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Mixed harmonic directional variational inequalities [1]

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Abstract

In this paper, we introduce and investigate some new classes of mixed harmonic directional variational inequalities. Several important new problems are obtained as special cases. The auxiliary principle technique is applied to suggest and analyze some hybrid inertial iterative methods for solving the mixed harmonic directional variational inequality. The convergence analysis of these iterative methods is also considered under some mild conditions. Some special cases are also pointed out. Results proved in this paper can be viewed as a significant refinement and improvement of the known results.

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1 Introduction

Variational inequalities were introduced by Stamapcchia [31] and Lions et al. [12] to study the elliptic equations subject to the obstacles. They proved that the optimality conditions of the energy(virtual work, potential) functions on the convex set can be studied via an the inequality. This inequality is called the variational inequality. Variational inequalities can be viewed as a novel extension and generalization of the variational principles. Variational inequality describes a broad spectrum of very interesting developments involving a link among various fields of mathematics,

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physics, economics, regional and engineering sciences. It is worth mentioning that this variational inequalities theory allows many diversified applications. In fact, this theory provides us the most natural, direct, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems. For the applications, motivations, generalizations, extensions, numerical methods, dynamical systems, sensitivity analysis, error bounds and related optimization problems, see [1, 2, 4, 9, 10, 12, 13, 14, 15, 16, 18, 20, 22, 23, 25, 26, 27, 28, 29, 31] and the references therein.

Noor and Noor [20] proved that the minimum of the differentiable harmonic convex function on the harmonic convex set can be characterized by the harmonic variational inequality. For the applications and generalizations of the harmonic variational inequalities, see [1, 2, 11, 17, 20, 21, 22, 23, 24] and the references therein.

Motivated and inspired by the ongoing research in this dynamic field, we introduce and study mixed harmonic directional variational inequalities, which is the main aim of this paper. We remark that the projection and resolvent type method cannot be used to suggest the iterative methods for solving such type of problems due to their nonlinear structure. To overcome this drawback, we use the auxiliary principle technique, which is mainly due to Lions and Stapacchia [12]. Glowinski et al. [10] applied this technique to study the existence of a solution of the mixed variational inequalities. Noor [13, 15, 18] and Noor et. [1, 2, 10, 13, 20, 21, 22, 25, 26] have used this technique to develop some iterative schemes for solving various classes of variational inequalities and equilibrium problems. We point out that this technique does not involve any projection and resolvent of the operator and is flexible. In this paper, we show that the auxiliary principle technique can be applied to suggest and analyze some new classes of inertial iterative methods for solving mixed harmonic directional variational inequalities. The inertial type methods was suggested by Polyak [30] to speed up the convergence of iterative methods. We also prove that the convergence of these new methods requires pseudomonotonicity, which is weaker conation than monotonicity. As special cases, one can obtain several known and new results for harmonic variational inequalities, variational inequalities and related optimization problems. Results obtained in this paper, represent an improvement and refinement of the known results for harmonic variational inequalities and their variant forms. It is perhaps part of the fascination of the subject that so many branches of pure and applied sciences are involved in the variational inequality theory.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let $j : \mathcal{H} \to R$ be a locally Lipschitz continuous function. First of all, we recall the following concepts and results from nonsmooth analysis, see [6, 8].

Definition 1 *[6] Let j be locally Lipschitz continuous at a given point $x \in \mathcal{H}$ and v be any other vector in $\mathcal{H}$. The Clarke's generalized directional derivative of j at x*

in the direction v, denoted by

$$j^0(x;v) = \lim_{t\to 0+} \sup_{h\to 0} \frac{j(x+h+tv) - j(x+h)}{t}.$$

The generalized gradient of j at x, denoted $\partial j(x)$, is defined to be subdifferential of the function $j^0(x;v)$ at 0. That is

$$\partial j(x) = \{w \in \mathcal{H} : \langle w,\ v\rangle \leq j^0(x;v),\ \forall v \in \mathcal{H}.\}$$

Lemma 1 *[6] Let j be a locally Lipschitz continuous at a given point $x \in \mathcal{H}$ with a constant L. Then*

