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The eigenstructure of Beta operators with Jacobi weights [1]

Sergiu-Vlad Paşca

Abstract

This paper investigates the eigenstructure of Beta-type operators belonging to a general family parameterized by $\alpha > -1$ and $\beta > -1$. Explicit expressions for the eigenvalues are derived, and a recurrence relation for the coefficients of the corresponding eigenpolynomials is provided.

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1 Introduction

This study deals with the eigenstructure of certain positive linear operators. First of all we will recall the results obtained by Cooper and Waldron [1] concerning the eigenstructures for the classical Bernstein operators

Bernstein operator $B_n : C[0,1] \to C[0,1]$, $n = 1, 2, \dots$, defined

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),$$

reproduces the linear polynomials, which are therefore eigenfunctions corresponding to the eigenvalue 1. Its eigenvalues are (see [1])

$$\lambda_k^{(n)} := \frac{n!}{(n-k)!} \cdot \frac{1}{n^k}, \quad k = 0, 1, \dots, n,$$

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and the corresponding eigenfunctions $p_k^{(n)}$ are monic polynomials of degrees k, which have k simple zeros in $[0,1]$.

Denote by $(x)_j := x(x+1)\dots(x+j-1), \quad j=1,2,\dots, \quad (x)_0 := 1$ the shifted factorial function and

$$S(k,j) = \frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}i^k, \quad 0 \le j \le k$$

the Stirling numbers of the second kind.

The eigenfunction for $\lambda_k^{(n)}$ is polynomial of degree k given by (see [1])

$$p_k^{(n)}(x) = \sum_{j=0}^{k} c(j,k,n)\cdot x^j = x^k - \frac{k}{2}x^{k-1} + \text{lower order terms},$$

where the coefficients can be computed using the recurrence formula

$$c(k,k,n) := 1, \quad c(k-1,k,n) := -k/2,$$

$$c(k-j,k,n) := \frac{1}{(n-k+1)_j - n^j}\sum_{i=0}^{j-1} n^i \cdot S(k-i,k-j)\cdot c(k-i,k,n),$$

$j = 2,\dots,k$.

2 Main Results

In this section we investigate the eigenstructure of the operators introduced by Muhlbach [5] and Lupaş [4]. In order to obtain these results we use a technique similar as in [1] and [2] .

Definition 1 *([3]) For $f \in C[0,1]$ and $x \in [0,1]$ these operators are defined as*

(i) in case $\alpha = \beta = -1$:

$$\mathcal{B}_n^{-1,-1}(f;x) = \begin{cases} f(0), & x = 0, \\ \dfrac{\displaystyle\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}f(t)dt}{B(nx,n(1-x))}, & x \in (0,1), \\ f(1), & x = 1; \end{cases}$$

(ii) in case $\alpha = -1$, $\beta > -1$:

$$\mathcal{B}_n^{-1,\beta}(f;x) = \begin{cases} f(0), & x = 0, \\ \dfrac{\displaystyle\int_0^1 t^{nx-1}(1-t)^{n(1-x)+\beta}f(t)dt}{B(nx,n(1-x)+\beta+1)}, & 0 < x \le 1; \end{cases}$$

(iii) in case $\alpha > -1$, $\beta = -1$:

$$\mathcal{B}_n^{\alpha,-1}(f;x) = \begin{cases} \dfrac{\displaystyle\int_0^1 t^{nx+\alpha}(1-t)^{n(1-x)-1} f(t)dt}{B(nx+\alpha+1, n(1-x))}, & 0 \le x < 1; \\ f(1), & x = 1; \end{cases}$$

(iv) in case $\alpha, \beta > -1$:

$$\mathcal{B}_n^{\alpha,\beta}(f;x) = \frac{\displaystyle\int_0^1 t^{nx+\alpha}(1-t)^{n(1-x)+\beta} f(t)dt}{B(nx+\alpha+1, n(1-x)+\beta+1)}, \quad 0 \le x \le 1.$$

Remark 1 *The case $\alpha = \beta = -1$ were considered by Muhlbach [5] and Lupaş [4]. For this particular case the operators are denoted by $\overline{\mathbb{B}}_n$. For $\alpha = \beta = 0$ the operators were introduced by Lupaş [4] and it is used the notation $\mathbb{B}_n$.*

In the following we study the eigenstructure of the operators $\mathcal{B}_n^{\alpha,\beta}$ for the case $\alpha, \beta > -1$.

