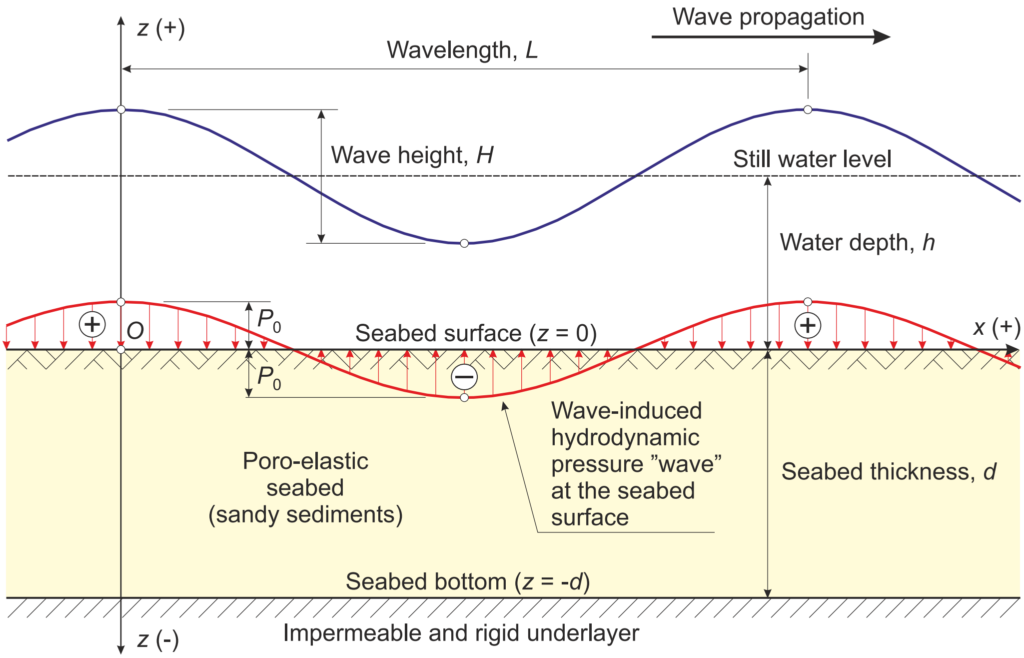


**Fig. 1.** Definition sketch for analyzing the wave-induced cyclic response of a poro-elastic sandy seabed of finite thickness (Magda, 2023)

$C_{32}$ given in the original equations (68) and (80) in Hsu and Jeng (1994, pp. 803–804). Jeng (2013) presented the following corrected expressions for both coefficients (the original notation is kept):

$$C_{11} = 2k^2h(\delta + k)(\delta - \delta\mu - k\mu)B_3 + \lambda\,(B_{11} + B_{12})\,, \tag{1a}$$

$$C_{32} = -2\delta^2 k\lambda\Big[(kh - \lambda)\left(\delta^2 - k^2\right)(1-\mu) - \delta^2(1-\mu) + k^2\mu + \lambda B_9\Big]. \tag{1b}$$

According to Jeng (2013), 'the influences of these typos on the numerical results are rather minor'. What does it mean qualitatively? Any analytical solution should be free of even minor typos! Sumer (2014) wrote that 'the end results are not significantly affected'. Is there any prove of it given by Sumer (2014)? Sumer (2014) emphasized also that 'these are only typing errors, the author's original code of computations is correct (Dong Jeng, 2012, personal communication)'.

### *An important digression*

At this point the author of the present paper would like to present a digression which is very important for the paper — it is high time to make this clear.

Unfortunately, an evident sign of our times is that more and more authors of scientific publications have a very strange mannerism, namely, calling all mathematical errors, appearing in their previous publications, typos in their successive publications. Maybe they are only typos for them but it must be stressed that these articles are very often used by common readers in their scientific studies. Therefore, it would be nice to remember that the term typos will always mean erroneous equations, solutions, theories! All books and papers, containing typos made particularly in more complex theories and mathematical solutions that cannot be simply reproduced by a statistical reader, are simply useless.

Many authors do not care about the quality of their papers and books. Publishers and reviewers are also to blame. For example, here is a fragment of the real reviewer's opinion on the paper, submitted to the Journal of Waterway, Port, Coastal, and Ocean Engineering (ASCE), reporting errors found in books by Jeng (2013, 2018): 'I will not try to defend Professor Jeng here. However, these (or some of them, at least) [author's note: errors] are because Professor Jeng was, most likely, being absent minded, or because of his inattentiveness or even because of typos'. As a matter of fact, a passer-by can be considered as a good example of an inattentive or absent minded human being but not, under no circumstances, a writer of a serious (at least in the theory) scientific book. Writing a book is a long-term undertaking process, requiring creativity and persistence, which should normally undergo a large number of check-points and reviewing steps. Of course, a writer is finally and fully responsible for the quality of a published book, but a respected publisher is always responsible for assigning reviewers of excellent quality.

Sumer (2014) was a lucky researcher who had initiated a private communication with Jeng in 2012 and had a possibility to follow the original computer code delivered to him. All other readers had not been so lucky, therefore these cases of typos must be called a spade. To put it in simple and understandable words, the analytical solution by Hsu and Jeng (1994) does not contain any typos — the solution is simply wrong. As it will be shown in the following, other works by Jeng are full of different typos and therefore they must be evaluated as erroneous ones. Many authors forget that they are fully responsible for any typos in their publications. These typos for common readers, who want to make use of them in their scientific research, are simply errors disqualifying the published solution entirely. One can also notice that the equation for coefficient $C_{01}$ in Sumer and Fredsøe (2002) differs from the appropriate equation in Sumer (2014) and Jeng and Hsu (1996). Is it a typing error or the result of another 'private communication' unknown to a common reader?

Spending usually a considerable amount of private money when purchasing any scientific book, it is expected to get a high quality product. Instead of it, the author of the present paper was sorely disappointed studying some parts of the books by Jeng (2013, 2018) in detail. Nowadays many books are published practically without any serious reviews. This strange procedure is already typical for many publishing companies, resulting in emergence of duds on the publishing market. On the other hand, it must be emphasized that the author alone pays for all the mistakes made in the book — ultimately, the author's name is printed on the cover.

And last but not least, it is worth nothing that 'Admoneri bonus gaudet; pessimus quisque correctorem asperrime patitur' (Lucius Annaeus Seneca the Younger, De Ira, Liber III, chapter 36, section 4). All scientists should always remember this epigraph and keep this leitmotiv lying always behind their research.

It is highly probable that Sumer and Fredsøe (2002) and Sumer (2014) did not noticed other mistakes in the solution by Hsu and Jeng (1994). This is because they only focused on the wave-induced pore-fluid pressure and the shear stress in their analyses. In addition, Jeng (2013, 2018) and Sumer (2014) assumed in their research only either sandy sediments characterized by the shear modulus $G = 10^4$ kPa or fully saturated soil conditions denoted by the degree of saturation $S_r = 1.0$, respectively. However, it is worth noting that the use of only these two selectively chosen values of the very meaningful soil parameters is highly insufficient to detect all potential mistakes appearing in the analytical solution under consideration. Checking and proving the quality of any analytical solution require comprehensive parameter studies and computational tests using wide ranges of values of the leading parameters. It can be easily proved, as done below, that even after introducing reportedly correct Eqs. (1a) and (1b) into the entire set of solution coefficients, the analytical solution obtained by Hsu and Jeng (1994) remains incorrect.