(i) *$\partial j(x)$ is a non-empty compact subset of $\mathcal{H}$ and $\|\xi\| \leq L$ for each $\xi \in \partial j(x)$.*

(ii) *For every $v \in \mathcal{H}, j^0(x;v) = \max\{\langle \xi,\ c\rangle : \xi \in \partial j(x)\}$.*

(iii) *The function $v \to j^0(x;v)$ is finite, positively homogeneous, subadditive, convex and continuous.*

(iv) *$j^0(x;-v) = (-j)^0(x;v)$.*

(v) *$j^0(x;v)$ is upper semi-continuous as a function of $(x;v)$.*

(vi) *$\forall x \in \mathcal{H}$, there exists a constant $\alpha > 0$ such that*

$$|j^0(x;v)| \leq \alpha\|v\|,\ \forall v \in \mathcal{H}.$$

If j is convex on C and locally Lipschitz continuous at $x \in C$, then $\partial j(x)$ coincides with the subdifferential $j'(x)$ of j at x in the sense of convex analysis, and $j^0(x;v)$ coincides with the directional derivative $j'(x;v)$ for each $v \in H$, that is, $j^0(x;v) = \langle j'(x), v\}$.

Definition 2 *[5] The set C_h is said to be a harmonic convex set, if*

$$\frac{uv}{v+\lambda(u-v)} \in C_h,\ \forall u,\ v \in C_h,\ \lambda \in [0,1].$$

For the applications of the harmonic means in circuit theory, risk analysis and related optimization programming, see [1, 3, 10, 28].

Definition 3 *[5] The function ϕ on the harmonic convex set C_h is said to be harmonic convex, if*

$$\phi\left(\frac{uv}{v+\lambda(u-v)}\right) \leq (1-\lambda)\phi(u) + \lambda\phi(v)\ ,\ \forall u,\ v \in C_h\ \lambda \in [0,1].$$

The function ϕ is said to be harmonic concave function, if and only if, $-\phi$ is harmonic convex function. For the applications, motivation, integral inequalities and other aspects, see [1, 2, 3, 5, 20, 21, 22, 24]. Consider the energy (virtual) functional

$$I[v] = F(v) - \phi(v), \tag{1}$$

where $F(v)$ and $\phi(v)$ are two harmonic convex functions.

We now consider the optimality conditions of the energy function $I[v]$ defined by (1) under suitable conditions.

Theorem 1 *Let F be a locally Lipschitz continuous harmonic convex function and ϕ be a harmonic convex function on the convex set C_h. If $u \in \mathcal{H}$ is the minimum of the functional $I[v]$ defined by* (1), *then*

$$F'\left(u; \frac{uv}{u-v}\right) + \phi(v) - \phi(u) \geq 0, \ \forall v, \ u \in \mathcal{H}. \tag{2}$$

Proof. Let $u \in \mathcal{H}$ be a minimum of the functional $I[v]$. Then

$$I[u] \leq I[v], \ \forall v \in \mathcal{H},$$

which implies that

$$F(u) - \phi(u) \leq F(v) - \phi(v) \ , \ \forall v \in \mathcal{H}. \tag{3}$$

Obviously, $\forall u, v \in \mathcal{H}, \lambda \in [0,1], v_t = \dfrac{uv}{(1-\lambda)v + \lambda u} \in \mathcal{H}.$

Taking $v = v_t$ in (3), we have

$$F(u) - \phi(u) \leq F(v_t) - \phi(v_t) \ , \ \forall v \in \mathcal{H}. \tag{4}$$

This implies that

$$0 \leq F\left(\frac{uv}{(1-\lambda)v + \lambda u}\right) - F(u) + \lambda(\phi(v) - \phi(u)) \geq 0 \ \forall v \in \mathcal{H}. \tag{5}$$

Dividing the above inequality by λ and taking limit as $\lambda \to 0$, we have

$$\begin{aligned} 0 &\leq \lim_{\lambda \to 0} \frac{F(\frac{uv}{(1-\lambda)v+\lambda u}) - F(u)}{\lambda} + \phi(v) - \phi(u) \\ &= F'\left(u; \frac{uv}{u-v}\right) + \phi(v) - \phi(u), \end{aligned}$$

which is the required (2).