As for $\alpha, \beta > -1$, for all $k \ge 0$,

$$\mathcal{B}_n^{\alpha,\beta} e_0 = e_0,$$

(1) $$\mathcal{B}_n^{\alpha,\beta} e_k(x) = \frac{(nx+\alpha+1)^{\overline{k}}}{(n+\alpha+\beta+2)^{\overline{k}}}, \quad x \in [0,1],\ k \ge 1.$$

From this we get

$$\mathcal{B}_n^{\alpha,\beta} e_k(x) = \frac{1}{(n+\alpha+\beta+2)^{\overline{k}}} \sum_{j=0}^{k} s_{k-j}(k,\alpha) n^j x^j,$$

where $s_{k-j}(k,\alpha)$ are elementary symmetric sums of the numbers $\alpha+1, \alpha+2, \ldots, \alpha+k$; in particular $s_0(k,\alpha) = 1$ and

$$s_1(k,\alpha) = (\alpha+1) + \cdots + (\alpha+k) = k\alpha + \frac{k(k+1)}{2}.$$

It follows that the numbers

$$\lambda_{n,k} := \frac{n^k}{(n+\alpha+\beta+2)^{\overline{k}}}, \quad k \ge 0,$$

are the eigenvalues of $\mathcal{B}_n^{\alpha,\beta}$, and to each of them there corresponds a monic eigenpolynomial $q_{n,k}$ with $\deg q_{n,k} = k$.

By direct computation it is easy to find the first eigenpolynomials of $\mathcal{B}_n^{\alpha,\beta}$. Starting from the relations

$$\mathcal{B}_n^{\alpha,\beta} q_{n,0} = \lambda_{n,0}^{\alpha,\beta} \cdot q_{n,0},$$
$$\mathcal{B}_n^{\alpha,\beta} q_{n,1} = \lambda_{n,1}^{\alpha,\beta} \cdot q_{n,1},$$
$$\mathcal{B}_n^{\alpha,\beta} q_{n,2} = \lambda_{n,2}^{\alpha,\beta} \cdot q_{n,2},$$

we get

$$\begin{aligned} q_{n,0}(x) &= 1, \\ q_{n,1}(x) &= x - \frac{\alpha+1}{\alpha+\beta+2}, \\ q_{n,2}(x) &= x^2 - \frac{2\alpha+3}{\alpha+\beta+3}x \\ &\quad + \frac{(\alpha^2+\alpha\beta+2\alpha n+4\alpha+\beta+3n+3)(\alpha+1)}{(\alpha+\beta+3)(\alpha^2+2\alpha\beta+2\alpha n+\beta^2+2n\beta+5\alpha+5\beta+5n+6)}. \end{aligned}$$

Let $k \geq 2$ and $n \geq 1$. We want to determine $q_{n,k} \in \Pi_k$ such that

$$\mathcal{B}_n^{\alpha,\beta} q_{n,k} = \lambda_{n,k} \cdot q_{n,k}.$$

We put

$$q_{n,k}(x) = \sum_{j=0}^{k} a(n,k,j)x^j, \quad \text{with} \quad a(n,k,k) = 1.$$

Hence

$$\mathcal{B}_n^{\alpha,\beta}(q_{n,k};x) = \sum_{j=0}^{k} a(n,k,j)\mathcal{B}_n^{\alpha,\beta}(e_j;x).$$

From (1) we derive

$$\begin{aligned} \mathcal{B}_n^{\alpha,\beta}(q_{n,k};x) &= \sum_{j=0}^{k} a(n,k,j)\frac{(nx+\alpha+1)^{\overline{j}}}{(n+\alpha+\beta+2)^{\overline{j}}} \\ &= \frac{n^k}{(n+\alpha+\beta+2)^{\overline{k}}}\sum_{j=0}^{k} a(n,k,j)x^j. \end{aligned} \tag{2}$$

From the definition of the Stirling numbers of the first kind $s(j,i)$, we obtain immediately

$$\begin{aligned} (nx+\alpha+1)^{\overline{j}} &= (nx+\alpha+1)(nx+\alpha+2)\dots(nx+\alpha+1+j-1) \\ &= \sum_{i=0}^{j} s(j,i)(-1)^{j-i}(nx+\alpha+1)^i, \end{aligned}$$

so that (2) becomes:

$$\sum_{j=0}^{k} a(n,k,j) \frac{\sum_{i=0}^{j} s(j,i)(-1)^{j-i}(nx+\alpha+1)^{i}}{(n+\alpha+\beta+2)^{\overline{j}}} \quad =$$

$$\frac{n^k}{(n+\alpha+\beta+2)^{\overline{k}}} \sum_{j=0}^{k} a(n,k,j)x^j.$$

$$\sum_{i=0}^{k} \left\{ \sum_{j=i}^{k} \frac{s(j,i)(-1)^{j-i}}{(n+\alpha+\beta+2)^{\overline{j}}} a(n,k,j) \right\} (nx+\alpha+1)^{i} \quad =$$

$$\sum_{i=0}^{k} \frac{a(n,k,i)n^k}{(n+\alpha+\beta+2)^{\overline{k}}} x^i.$$

This leads to

$$\sum_{i=0}^{k} \left\{ \sum_{j=i}^{k} \frac{s(j,i)(-1)^{j-i}}{(n+\alpha+\beta+2)^{\overline{j}}} a(n,k,j) \right\} \sum_{\nu=0}^{i} \binom{i}{\nu} n^{\nu} x^{\nu} (\alpha+1)^{i-\nu} =$$

$$\sum_{\nu=0}^{k} \frac{a(n,k,\nu)n^k}{(n+\alpha+\beta+2)^{\overline{k}}} x^{\nu}.$$