Computational examples performed for the purpose of the present paper are based on the following North Sea wave and soil data, adopted initially by Mei and Foda (1981) and also used later by Hsu and Jeng (1994): water depth $h = 70$ m, wave period $T = 15$ s, wavelength $L = 324$ m (sinusoidal wave theory), seabed thickness $d = 25$ m ($d/L = 0.077$), shear modulus of soil $G = 10^4$ kPa, porosity of soil $n = 0.3$, Poisson's ratio of soil $\nu = 0.333$, pore-fluid mobility $K_m = 10^{-8}$ m$^3 \cdot$ s/kg (fine sand) and $K_m = 10^{-6}$ m$^3 \cdot$ s/kg (coarse sand), coefficient of soil permeability $K = 10^{-4}$ m/s (fine sand), and $K = 10^{-2}$ m/s (coarse sand), respectively to $K_m$-values, density of seawater $\rho_w = 1000$ kg/m$^3$, acceleration due to the gravity of Earth $g = 10$ m/s$^2$, unit weight of seawater $\gamma_w = 10$ kN/m$^3$, and compressibility of seawater $\beta_w = 4.4 \times 10^{-7}$ m$^2$/kN (bulk modulus of seawater $K_w = 1/\beta_w = 2.27 \times 10^6$ kPa).

Using the above given set of input-data, with the shear modulus of soil varying in the range $G = 10^3$–$10^5$ kPa, and taking the coefficient of soil permeability $K = 10^{-2}$ m/s (coarse sand; $K_x = K_z = K$ for hydraulically isotropic soil conditions), comparative computations have been performed and their selected results are presented in the form of vertical distributions of the dimensionless amplitude of oscillations of the wave-induced seabed response parameters: soil vertical displacement, $\bar{u}_z = \widetilde{u}_z/[P_0/(2Gk)]$, in Fig. 2, and effective vertical normal stress, $\bar{\sigma}_z = \widetilde{\sigma}_z/P_0$, in Fig. 3, where $P_0$ denotes the amplitude of the wave-induced hydrodynamic pressure at the seabed surface (see Fig. 1).

For a very dense seabed, modelled by the shear modulus of soil $G = 5 \times 10^4$ kPa and $G = 10^5$ kPa, it is expected that there should be only small differences in the vertical profile of the amplitude of soil vertical displacement oscillations. Additionally, the reaction of a stiffer seabed on one and the same surface water-wave loading should be logically reflected in smaller values of the vertical component of soil displacement. Unfortunately, the trend in the results illustrated in Fig. 2 is opposite where the soil vertical displacement (in its dimensionless form) becomes larger and larger for increasing stiffness of the seabed. Simultaneously, the dimensioning parameter $P_0/(2Gk)$ becomes smaller deepening additionally the discrepancy between the computed results and the expected correct solution, when presenting them in their dimensional forms.

The upper limit of $|\bar{u}_z|$ at $z = 0$ can be easily derived based on the 'infinite seabed thickness' analytical solution by Yamamoto et al. (1978) who considered two special cases for both components of the soil displacement (here, only the vertical component is considered). The first case was obtained for completely saturated soils ($G\beta \to 0$, denoting a relatively small $G$-value)

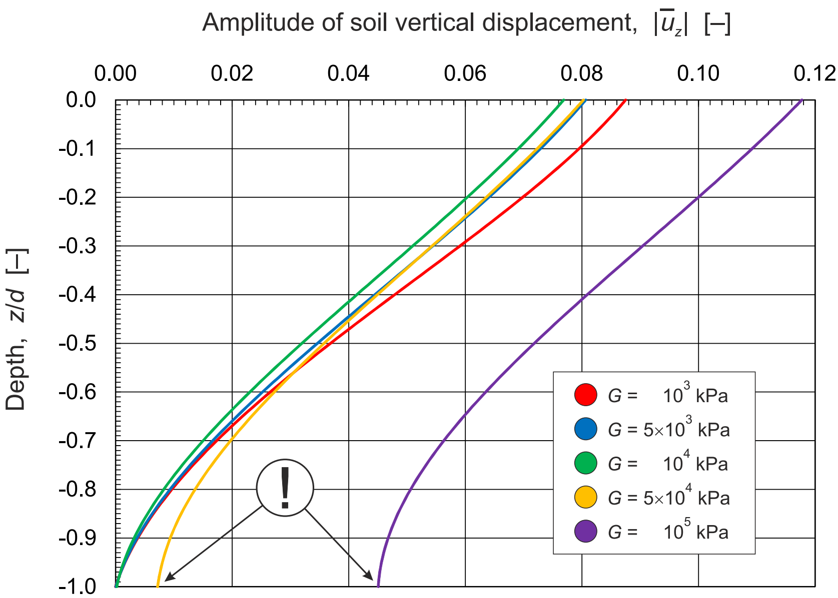


**Fig. 2.** Vertical distribution of the amplitude (dimensionless) of wave-induced soil vertical displacement oscillations in the seabed of finite thickness and for different soil stiffness conditions (input-data: $S_r$ = 1.000, $K = 10^{-2}$ m/s (coarse sand), and $G = 10^3$–$10^5$ kPa) — computations after the analytical solution by Hsu and Jeng (1994)

$$\widetilde{u_z} = [\exp(-kz) + kz\exp(-kz)]\frac{P_0}{2Gk}\exp[i(kx + \omega t)] \tag{2}$$

and the second one was obtained for partially saturated dense sand and sandstone ($G\beta \to \infty$, where $G$ takes relatively large values). Unfortunately, it has been found that the second solution (see the original equation (3.18b) in Yamamoto et al. (1978, p. 200)) is erroneous and the correct one should be written as (the original notation is kept; $w \equiv \widetilde{u_z}$)

$$w = \left\{\left[1 + \frac{1 + (1-2\nu)(-\lambda'' + i\omega'')}{-\lambda'' + i(1+m)\omega''} + \left(1 + \frac{i\lambda''}{\omega''}\right)\lambda z\right]\exp(-\lambda z) + \right.$$
$$\left. + \frac{i}{\omega''}(1 + \lambda'')\exp(-\lambda' z)\right\}\frac{p_0}{2\lambda G}\exp[i(\lambda x + \omega t)]. \tag{3}$$

In the case of $G\beta \to 0$ [see Eq. (2)] it becomes obvious that the relative and dimensionless amplitude $|\bar{u}_z|$ is equal to unity at the seabed surface ($z = 0$). Surprisingly, after some mathematical manipulations performed on Eq. (3) it can be shown that $|\bar{u}_z|$ approximates again unity for $z = 0$ also in the case of $G\beta \to \infty$. On the other hand, when a finite thickness of the poro-elastic seabed layer is considered, it is clear that