Since F is a locally Lipschitz continuous harmonic convex function, *so*

$$F\left(\frac{uv}{v + \lambda(u-v))}\right) \ \leq \ F(u) + \lambda(F(v) - F(u)) \ , \ \forall u, v \in \mathcal{H},$$

from which, we have

$$F(v) - F(u) \geq \lim_{\lambda \to 0} \frac{F(\frac{uv}{(1-\lambda)v+\lambda u}) - F(u)}{\lambda} = F'\left(u; \frac{uv}{u-v}\right) \tag{6}$$

Now using (6), we obtain

$$\begin{aligned} I[v] - I[u] &= F(v) - F(u) + \phi(v) - \phi(u) \\ &\geq F'\left(u; \frac{uv}{u-v}\right) + \phi(v) - \phi(u) \geq 0. \end{aligned}$$

Consequently, it follows that $u \in \mathcal{H}$, such that

$$F(u) - \phi(u) \leq F(v) - \phi(v) \ , \ \forall v \in \mathcal{H},$$

which shows that $u \in \mathcal{H}$ is the minimum of the function $I[v]$ defined by (1).

Remark 1 *The inequality of the type (2) is called the mixed harmonic variational inequality.*

In many applications, the inequalities of the type (2) may not arise as a minimum of the difference of the two locally Lipschitz continuous harmonic convex functions. These facts motivated us to consider more general harmonic variational inequality, which contains the inequalities (2) as a special case.

For given nonlinear continuous bifunction $B(\cdot,\cdot), H \times H \to H$, and ϕ be a continuous function we consider the problem of finding $u \in \mathcal{H}$ such that

$$B\left(u; \frac{uv}{u-v}\right) + \phi(v) - \phi(u) \geq 0, \ \forall v \in \mathcal{H}. \tag{7}$$

which is called the mixed harmonic directional variational inequality.

We now discuss some new and known classes of harmonic directional variational inequalities and related optimization problems.

Special Cases

(I) if $B\left(u; \frac{uv}{u-v}\right) = \left\langle Tu, \frac{uv}{u-v}\right\rangle$, then the problem (7) reduces to finding $u \in C_h$ such that

$$\left\langle Tu, \ \frac{uv}{u-v}\right\rangle + \phi(v) - \phi(u) \geq 0, \forall v \in C_h, \tag{8}$$

which is called the mixed harmonic variational inequality.

(II) If $B\left(u; \frac{uv}{u-v}\right) = F'\left(u; \frac{uv}{u-v}\right)$, where $\phi'(u)$ denotes derivative of the harmonic convex function $\phi(u)$ in the direction $\frac{uv}{u-v}$, then problem (7) reduces to finding $u \in \mathcal{H}$, such that

$$F'\left(u; \frac{uv}{u-v}\right) + \phi(v) - \phi(u) \geq 0, \ \forall v \in \mathcal{H}, \tag{9}$$

which is called the mixed harmonic directional variational inequality.

(III) If ϕ is the indicator function of a closed convex set C_h in $\mathcal{H}$, then problem (8) reduces to finding $u \in C_h$ such that

$$B\left(u; \frac{uv}{u-v}\right) \geq \ 0, \ \forall v \in C_h, \tag{10}$$

is called the harmonic directional variational inequality, introduced and studied by Noor and Noor [20]

(IV) If $C_h^\star = \{u \in \mathcal{H} : \left\langle u, \ \frac{uv}{u-v}\right\rangle \geq 0, \forall v \in C_h\}$ is a polar harmonic convex cone of the harmonic convex C_h, then problem (10) is equivalent to fining $u \in \mathcal{H}$, such that

$$\frac{uv}{u-v} \in C_h, Tu \in C_h^\star, \left\langle Tu, \ \frac{uv}{u-v}\right\rangle = 0, \tag{11}$$

is called the harmonic complementarity problem. For the applications, numerical methods and other aspects of complementarity problems, see [7, 16, 18, 28] and the references therein.