Denote

$$E_{i,k} := \sum_{j=i}^{k} \frac{s(j,i)(-1)^{j-i}}{(n+\alpha+\beta+2)^{\overline{j}}} a(n,k,j).$$

Therefore,

$$\sum_{i=0}^{k} \sum_{\nu=0}^{i} E_{i,k} \binom{i}{\nu} n^{\nu} x^{\nu} (\alpha+1)^{i-\nu} = \sum_{\nu=0}^{k} \frac{a(n,k,\nu)n^k}{(n+\alpha+\beta+2)^{\overline{k}}} x^{\nu}.$$

Changing the order of summation we can write

$$\sum_{\nu=0}^{k} \left(\sum_{i=\nu}^{k} E_{i,k} \binom{i}{\nu} n^{\nu} (\alpha+1)^{i-\nu} \right) x^{\nu} = \sum_{\nu=0}^{k} \frac{a(n,k,\nu)n^k}{(n+\alpha+\beta+2)^{\overline{k}}} x^{\nu}.$$

Equating the coefficients of x^{ν} one gets

$$\sum_{i=\nu}^{k} E_{i,k}\binom{i}{\nu} n^{\nu}(\alpha+1)^{i-\nu} = \frac{a(n,k,\nu)n^k}{(n+\alpha+\beta+2)^{\overline{k}}},$$

for all $\nu = 0, 1, \cdots, k$. Recalling the signification of $E_{i,k}$ it follows that

$$\sum_{i=\nu}^{k}\sum_{j=i}^{k} \frac{s(j,i)(-1)^{j-i}}{(n+\alpha+\beta+2)^{\overline{j}}} a(n,k,j)\binom{i}{\nu} n^{\nu}(\alpha+1)^{i-\nu} = \frac{a(n,k,\nu)n^k}{(n+\alpha+\beta+2)^{\overline{k}}},$$

for all $\nu = 0, 1, \cdots, k$.

Once again changing the order of summation we have

$$\sum_{j=\nu}^{k}\sum_{i=\nu}^{j} \frac{s(j,i)(-1)^{j-i}}{(n+\alpha+\beta+2)^{\overline{j}}} a(n,k,j)\binom{i}{\nu} n^{\nu}(\alpha+1)^{i-\nu} = \frac{a(n,k,\nu)n^k}{(n+\alpha+\beta+2)^{\overline{k}}}.$$

Denote

$$U_{j,\nu} := \sum_{i=\nu}^{j} s(j,i)(-1)^{j-i}\binom{i}{\nu}(\alpha+1)^{i-\nu}.$$

Therefore,

$$\sum_{j=\nu}^{k} \frac{U_{j,\nu} n^{\nu}}{(n+\alpha+\beta+2)^{\overline{j}}} a(n,k,j) = \frac{a(n,k,\nu)n^k}{(n+\alpha+\beta+2)^{\overline{k}}},$$

for all $\nu = 0, 1, \cdots, k$.

This can be written as

$$a(n,k,\nu)\left[\frac{n^k}{(n+\alpha+\beta+2)^{\overline{k}}} - \frac{U_{\nu,\nu}n^{\nu}}{(n+\alpha+\beta+2)^{\overline{\nu}}}\right] = \sum_{j=\nu+1}^{k} \frac{U_{j,\nu}n^{\nu}}{(n+\alpha+\beta+2)^{\overline{j}}} a(n,k,j),$$

i.e.,

$$\text{(3)}\quad a(n,k,\nu) = \frac{\displaystyle\sum_{j=\nu+1}^{k} U_{j,\nu}(n+\alpha+\beta+2+j)\dots(n+\alpha+\beta+2+k-1)a(n,k,j)}{n^{k-\nu} - U_{\nu,\nu}(n+\alpha+\beta+2+\nu)\dots(n+\alpha+\beta+2+k-1)},$$

for $\nu \in \{k-1, k-2, \dots, 0\}$.

Recall that n and k are given and $a(n,k,k) = 1$. So we have obtained the following result.

Proposition 1 *The relation (3) represents a reccurence relation for computing $a(n,k,\nu)$, $\nu = k-1, k-2, \dots, 0$.*

In particular, using

$$s(k,k-1) = -\frac{k(k-1)}{2},$$

we get

$$a(n,k,k-1) = -\frac{k(k-1+2(\alpha+1))}{2(\alpha+\beta+2+k-1)}.$$

Remark 2 *In the limiting case when $\alpha \to -1$, $\beta \to -1$ the results recover several results from [2]. On the other hand the authors of [2] determined the limit of $a(n,k,\nu)$, when $n \to \infty$, in the case $\alpha = \beta = -1$.*

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Lucian Blaga University of Sibiu,
Faculty of Science
Department of Mathematics and Informatics,
Romania,
e-mail: sergiu.pasca@ulbsibiu.ro