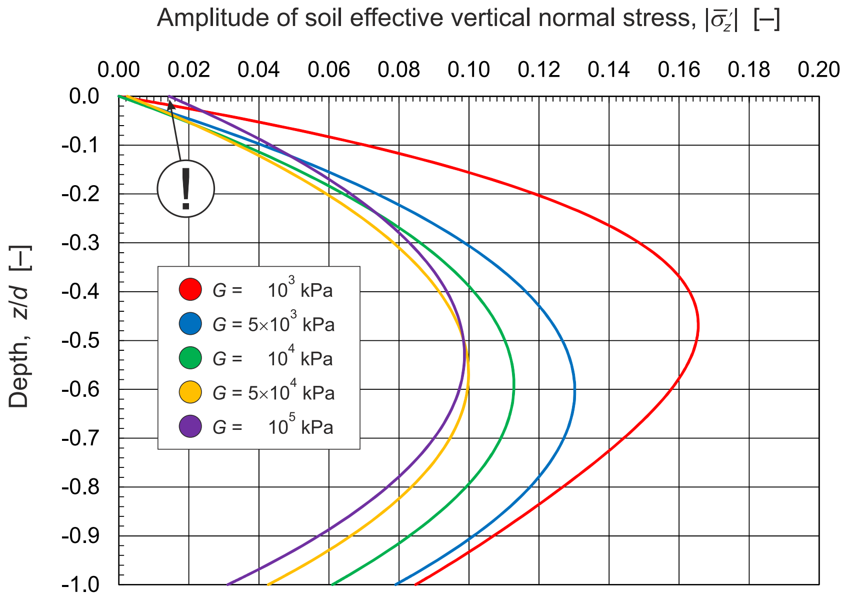


**Fig. 3.** Vertical distribution of the amplitude (dimensionless) of wave-induced effective vertical normal stress oscillations in the seabed of finite thickness and for different soil stiffness conditions (input-data: $S_r$ = 1.000, $K = 10^{-2}$ m/s (coarse sand), and $G = 10^3$–$10^5$ kPa) — computations after the analytical solution by Hsu and Jeng (1994)

the vertical component of soil displacement will undergo a certain reduction ($|\bar{u}_z| < 1$) with respect to the ‘infinite seabed thickness’ solution, whereas the amount of reduction will depend on the relative thickness of seabed layer $kd$. It means that $|\bar{u}_z| = 1$ can be treated as the upper bound for the wave-induced soil vertical displacement in the case of finite seabed thickness. Unfortunately, the results of computations — performed using the analytical solution by Hsu and Jeng (1994) and presented in Fig. 4 — do not confirm this obvious restriction because $|\bar{u}_z| > 1$ for $G > 8 \times 10^4$ kPa which cannot be accepted from the physics of this phenomenon.

Moreover, when formulating the boundary value problem for the seabed of finite thickness, the following two boundary condition, among others, are always assumed (Richwien and Magda, 1994; Magda, 1995; Hsu and Jeng, 1994; Jeng and Hsu, 1996; Jeng, 2013, 2018):

$$\bar{u}_z \overset{\text{def}}{=} \frac{\widetilde{u}_z}{\frac{P_0}{2Gk}} = 0 \qquad \text{at} \quad z = -d, \tag{4a}$$

$$\bar{\sigma}'_z \overset{\text{def}}{=} \frac{\widetilde{\sigma}'_z}{P_0} = 0 \qquad \text{at} \quad z = 0. \tag{4b}$$

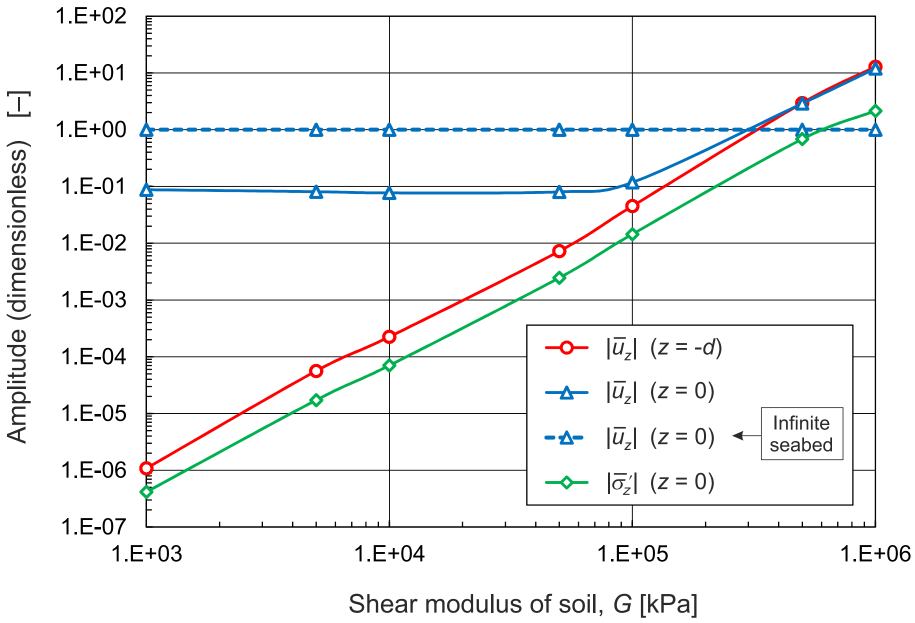


**Fig. 4.** Influence of soil stiffness on the amplitudes (dimensionless) of the wave-induced soil vertical displacement and effective vertical normal stress oscillations at the upper ($z = 0$) and lower ($z = -d$) edges of the seabed of finite thickness — computations after the analytical solution by Hsu and Jeng (1994)

The boundary condition given in Eq. (4b) is also a fixed point in all theories assuming an infinite thickness of a compressible seabed, as done by e.g.: Madsen (1978), Yamamoto et al. (1978), Okusa (1985). Unfortunately, Figs. 2 and 3 indicate very clearly that the above-mentioned boundary conditions are not preserved any more. Figure 4, in particular, confirms these unacceptable behaviour of the solution recorded for a wide range of the shear modulus of soil $G = 10^3$–$10^6$ kPa. One can easily detect unexpected linear increases in $|\bar{u}_z|$ at $z = -d$ and in $|\bar{\sigma}'_z|$ at $z = 0$ for increasing values of $G$. Taking the wave-induced effective normal vertical stress as an example, the results presented in Fig. 3 can be confronted with the results illustrated in Fig. 5, obtained by the author using the analytical solution by Mei and Foda (1981), where the problem with the upper boundary condition (see Eq. (4b)) does not exist at all.

Of course, one can question the use of such large values of the shear modulus of soil in computations. However, it must be stresses very emphatically that the use of $G$-values higher by one order, or even more, than the upper limit of practical $G$-values is fully justified and has a very simple explanation. The use of extreme values of a certain physical parameter (here $G$) is a typical procedure in order to:

– perform a computational test of mathematical correctness of the analytical solution obtained,

– model the seabed as an incompressible medium, which is of great meaning in some practical analyses, e.g. related with submarine pipelines buried in seabed sediments where the upper limit of the hydrodynamic uplift force acting on a pipeline is sought. In general, the smaller compressibility of soil (larger $G$-values), the larger vertical gradients of the wave-induced pore-fluid pressure, at least to a certain extend. Therefore, larger values of the hydrodynamic uplift force are induced (Magda, 2000).

Although the corresponding values of the shear modulus of soil used in the above presented comparative analysis seem quite high (up to $G = 10^5$ kPa), in fact there is no any formal restriction in the mathematical formulation of the problem, presented by Hsu and Jeng (1994), forbidding the use of extremely high values of the shear modulus of soil in computational examples. If the analytical solution by Hsu and Jeng (1994) were correct, this solution should work properly also for $G = 10^6$ kPa, or even higher, modelling simultaneously the seabed as a completely incompressible medium. Meanwhile, this is not the case, as proved in the present paper.