(V) If $C_h = \mathcal{H}$, then problem (10) is equivalent to fining $u \in \mathcal{H}$, such that

$$\left\langle Tu, \ \frac{uv}{u-v}\right\rangle = 0, \forall v \in \mathcal{H}, \tag{12}$$

which is called the weak formulation of the harmonic boundary value problem.

(VI) For $Tu = T|u|$, the problem (12) reduces to finding $u \in \mathcal{H}$ such that

$$\left\langle T|u|, \ \frac{uv}{u-v}\right\rangle = 0, \ \forall v \in H, \tag{13}$$

which is called the system of absolute value harmonic equations.

(VII) If $\langle Tu = u - f$, then problem (12) reduces to finding $u \in \mathcal{H}$ such that

$$\left\langle u, \ \frac{uv}{v-u}\right\rangle = \left\langle f, \ \frac{uv}{v-u}\right\rangle, \ \forall v \in \mathcal{H}, \tag{14}$$

which is called the Riesz-Frechet representation theorem.

Remark 2 *For diffe rent and suitable choice of the bifunction, operators and the spaces, one can obtained several new and known classes of the harmonic variational inequalities and related optimization problems. This show that the problem (7) is quite general and unified one. Due to the structure and nonlinearity involved, one has to consider its own. It is an open problem to develop unified numerical implementation numerical methods for solving the harmonic variational inequalities*

3 Iterative Methods and Convergence Analysis

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can be applied for solving the harmonic variational inequalities. The inertial type iterative methods were suggested by Polyak [30] to speed the convergence analysis of the iterative methods. Alvarez [4] analyzed the weak convergence of the relaxed and inertial hybrid projection proximal point algorithm for maximal monotone operators in Hilbert space. For the applications of the inertial type methods for solving variational inequalities, variational inclusions and their variant forms, see [18, 25] and the references therein. To overcome these drawbacks, we apply the auxiliary principle technique, which is mainly due to Lions et al [12] and Glowinski et al. [10] as modified in [15, 18, 29] to suggest and analyze some inertial iterative methods for solving mixed harmonic directional variational inequalities (7).

For a given $u \in \mathcal{H}$ satisfying (7), consider the problem of finding $w \in \mathcal{H}$ such that

$$\rho B\left(w + \eta(u - w); \frac{vw}{w - v}\right) + \langle w - u, v - w\rangle + \rho(\phi(v) - \phi(w)) \geq 0, \quad \forall v \in \mathcal{H}, \tag{15}$$

where $\rho > 0, \eta \in [0, 1]$ are constants.

Inequality of type (15) is called the auxiliary harmonic directional variational inequality.

If $w = u$, then w is a solution of (7). This simple observation enables us to suggest the following iterative method for solving (7).

Algorithm 1 *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho B(u_{n+1} + \eta(u_n - u_{n+1} \ \frac{vu_{n+1}}{v - u_{n+1}}) + \langle u_{n+1} - u_n,\ v - u_{n+1}\rangle \\ \geq -\rho(\phi(v) - \phi(u_{n+1}))\ ,\ \forall v \in \mathcal{H}. \end{aligned} \tag{16}$$

Algorithm 1 is called the hybrid proximal point algorithm for solving the mixed harmonic directional variational inequalities (7).

Special Cases

We now consider some cases of Algorithm 1.

(I) For $\eta = 0$, Algorithm 1 reduces to:

Algorithm 2 *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B\left(u_{n+1}; \frac{vu_{n+1}}{v - u_{n+1}}\right) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in \mathcal{H}. \tag{17}$$

(II) If $\eta = 1$, then Algorithm 1 reduces to:

Algorithm 3 *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B\left(u_n; \frac{vu_{n+1}}{v - u_{n+1}}\right) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho(\phi(v) - \phi(u_{n+1})) \geq 0, \forall v \in \mathcal{H}.$$

(III) If $\eta = \frac{1}{2}$, then Algorithm 1 collapses to:

Algorithm 4 *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B\left(\frac{u_{n+1} + u_n}{2}; \frac{vu_{n+1}}{v - u_{n+1}}\right) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in \mathcal{H}.$$

which is called the mid-point proximal method for solving the problem (7).