Figure 6 presents the original figure 6 in Hsu and Jeng (1994, p. 798), illustrating computational results obtained for a fully saturated seabed created by coarse sand sediments. It is not so easy to follow these results because Hsu and Jeng (1994) did

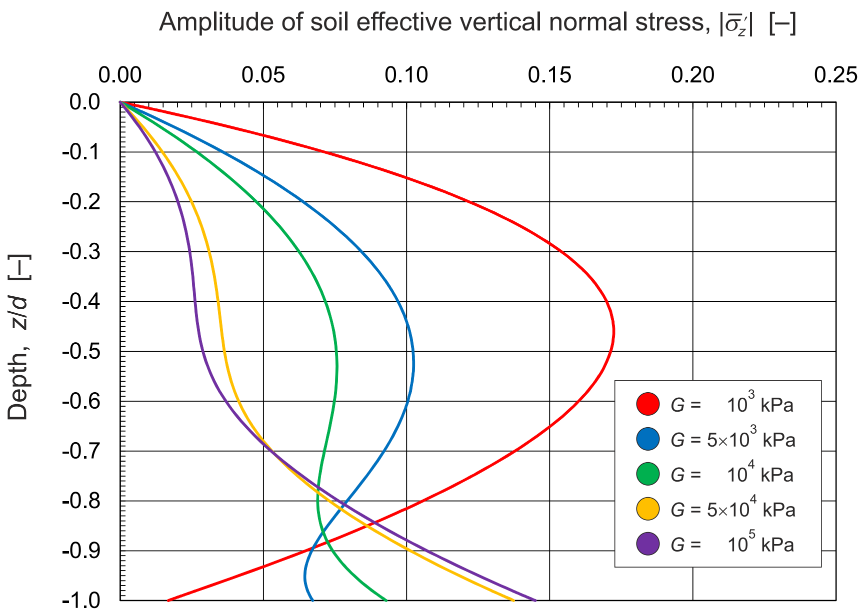


**Fig. 5.** Vertical distribution of the amplitude (dimensionless) of wave-induced effective vertical normal stress oscillations in the seabed of finite thickness and for different soil stiffness conditions (input-data: $S_r = 1.000$, $K_m = 10^{-6}$ m$^3 \cdot$ kg/s, and $G = 10^3$–$10^5$ kPa) — computations after the analytical solution by Mei and Foda (1981)

not show values for $n$ and $\nu$; presumably they are: $n = 0.3$ and $\nu = 0.33$. It becomes clear that the pore-fluid pressure results, calculated by Hsu and Jeng (1994) after the boundary-layer approximation by Mei and Foda (1981), are not correct because they do not account for the no-flux correction. This finding can be confirmed by comparison of vertical distributions of the wave-induced pore-fluid pressure amplitude presented in Figs. 6 and 7.

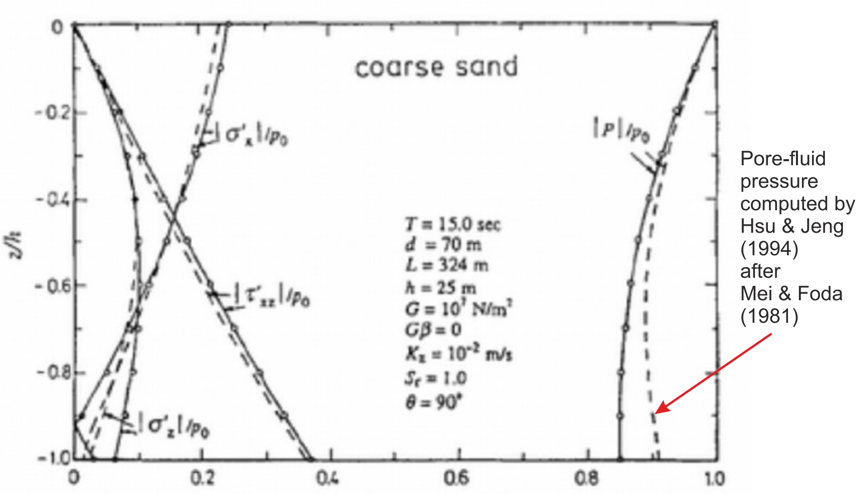


**Fig. 6.** Vertical distributions of amplitudes (dimensionless) of the wave-induced seabed response parameters cyclically oscillating in the seabed of finite thickness (saturated isotropic coarse sand, $S_r = 1.0$, $G\beta = 0$; dashed lines — solution by Mei and Foda (1981), solid lines — solution by Hsu and Jeng (1994)) [based on Hsu and Jeng (1994)]

On the other hand, Fig. 8 presents the original figures 7 and 8 from Hsu and Jeng (1994, p. 799), illustrating computational results obtained for a partly saturated seabed created by fine and coarse sand sediments, respectively. The results presented in Fig. 8 has been confronted with the results of the author's own computations based on the boundary-layer approximation by Mei and Foda (1981) and illustrated in Fig. 9. The computational example, presented in the original figures 3(a) and 3(b) in Mei and Foda (1981, p. 620), has been recomputed in order to check, by the way, the quality of the author's derivation of the no-flux correction term. A very good agreement, achieved by the author for both cases: completely saturated soil ($S_r = 1.0$ and $G\beta = 0.0044$), denoting a practically incompressible pore-fluid [see Fig. 9(a)], and partially saturated soil ($G\beta = 1.0$), denoting a highly compressible pore-fluid due to air entrainment (see Fig. 9(b)).

The relative compressibility in the case of partly saturated soil ($G\beta = 1.0$) implies that the pore-fluid compressibility $\beta = 1/G = 10^{-4}$ m$^2$/kN. And what is the corresponding value of the degree of soil saturation? This question can be answered using the pore-fluid compressibility formula proposed by Verruijt (1969), commonly applied by many other researchers, e.g.: Madsen (1978), Yamamoto et al. (1978), Richwien and Magda (1994), Hsu and Jeng (1994), Jeng (2013, 2018), after its transformation to the following form

$$S_r = 1 - (\beta - \beta_w)\, P_h, \tag{5}$$

where: $S_r$ is the degree of soil saturation [–], $\beta$ is the compressibility of pore-fluid [m$^2$/kN], $\beta_w$ is the compressibility of pure water (without any air bubbles entrapped in soil pores) [m$^2$/kN], $P_h = p_{at} + p_h$ is the absolute hydrostatic pressure at the computation point in the seabed (usually at the seabed surface) [kPa], $p_{at}$ is the atmospheric pressure [kPa], $p_h$ is the hydrostatic pressure at the computational point in the seabed (usually at the seabed surface for which $p_h = \gamma_w h$) [kPa], $\gamma_w$ is the unit weight of seawater [kN/m$^3$], $h$ is the water depth [m]. Taking the atmospheric pressure $p_{at} = 101.325$ kPa, it can be easily calculated from Eq. (5) that the case of partly saturated seabed sediments, considered by Mei and Foda (1981), and used in the author's own computations, is correlated with the degree of saturation of relatively low value $S_r = 0.920$. At first glance, the results presented in Figs. 8 and 9 seem very similar, even identical. Therefore, one could suppose that the results computed by Hsu and Jeng (1994) were obtained for the above-calculated degree of saturation

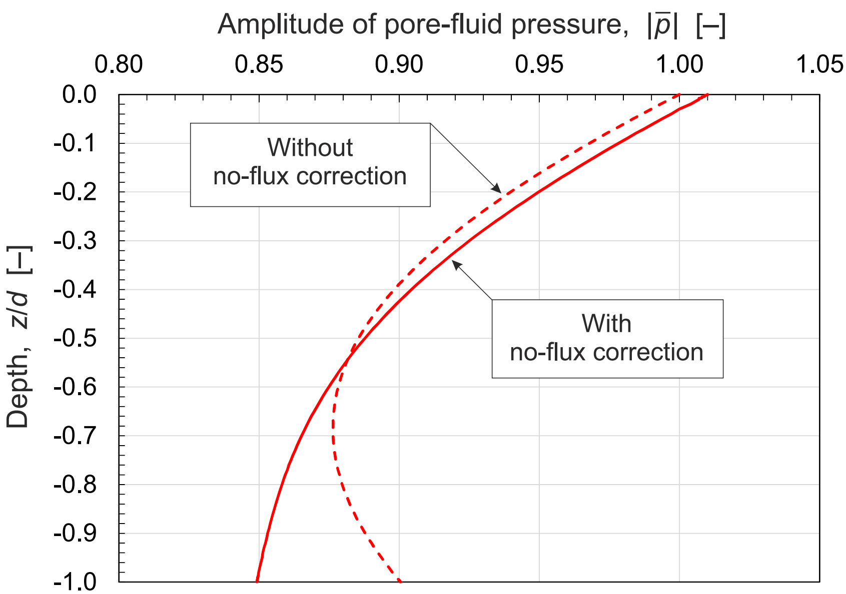


**Fig. 7.** Vertical distribution of the amplitude (dimensionless) of wave-induced pore-fluid pressure oscillations in the seabed of finite thickness (input-data: $G\beta = 0.0044 \approx 0$ (fully saturated soil) and $K_m = 10^{-6}$ m$^3$s/kg (coarse sand)) – computations based on the boundary-layer approximation by Mei and Foda (1981)

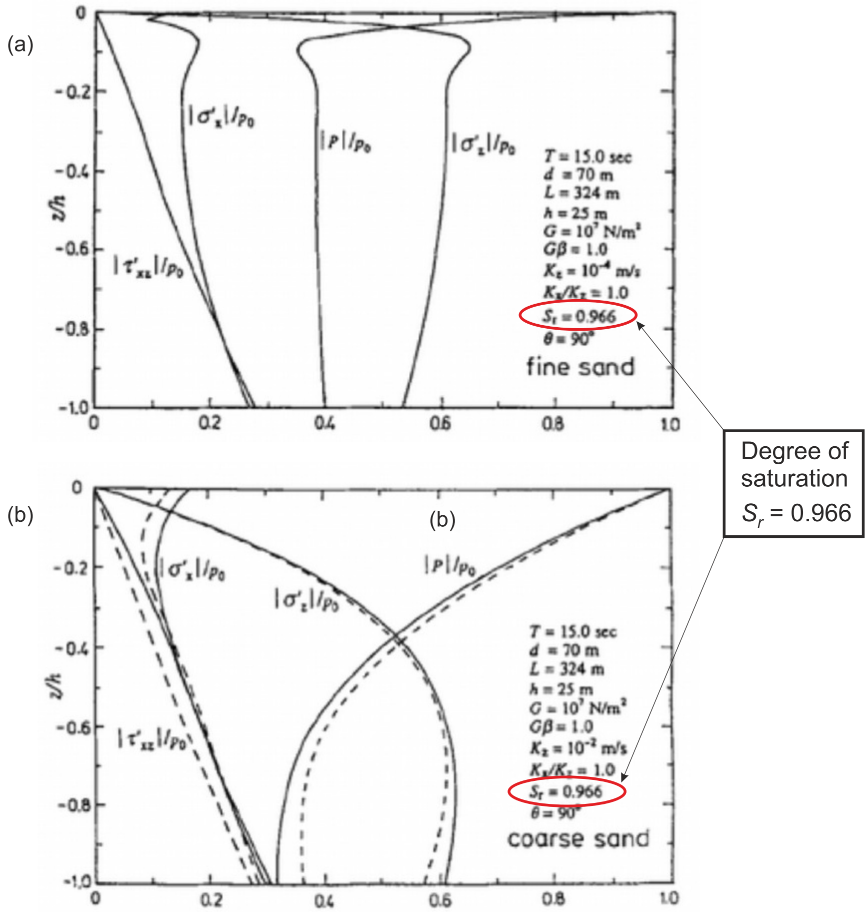


**Fig. 8.** Vertical distributions of the amplitude (dimensionless) of wave-induced seabed response parameters cyclically oscillating in the seabed of finite thickness; (a) $K = 10^{-4}$ m/s for fine sand, (b) $K = 10^{-2}$ m/s for coarse sand (unsaturated seabed sediments, $S_r = 0.966$, $G\beta = 1$; dashed lines — solution by Mei and Foda (1981), solid lines — solution by Hsu and Jeng (1994)) [based on Hsu and Jeng (1994)]

$S_r = 0.920$. Unfortunately, the legends in Fig. 8 inform about much higher soil saturation ($S_r = 0.966$) used by Hsu and Jeng (1994) in their computations. This comparison of soil saturation conditions, indicating a significant difference in the degree of saturation, must put into question the correctness of the computational analysis presented by Hsu and Jeng (1994).

## 2.2. Theory by Jeng

The analytical solution to the wave-induced response of a poro-elastic seabed loaded by a sinusoidal progressive surface water-wave, obtained originally by Hsu and Jeng

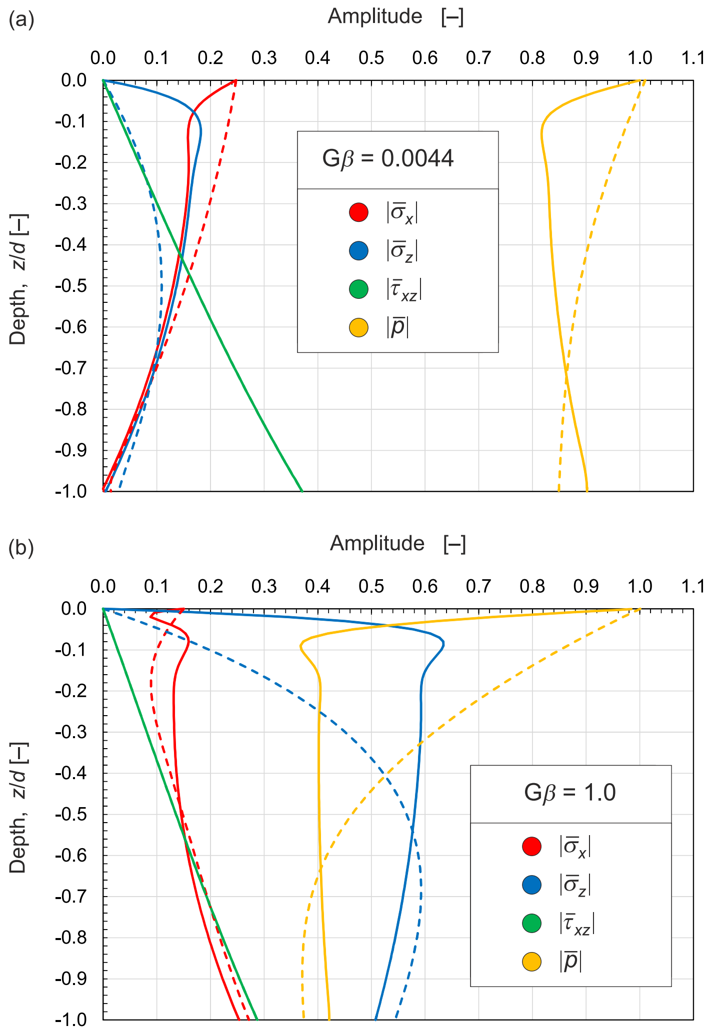


**Fig. 9.** Vertical distributions of the amplitudes (dimensionless) of wave-induced seabed response parameters cyclically oscillating in the seabed of finite thickness; (a) $G\beta = 0.0044 \approx 0$, (b) $G\beta = 1.0$ (input-data: $K_m = 10^{-8}$ m$^3 \cdot$ s/kg for fine sand (solid line), $K_m = 10^{-6}$ m$^3 \cdot$ s/kg for coarse sand (dashed line)) — computation after the boundary-layer approximation by Mei and Foda (1981)

(1994), was repeated in two books by Jeng (2013, 2018). Unfortunately, many changes in the final solution equations and solution coefficients introduced in Jeng (2013, 2018) with respect to the original paper by Hsu and Jeng (1994) have been found. For example, the equation for the two-dimensional wave-induced effective vertical normal stress (see the original equation (3.81) in Jeng (2013, p. 49) and the original equation (A.6) in Jeng (2018, p. 316)) was changed without giving any reason for such action. In addition, Jeng (2013) changed the expressions for the following solution coefficients: $C_{01}$, $C_{03}$, $C_{22}$, $C_{51}$, $C_{52}$, $C_{64}$, $B_5$, $B_9$, $B_{11}$, $B_{12}$, $B_{13}$, $B_{18}$, with respect to Hsu and Jeng (1994) without any comments (for example, a three-fold modification has been found in the expression for coefficient $C_{52}$). Jeng (2013) gave up including the expressions for other four solution coefficients: $C_{12}$, $C_{21}$, $C_{37}$, and $C_{53}$, which makes the analytical solution incomplete. As far as Jeng (2018) is concerned, the solution coefficients $C_{12}$ and $C_{64}$ were subject to change with respect to Hsu and Jeng (1994) again without giving any explanation. Consequently, the analytical solutions presented in Jeng (2013, 2018) differ from each other.