(IV) If $B\left(u; \frac{uv}{u - v}\right) = \left\langle Tu, \frac{uv}{u - v}\right\rangle$, then Algorithm 1 reduces to:

Algorithm 5 *For a given $u_0 \in C_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho\left\langle T(u_{n+1} + \eta(u_n - u_{n+1}; \frac{vu_{n+1}}{v - u_{n+1}}\right\rangle + \langle u_{n+1} - u_n,\ v - u_{n+1}\rangle$$
$$+\rho(\phi(v) - \phi(u_{n+1})) \geq 0, \forall v \in \mathcal{H}$$

for solving mixed harmonic variational inequality.

Lemma 2 $\forall u, v \in \mathcal{H}$,

$$2\langle u,\ v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2. \tag{18}$$

Definition 4 *A bifunction $B(\cdot;\cdot)$ is said to be psedomontone harmonic with respect to the function ϕ if*

$$B\left(u; \frac{uv}{u-v}\right) + \phi(v) - \phi(u), \ \forall u, v \in \mathcal{H}$$

implies that

$$-B\left(v; \frac{uv}{v-u}\right) - \phi(v) + \phi(u) \geq 0, \ \forall v \in \mathcal{H}.$$

We now consider the convergence criteria of Algorithm 2.

Theorem 2 *Let $u \in \mathcal{H}$ be a solution of (7) and let u_{n+1} be the approximate solution obtained from Algorithm 2. Let the bifunction $B(\cdot,\cdot)$. be pseudomonotone with respect to the function $\phi(\cdot)$ Then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{19}$$

Proof. Let $u \in \mathcal{H}$ be a solution of (7). Then

$$-B\left(v; \frac{uv}{v-u}\right) - \phi(v) + \phi(u) \geq 0, \ \forall v \in \mathcal{H}. \tag{20}$$

since the bifunction $B(\cdot,\cdot)$. pseudomonotone with respect to the bifunction $\phi(\cdot,\cdot)$. Now taking $v = u_{n+1}$ in (20), we have

$$-B\left(u_{n+1}; \frac{uu_{n+1}}{u_{n+1}-u}\right) - \phi(u_{n+1}) - \phi(u) \geq 0, \ \forall v \in \mathcal{H}. \tag{21}$$

Taking $v = u$ in (17), we get

$$\rho B\left(u_{n+1}; \frac{uu_{n+1}}{u-u_{n+1}}\right) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle + \rho(\phi(u) - \phi(u_{n+1})) \geq 0, \forall v \in \mathcal{H}.$$

which can be written as

$$\begin{aligned} \langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho B\left(u_{n+1}; \frac{uu_{n+1}}{u-u_{n+1}}\right) \\ -\rho(\phi(u) - \phi(u_{n+1})) \geq 0, \end{aligned} \tag{22}$$

where we have used (21).
Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (18), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_{n+1} - u_n\|^2. \tag{23}$$

Combining (22) and (23), we have

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result (19).

Theorem 3 *Let H be a finite dimensional space and all the assumptions of theorem 2 hold. Then the sequence $\{u_n\}_0^\infty$ given by Algorithm 2 converges to a solution u of (7).*

Proof. Let $u \in \mathcal{H}$ be a solution of (7). From (19), it follows that the sequence $\{||u - u_n||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, *we* have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \le \|u_0 - u\|^2,$$

which implies that Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (18), we obtain

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{24}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_0^\infty$; whose subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_0^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (17), taking the limit $n_j \to \infty$ and using (24), we have

$$B\left(\hat{u}; \frac{\hat{u}v}{v - \hat{u}}\right) + \phi(v) - \phi(\hat{u}) \ge 0, \ \forall v \in \mathcal{H},$$

which implies that $\hat{u}$ solves the mixed harmonic directional variational inequality (7) and

$$\|u_{n+1} - u\|^2 \le \|u_n - u\|^2$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

the required result.