To make matters worse, Jeng (2018, p. 317) stated: '... there were two typos in the coefficients ($C_{11}$ and $C_{32}$) originally presented by Hsu and Jeng (1994). Although the influence of these typos on the numerical results is rather minor, these have been corrected in the coefficients listed as follows'. However, it can be easily checked that the equations for coefficients $C_{11}$ and $C_{32}$ in Jeng (2018) are surprisingly the same as those given in Hsu and Jeng (1994)! It would be also desirable to remember that typos, made in mathematical expressions by authors of scientific papers and books, always mean errors for a common reader who, expecting to hold a high quality work in hands, is completely unaware of it. A comprehensive and comparative list of all the questionable formulas for the solution coefficients related to the analytical solutions presented by Hsu and Jeng (1994) and Jeng (2013, 2018) has been included into the Appendix section of the present paper, indicating a large number of errors, incompletenesses, and incompatibilities.

Jeng (2013, 2018) tried to reproduce some computational results obtained by Mei and Foda (1981), showing — in the original figure 3.6 in Jeng (2013, p. 58) and figure 2.2 in Jeng (2018, p. 51) — the meaningful differences in the vertical distribution of pore-fluid pressure amplitude for the case of fully saturated seabed sediments (coarse sand). The explanation of these differences is exactly the same as previously given in the case of solution by Hsu and Jeng (1994) — the flux-correction term in the wave-induced pore-fluid pressure was not taken into account by Jeng (2013, 2018). This was probably due to serious mistakes made by Jeng (2018) who tried to present a practical dimensional form of the final equations of the boundary-layer approximation by Mei and Foda (1981) for the purpose of carrying out comparative computations. Unfortunately, all together 12 errors have been detected in the original equations (2.63)–(2.66) and (2.68) in Jeng (2018, pp. 47–48). Such a large number of errors made in only 5 mathematically simple equations gives their author a very bad rating. A copy-and-paste technique should be used with caution! Certainly, these errors do not look good for the reputation and professionalism of the publisher and the reviewers of the book as well.

As far as the case of partially saturated seabed is concerned, Jeng (2013, 2018) presented the original figure 3.6 in Jeng (2013, p. 58) and figures 2.4 and 2.5 in Jeng (2018, p. 52) which are very similar to the original figures 7 and 8 in Hsu and Jeng (1994, p. 799). This time, however, the results were correlated with the degree of saturation $S_r = 0.932$, which is much smaller than $S_r = 0.966$ reported initially by Hsu and Jeng (1994) (see Fig. 8). There is no rational explanation for such large difference in $S_r$; it is not possible to get the same results for two significantly different values of $S_r$. Anyway, saturation conditions described by $S_r = 0.932$ in Jeng (2013, 2018) still differ from $S_r = 0.920$ (obtained either for $\beta = 4.4 \times 10^{-7}$ m$^2$/kN used by Mei and Foda (1981) or for $\beta = 5 \times 10^{-7}$ m$^2$/kN used by Hsu and Jeng (1994), Jeng and Hsu (1996), and Jeng (2013, 2018)), which has been already estimated in the present paper as the best fit for the relative compressibility $G\beta = 1.0$ and the compatibility with the results presented by Mei and Foda (1981).

A careful study of section 2.3.3 in Jeng (2018) has also indicated that there is no difference between the wave number and the coefficient of permeability, where the same symbol $k$ was used in both cases. As far as the soil permeability is concerned, there is another serious mistake found in Jeng (2018) where the mobility of pore-fluid, used thoroughly by Mei and Foda (1981), is simply replaced by the coefficient of soil permeability. It is important to emphasized that the coefficient of permeability is not an exclusive property of the soil alone, since it depends on properties of both the soil (total porosity, distribution of the pore size, tortuosity and pore geometry) and the pore-fluid (fluid density and viscosity), as presented by Al-Doury (2010). For example, assuming the unit weight of seawater $\gamma \approx 10$ kN/m$^3$ = 10 000 N/m$^3$, and if the traditional parameter like the coefficient of permeability takes the following values: $K = 10^{-2}$ m/s (for coarse sand) and $K = 10^{-4}$ m/s (for fine sand), the values of pore-fluid mobility $K_m = K/\gamma = 10^{-6}$ m$^3 \cdot$ s/kg and $10^{-8}$ m$^3 \cdot$ s/kg, respectively, must be used in computations according to some more sophisticated theories like that one by Mei and Foda (1981). The term permeability is commonly used as a synonym of the hydraulic permeability which is usually confusing and incorrect from the scientific point of view. An interesting discussion on this important scientific and engineering issue can be found in Al-Doury (2010). The term hydraulic permeability is used traditionally in geology or hydrogeology. However in soil mechanics and geotechnical engineering this proportionality constant is called the coefficient of permeability (also known as the hydraulic conductivity). Just for simplicity but unfortunately not for clarity, the mobility is called the permeability in Mei and Foda (1981, p. 600). This simplification causes very often misunderstandings and can lead to critical errors, as found in the book by Jeng (2018), putting into question his comparison analysis with the results obtained by Mei and Foda (1981).

## 3. Conclusions

It took weeks of painstaking research to write this paper. Some selected sophisticated theories of the wave-induced cyclic response of a poro-elastic seabed subject to progressive sinusoidal surface water-wave loading, together with their analytical solutions obtained for the special case of finite thickness of the poro-elastic seabed layer, have been thoroughly examined, finally leading to the following conclusions:

1. The computational analysis has confirmed that the analytical solution by Hsu and Jeng (1994), even after assuming the corrected forms of coefficients $C_{11}$ and $C_{32}$, gives reasonable results only for more compressible soils. However, introducing less compressible (i.e. stiffer) seabed sediments (e.g. $G = 5 \times 10^4$–$10^5$ kPa for a very dense sand) in combination with higher soil permeability ($K = 10^{-2}$ m/s for coarse sand), the results obtained for the soil vertical displacement and the effective vertical normal stress in the seabed make no sense, as illustrated in Figs. 2 and 3. Especially, the results presented in Fig. 4 have confirmed that the boundary conditions assumed for the wave-induced vertical displacement of the soil skeleton (see Eq. (4a)) and the effective vertical normal stress in the soil matrix (see Eq. (4b)) are not valid anymore, questioning thereby the formal correctness of the analytical solution by Hsu and Jeng (1994).
2. The analytical solution by Hsu and Jeng (1994) was repeated in two books by Jeng (2013, 2018) together with many subsequent inconsistent changes. Unfortunately, these books contain many incomplete data, substantive errors, and confusions, mainly regarding the solution coefficients (see the Appendix section for the reference).
3. Some important uncertainties have been found with respect to soil saturation conditions. The degree of saturation, used in the above discussed solutions, differs from $S_r = 0.966$ (Hsu and Jeng, 1994) to $S_r = 0.932$ (Jeng, 2013, 2018) when trying to reproduce one and the same graphical results presented by Mei and Foda (1981). It is worth noting that the author's own computations have indicated the best match obtained for $S_r = 0.920$; this value was also confirmed in simple use of Eq. (5).

As it has been proved in the present paper, the above considered analytical solutions are not free of their practical limitations (with respect to compressibility of the soil skeleton), incompletenesses, and errors, concerning the solutions coefficients. The research work presented above should serve as an ever-timely reminder to our community of the extreme importance of rigor in accurate and reliable writing and checking (reviewing) scientific articles and books, which obviously should be free from errors that prevent the safe use of published solutions, especially when applied to engineering practice.

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## Appendix

Regarding the most general analytical solutions to the wave-induced cyclic response of a poro-elastic and partly saturated seabed of finite thickness, published by Hsu and Jeng (1994) and repeated by Jeng (2013, 2018), below please find the list of questionable equations together with comparisons of the corresponding formulas (please note that the original notation is kept).

*A.1. The wave-induced effective vertical normal stress (2D solution)*

– Hsu and Jeng (1994, equation (44), p. 793):

$$\sigma_z' = p_0 \left\{ \left[ (C_1 + C_2 kz) - \frac{2\lambda}{1-\mu} C_2 \right] e^{kz} + \left[ (C_3 + C_4 kz) + \frac{2\lambda}{1-\mu} C_4 \right] e^{-kz} + \right.$$
$$\left. + \frac{1}{1-2\mu} \left[ \delta^2(1-\mu) - k^2\mu \right] \left( C_5 e^{\delta z} + C_6 e^{-\delta z} \right) \right\} e^{i(kx+\omega t)}, \qquad \text{(A-1a)}$$

– Jeng (2013, equation (3.57), p. 45) (assuming 2D case: $n = 0$ and $\cos(nky) = 1$) and Jeng (2018, equation (A.6), p. 316):

$$\sigma_z' = p_0 \left\{ \left[ (C_1 + C_2 kz) - \frac{2\lambda}{1-\mu} C_2 \right] e^{kz} + \left[ (C_3 + C_4 kz) - \frac{2\lambda}{1-\mu} C_4 \right] e^{-kz} + \right.$$
$$\left. + \frac{1}{1-2\mu} \left[ \delta^2(1-\mu) - k^2\mu \right] \left( C_5 e^{\delta z} + C_6 e^{-\delta z} \right) \right\} e^{i(kx+\omega t)}. \qquad \text{(A-1b)}$$

*A.2. Solution coefficient $C_{01}$*

– Hsu and Jeng (1994, equation (60), p. 803) and Jeng (2018, equation (A.11), p. 317):

$$C_{01} = -2\delta \left[ \left( k^2\mu - \delta^2 + \delta^2\mu \right)^2 + k^4(1-2\mu)^2 + 2k^2h^2(1-\mu)^2 \left( \delta^2 - k^2 \right)^2 \right] +$$
$$+ 4k^2h \left( \delta^4 - k^4 \right) (1-2\mu)(1-\mu) + \lambda B_8, \qquad \text{(A-2a)}$$

– Jeng (2013, equation (3.124), p. 73):

$$C_{01} = -2\delta \left[ \left( k^2\mu - \delta^2 + \delta^2\mu \right)^2 + k^4(1-2\mu)^2 + 2k^2h^2(1-\mu)^2 \left( \delta^2 - k^2 \right)^2 \right] +$$
$$+ 4k^2h \left( \delta^4 - k^4 \right) (1-2\mu) + \lambda B_8. \qquad \text{(A-2b)}$$

*A.3. Solution coefficient $C_{03}$*

– Hsu and Jeng (1994, equation (62), p. 803) and Jeng (2018, equation (A.13), p. 317):

$$C_{03} = (\delta + k)^2 (\delta - \delta\mu - k\mu - k\lambda) (B_2 + \lambda B_{10}), \qquad \text{(A-3a)}$$

– Jeng (2013, equation (3.126), p. 73):

$$C_{03} = (\delta + k)^2 (\delta - \delta\mu - k\mu k\lambda) (B_2 + \lambda B_{10}). \qquad \text{(A-3b)}$$

*A.4. Solution coefficient* $C_{11}$

– Hsu and Jeng (1994, equation (68), p. 803) and Jeng (2018, equation (A.19), p. 318):

$$C_{11} = 2k^2h(\delta + k)(\delta - \delta\mu - k\mu)B_3 + \lambda\left(B_{11} + \lambda B_{12}\right), \tag{A-4a}$$

– Jeng (2013, equation (3.132), p. 73):

$$C_{11} = 2k^2h(\delta + k)(\delta - \delta\mu - k\mu)B_3 + \lambda\left(B_{11} + B_{12}\right). \tag{A-4b}$$

*A.5. Solution coefficient* $C_{12}$

– Hsu and Jeng (1994, equation (69), p. 803):

$$C_{12} = 4\delta k^3h(1 - 2\mu)\left(\delta^2 - \delta^2\mu - k^2\mu\right) + \lambda B_{13}, \tag{A-5a}$$

– Jeng (2013) — $C_{12}$ non-available,
– Jeng (2018, equation (A.20), p. 318):

$$C_{12} = 4\delta k^2h(1 - 2\mu)\left(\delta^2 - \delta^2\mu - k^2\mu\right) + \lambda B_{13}. \tag{A-5b}$$

*A.6. Solution coefficient* $C_{21}$

– Hsu and Jeng (1994, equation (74), p. 804) and Jeng (2018, equation (A.25), p. 318):

$$C_{21} = C_{03} + (\delta + k)\left[(\delta + k)(\delta - \delta\mu - k\mu)B_5 + \lambda\left(B_{14} + B_{15}\right)\right], \tag{A-6}$$

– Jeng (2013) — $C_{21}$ non-available.

*A.7. Solution coefficient* $C_{22}$

– Hsu and Jeng (1994, equation (75), p. 804) and Jeng (2018, equation (A.26), p. 318):

$$C_{22} = 4\delta k^2(1 - 2\mu)\left[2\delta^2(1 - \mu) - 2k^2\mu - kh(1 - \mu)\left(\delta^2 - k^2\right)\right] + \lambda B_{16}, \tag{A-7a}$$

– Jeng (2013, equation (3.137), p. 74):

$$C_{22} = 4\delta k^2(1 - 2\mu)\left[2\delta^2(1 - \mu) - 2k^2\mu - kh(1 - \mu)\left(\delta^2 - k^2\right)\right]. \tag{A-7b}$$

*A.8. Solution coefficient* $C_{32}$

– Hsu and Jeng (1994, equation (80), p. 804) and Jeng (2018, equation (A.31), p. 318):

$$C_{32} = -2\delta k^2\lambda\left[(kh - \lambda)\left(\delta^2 - k^2\right)(1 - \mu) - \delta^2(1 - \mu) + k^2\mu + \lambda B_9\right], \tag{A-8a}$$

– Jeng (2013, equation (3.142), p. 74):

$$C_{32} = -2\delta^2k\lambda\left[(kh - \lambda)\left(\delta^2 - k^2\right)(1 - \mu) - \delta^2(1 - \mu) + k^2\mu + \lambda B_9\right]. \tag{A-8b}$$

*A.9. Solution coefficient $C_{37}$*

– Hsu and Jeng (1994, equation (84), p. 804 and equation (67), p. 803) and Jeng (2018, equation (A.35), p. 318 and equation (A.18), p. 317):

$$C_{37} = C_{10}, \tag{A-9a}$$

$$C_{10} = -\lambda(\delta - k)\left[-\delta(\delta - 2k)(1 - \mu) + k^2\mu\right](B_1 + \lambda B_7)\,, \tag{A-9b}$$

– Jeng (2013) — $C_{37}$ non-available.