We again consider the modified auxiliary principle technique [15] to suggest some hybrid inertial proximal point methods for solving the problem (7).

For a given $u \in \mathcal{H}$ satisfying (7), consider the problem of finding $w \in \mathcal{H}$, such that

$$\begin{aligned} \rho B\left(w + \eta(u - w); \frac{uw}{u - w}\right) + \ \langle M(w) - M(u) + \alpha(u - u), \ v - w\rangle \\ +\rho(\phi(v) - \phi(w)) \ge 0, \ \forall v \in \mathcal{H}, \end{aligned} \tag{25}$$

where $\rho > 0, \alpha, \eta, \in [0, 1]$ are constants and M is an arbitrary operator.

Clearly, for $w = u, w$ is a solution of (7). This fact motivated us to to suggest the following inertial iterative method for solving (7).

Algorithm 6 *For given $u_0, u_1 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho B\left(u_{n+1} + \eta(u_n - u_{n+1}); \frac{vu_{n+1}}{v - u_{n+1}}\right) + \langle M(u_{n+1}) - M(u_n) + \alpha\langle u_n - u_{n-1}, v - u_{n+1}\rangle \\ \ge -\rho(\phi(v) - \phi(u_{n+1})) \ , \ \forall v \in \mathcal{H}. \end{aligned}$$

which is known as the inertial iterative method.

Note that, for $\alpha = 0M = I$, the identity operator, Algorithm 6 is exactly the Algorithm 1. Using essentially the technique of Theorem 2, Alshejari et al. [1, 2] and Noor et al. [28], one can study the convergence analysis of Algorithm 6.

If $\eta = \frac{1}{2}$, the Algorithm 6 reduces to:

Algorithm 7 *For given $u_0, u_1 \in C_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B\left(\frac{u_{n+1}+u_n}{2}; \frac{vu_{n+1}}{v-u_{n+1}}\right) + M(u_{n+1}) - M(u_n) + \alpha\langle u_n - u_{n-1},\ v - u_{n+1}\rangle \\ \geq -\rho(\phi(v) - \phi(u_{n+1}))\ ,\ \forall v \in \mathcal{H}.$$

which is known as the inertial mid-point iterative method.

Remark 3 *For different and appropriate values of the parameters η, α, the bifunction $B(\cdot,\cdot)$ operator M and spaces, one can obtain a wide class of inertial type iterative methods for solving the harmonic variational inequalities and related optimization problems.*

4 Conclusion

Some new classes of mixed harmonic directional variational inequalities are introduced and discussed. Several important problems such as harmonic complementarity problems, system of harmonic absolute value problems and related problems can be obtained as special cases. We have applied the auxiliary principle technique to suggest several inertial type methods for solving harmonic directional variational inequalities with suitable modifications. The convergence analysis of these new methods are analyzed under weaker conditions. We have only considered the theoretical aspects of the hybrid inertial iterative methods. It is an interesting problem to develop implementable numerical methods for solving the harmonic variational inequalities. One can explore the applications of the harmonic variational inequalities in different branches of pure and applied sciences.

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References

[1] A. A. Alshejari, M. A. Noor, K. I. Noor, *Inertial algorithms for bifunction harmonic variational inequalities*, Int. J. Anal. Appl., vol. 22, 2024, ID=46, 1-19, doi.org/10.28924/2291-8639-22-2024-46.

[2] A. A. Alshejari, M. A. Noor, K. I. Noor, *New auxiliary principle technique for generalharmonic diectional variational inequalities*, Int. J. Anal. Appl., vol. 22

[3] F. Al-Azemi, O. Calin, *Asian options with harmonic average*, Appl. Math. Inform. Sci., vol. 9, 2015, 1-9

[4] F. Alvarez, *Weak convergence of a relaxed and inertial hybrid projection- proximal point algorithm for maximal monotone operators in Hilbert space*, SIAM J. Optim., vol. 14, 2003, 773-782

[5] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl., vol. 335, 2007, 1294-1308

[6] F. H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York, 1983

[7] R. W. Cottle, Jong-S. Pang, R. E. Stone, *The Linear Complementarity Problem*, SIAM Publication, Philadelphia, USA, 1992

[8] G. Cristescu, L. Lupsa, *Non-Connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, Holland, 2002

[9] F. Giannessi, A. Maugeri, M. S. Pardalos, *Equilibrium Problems*: *Nonsmooth Optimization and va riational Inequality Models*, Kluwer Academic, Dordrecht, Holland, 2001

[10] R. Glowinski, J-L. Lions, R. Tremolieres, *Numerical Analysis of va riational Inequalities*, North-Holland, Amsterdam, Holland, 1981

[11] J. K. Kim, A. Lakhnotra, T. Ram, *General mixed harmonic variational inequalities*, Nonlinear Functional Analysis and Applications, vol. 29, no. 2, 2024, 517-526 https://doi.org/10.22771/nfaa.2024.29.02.12

[12] J. L. Lions, G. Stampacchia, *va riational inequalities*, Commun. Pure Appl. Math., vol. 20, 1967, 492-512

[13] M. A. Noor, *On va riational Inequalities*, PhD Thesis, Brunel University, London, UK, 1975

[14] M. A. Noor, *Fixed point approach for complementaritty problems*, J. Math. Anal. Appl., vol. 133, 1988, 437-448

[15] M. A. Noor, *General algorithm for variational inequalities*, J. Optim. Theory Appl., vol. 73, 1992, 409-413

[16] M. A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., vol. 251, 2000, 217-229

[17] M. A. Noor, K. I. Noor, *Some novel aspects and applications of Noor iteration and Noor orbits*, J. Advanced Math. Studies, vol. 17, no. 3, 2024, 276-284

[18] M. A. Noor, *Some developments in general variational inequalities*, Appl. Math. Comput., vol. 152, 2004, 199-277

[19] M. A. Noor, K. I. Noor, *Some new trends in mixed variational inequalities*, J. Advanc. Math. Studies, vol. 15, no. 2, 2022, 105-140

[20] M. A. Noor, K. I. Noor, *Harmonic variational inequalities*, Appl. Math. Inform. Sci., vol. 10, no. 5, 2016, 1811-1814

[21] M. A. Noor, K. I. Noor, *Some imlpicit shemses for harmonic variational inequalities*, Inter. J. Anal. Appl., vol. 12, no. 1, 2016, 10-14

[22] M. A. Noor, K. I. Noor, *Some New classes of harmonic hemivariational inequalities*, Earthline J. Math. Sci., vol. 13, no. 2, 2023, 473-495

[23] M. A. Noor, K. I. Noor, *Auxiliary principle technique for solving trifunction harmonic variational inequalities*, RAD HAZU Matemat.Znanost, vol. 28/558, 2024, In Press

[24] M. A. Noor, K. I. Noor, S. Iftikhar, *Strongly generalized harmonic convex functions and integral inequalities*, J. Math. Anal., vol. 7, no. 3, 2016, 66-77

[25] M. A. Noor, K. I. Noor, M. Th. Rassias, *New trends in general variational inequalitiess*, Acta Applicand. Math., vol. 170, no. 1, 2020, 981-1046

[26] M. A. Noor, K. I. Noor, Th. M. Rassias, *Some aspects of variational inequalities*, J. Comput. Appl. Math., vol. 47, 1993, 285-312

[27] M. A. Noor, K. I. Noor, Th. M. Rassias, Set-valued resolvent equations and mixed variational inequalities, J. Math. Anal. Appl., vol. 220, 1998, 741-759

[28] M. A. Noor, W. Oettli, *On general nonlinear complementarity problems and quasi equilibria*, Le Math.(Catania), vol. 49, 1994, 313-331

[29] M. Patriksson, *Nonlinear Programming and va riational Inequalities*: *A Unified Approach*, Kluwer Academic Publishers, Dordrecht, Holland, 1999

[30] B. T. Polyak, *Some methods of speeding up the convergence of iterative methods*, USSR Comput. Math. Math. Phys., vol. 4, 1964, 1-17

[31] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*, Comptes Rendus de l'Academie des Sciences, Paris, vol. 258, 1964, 4413-4416

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