*A.10. Solution coefficient $C_{51}$*

– Hsu and Jeng (1994, equation (92), p. 804) and Jeng (2018, equation (A.43), p. 319):

$$C_{51} = -4k^2h(1 - 2\mu)B_3 + 2\lambda\left[B_{22} - 2k^4h(1 - 2\mu)\right], \tag{A-10a}$$

– Jeng (2013, equation (3.153), p. 74):

$$C_{51} = -4k^2h(1 - 2\mu)B_3 + 2\lambda\left[B_{22} - k^4h(1 - 2\mu)\right]. \tag{A-10b}$$

*A.11. Solution coefficient $C_{52}$*

– Hsu and Jeng (1994, equation (93), p. 804) and Jeng (2018, equation (A.43), p. 319):

$$C_{52} = -2k(1 - \lambda - 2\mu)\left[kh(\delta + k)(\delta - \delta\mu - k\mu) + \lambda B_{23}\right], \tag{A-11a}$$

– Jeng (2013, equation (3.154), p. 74):

$$C_{52} = -2k(1 - \lambda - 2\mu)\left[-kh(\delta - k)(\delta - \delta\mu + k\mu) + \lambda B_{23}\right]. \tag{A-11b}$$

*A.12. Solution coefficient $C_{53}$*

– Hsu and Jeng (1994, equation (94), p. 804) and Jeng (2018, equation (A.45), p. 319):

$$C_{53} = \lambda(\delta + k)(B_2 + \lambda B_{10})\,, \tag{A-12}$$

– Jeng (2013) — $C_{53}$ non-available.

*A.13. Solution coefficient $C_{64}$*

– Hsu and Jeng (1994, equations (97) and (94), p. 804):

$$C_{64} = C_{53}, \tag{A-13a}$$

$$C_{53} = \lambda(\delta + k)(B_2 + \lambda B_{10})\,, \tag{A-13b}$$

– Jeng (2013, equation (3.157), p. 74):

$$C_{64} = -C_{53}, \tag{A-13c}$$

$C_{53}$ non-available,

– Jeng (2018, equations (A.48) and (A.45), p. 319):

$$C_{64} = -C_{53}, \tag{A-13d}$$
$$C_{53} = \lambda(\delta + k)\,(B_2 + \lambda B_{10})\,. \tag{A-13e}$$

*A.14. Solution coefficient $B_5$*

– Hsu and Jeng (1994, equation (107), p. 805) and Jeng (2018, equation (A.57), p. 319):

$$B_5 = 2\delta kh(\delta - k)(1 - \mu), \tag{A-14a}$$

– Jeng (2013, equation (3.166), p. 75):

$$B_5 = 2\delta kh(\delta k)(1 - \mu). \tag{A-14b}$$

*A.15. Solution coefficient $B_9$*

– Hsu and Jeng (1994, equation (111), p. 508) and Jeng (2018, equation (A.61), p. 320):

$$B_9 = -\delta^2(1 - \mu) + k^2(2 - \mu), \tag{A-15a}$$

– Jeng (2013, equation (3.170), p. 75):

$$B_9 = -\delta^2(1\mu) + k^2(2 - \mu). \tag{A-15b}$$

*A.16. Solution coefficient $B_{11}$*

– Hsu and Jeng (1994, equation (113), p. 805) and Jeng (2018, equation (A.63), p. 320):

$$\begin{aligned} B_{11} = {} & k^5(1 - \lambda - 2\mu) - 2k^6 h\mu + \\ & - \delta^2 k^3(1 - \mu)\,[1 + 2kh(1 - 2\mu) + k\lambda(5 - 4\mu)] + \\ & + 2\delta^4 k(1 - \mu)^2(1 + 2kh + 2\lambda), \end{aligned} \tag{A-16a}$$

– Jeng (2013, equation (3.172), p. 75):

$$\begin{aligned} B_{11} = {} & k^5(1 - \lambda - 2\mu)02k^6 h\mu + \\ & - \delta^2 k^3(1 - \mu)\,[1 + 2kh(1 - 2\mu) + k\lambda(5 - 4\mu)] + \\ & + 2\delta^4 k(1 - \mu)^2(1 + 2kh + \lambda). \end{aligned} \tag{A-16b}$$

*A.17. Solution coefficient $B_{12}$*

– Hsu and Jeng (1994, equation (114), p. 805) and Jeng (2018, equation (A.64), p. 320):

$$B_{12} = \delta k^4 \left[2kh\left(1-\mu-\mu^2\right) + \lambda\left(2+\mu-2\mu^2\right) - \left(2-6\mu+3\mu^2\right)\right] + \\ - \delta^3 k^2 (1-\mu)(2kh+3\lambda+2\mu) + \delta^5 (1-\mu)^2 (1+2kh+2\lambda), \tag{A-17a}$$

– Jeng (2013, equation (3.173), p. 75):

$$B_{12} = \delta k^4 \left[2kh\left(1-\mu-mu^2\right) + \lambda\left(2+\mu-\mu^2\right) - \left(2-6\mu+3\mu^2\right)\right] + \\ - \delta^3 k^2 (1-\mu)(2kh+3\lambda+2\mu) + \delta^5 (1-\mu)^2 (1+2kh+2\lambda). \tag{A-17b}$$

*A.18. Solution coefficient $B_{13}$*

– Hsu and Jeng (1994, equation (115), p. 805) and Jeng (2018, equation (A.64), p. 320):

$$B_{13} = 2k^4\delta \left[kh\left(3-5\mu+4\mu^2\right) - \lambda\left(5-10\mu+4\mu^2\right) - \mu(3-4\mu)\right] + \\ + k^2\delta^3(1-\mu)\left[(3-4\mu) - 2kh(5-4\mu) + 2\lambda(1-2\mu)\right], \tag{A-18a}$$

– Jeng (2013, equation (3.174), p. 75):

$$B_{13} = 2k^4\delta \left[kh\left(2-5\mu+4\mu^2\right) - \lambda\left(5-10\mu+4\mu^2\right) - \mu(3-4\mu)\right] + \\ + k^2\delta^3(1-\mu)\left[(3-4\mu) - 2kh(5-4\mu) + 2\lambda(1-2\mu)\right]. \tag{A-18b}$$

*A.19. Solution coefficient $B_{18}$*

– Hsu and Jeng (1994, equation (120), p. 805) and Jeng (2018, equation (A.70), p. 320):

$$B_{18} = -k^4\mu(1-\lambda-2\mu) + 2\delta k^4 h(1-\mu-\mu^2) + \\ + \delta^2 k^2 (1-\mu)[1+\lambda(5-4\mu)] - 2\delta^3 k^2 h(1-\mu) + \\ - 2\delta^4 (1+2\lambda)(1-\mu)^2 + 2\delta^5 h(1-\mu)^2, \tag{A-19a}$$

– Jeng (2013, equation (3.179), p. 76):

$$B_{18} = -k^4\mu(1-\lambda-2\mu) + 2\delta k^4 h(1-\mu-\mu^2) + \\ - \delta^2 k^2 (1-\mu)[1+\lambda(5-4\mu)] - 2\delta^3 k^2 h(1-\mu) + \\ - 2\delta^4 (1+2\lambda)(1-\mu)^2 + 2\delta^5 h(1-\mu)^2. \tag{A-19b}$$